Ph 206 LECTURE 4

ELECTROSTATIC POTENTIAL PROBLEMS

WE NOW EXAMINE SOME OF THE METHODS AVAILABLE FOR CALCULATING THE ELECTRIC FIELD $\bar{E}$ IN VARIOUS CIRCUMSTANCES. DIRECT CALCULATION OF $\bar{E}$ CAN OFTEN BE MADE IN SIMPLE CASES OF HIGH SYMMETRY BY THE INTEGRAL FORM OF GAUSS' LAW:

$$\int_{\text{SURFACE}} \bar{E} \cdot d\bar{S} = 4\pi \, Q_{\text{INSIDE}}$$

HERE WE CONCENTRATE ON MORE COMPLICATED SITUATIONS, AND WILL FIRST CALCULATE THE POTENTIAL ϕ. THEN WE CAN ALWAYS CALCULATE $\bar{E} = -\bar{\nabla}\phi$ AS AN 'ELEMENTARY' ALGEBRAIC EXCERCISE. SINCE ϕ IS A SCALAR FUNCTION, THIS APPROACH IS TYPICALLY QUICKER THAN DIRECT CALCULATIONS OF THE VECTOR $\bar{E}$.

WE ALREADY HAVE A FORMAL SOLUTION FOR THE POTENTIAL:

$$\phi = \int \frac{\rho}{r} \, d\text{vol}$$

WHERE ρ INCLUDES ALL CHARGES (FREE OR BOUND IN MOLECULAR DIPOLES..) AND THE INTEGRAL MUST BE TAKEN OVER ALL SPACE.

TYPICALLY WE ARE INTERESTED IN $\bar{E}$ AND ϕ ONLY IN A LIMITED REGION OF SPACE — OFTEN DEFINED BY PHYSICAL BOUNDARIES WHICH ARE CONDUCTORS. CAN WE AVOID THE NEED OF KNOWING ρ OVER ALL SPACE, AND GET BY WITH INFORMATION RESTRICTED TO THE VOLUME OF INTEREST?

IT IS VERY REASONABLE THAT THE ANSWER IS YES, AS CAN BE DEMONSTRATED WITH SOME THEOREMS DUE TO GREEN:

$$: \int \bar{\nabla} \cdot (\phi \bar{\nabla} \psi - \psi \bar{\nabla} \phi) \, d\text{vol}$$

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, d\text{vol} = \int_S (\phi \bar{\nabla} \psi - \psi \bar{\nabla} \phi) \cdot d\bar{S} \equiv \int_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) dS$$

JACKSON (1.8)

[C.F. BECKER (10-17)]. AS SUGGESTED IN PROB ①, SET 1, CHOOSE

$\psi = \dfrac{1}{R} = \dfrac{1}{|\bar{r} - \bar{r}'|}$, PUT $\phi = \phi(\bar{r}')$ AND INTEGRATE OVER $\bar{r}'$:

NOTING THAT $\nabla^2 \psi = \nabla^2 \dfrac{1}{R} = -4\pi \, \delta(R) = -4\pi \, \delta(\bar{r} - \bar{r}')$

AND THAT $\nabla^2 \phi = \nabla \cdot (\nabla \phi) = -\bar{\nabla} \cdot \bar{E} = -4\pi \rho(\bar{r}')$, WE GET

$$\int_V \left[-4\pi \, \phi(\bar{r}') \, \delta(\bar{r} - \bar{r}') + \frac{4\pi}{R} \rho(\bar{r}')\right] d\text{vol}' = \int_S \left[\phi \frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R} \frac{\partial \phi}{\partial n'}\right] dS'$$

SO FOR THE CASE WHERE POINT $\bar{r}$ LIES WITHIN THE VOLUME,

$$\phi(\bar{r}) = \int_V \frac{\rho(\bar{r}')}{R}\, dvol' + \frac{1}{4\pi}\int_S \left[\frac{1}{R}\frac{\partial \phi}{\partial n'} - \phi \frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right] ds'.$$

THUS TO CALCULATE ϕ ANYWHERE IN THE VOLUME WE NEED TO KNOW:

- THE CHARGE DENSITY ρ IN THE VOLUME
- THE POTENTIAL ϕ ON THE SURFACE
- THE DERIVATIVE $\frac{\partial \phi}{\partial n}$ = NORMAL COMPONENT OF $\bar{E}$ ON THE SURFACE $= \pm 4\pi\sigma$

NOTE THAT FOR A CONDUCTING SURFACE THE NORMAL COMPONENT OF $\bar{E}$ IS JUST THE SURFACE CHARGE DENSITY : $4\pi\sigma = E_n$.

ELECTROSTATIC POTENTIAL PROBLEM ARE OFTEN CALLED <u>BOUNDARY VALUE PROBLEMS</u>.

IN GENERAL, THE BOUNDARY VALUES OF ϕ AND $\frac{\partial \phi}{\partial n}$ ARE NOT INDEPENDENT, AND IT IS SUFFICIENT TO KNOW ONLY ONE OR THE OTHER TO OBTAIN A UNIQUE SOLUTION FOR ϕ THROUGHOUT THE VOLUME. (SEE BECKER SEC. 20)

TO SEE THIS SUPPOSE THE CONTRARY. I.e., WE HAVE TWO SOLUTIONS, ϕ_1 AND ϕ_2, BOTH OF WHICH SATISFY $\nabla^2\phi = -4\pi\rho$ IN THE VOLUME, AND WHICHEVER BOUNDARY CONDITION IS GIVEN. LET $U = \phi_1 - \phi_2$, SO $\nabla^2 U = 0$ INSIDE

THEN WE INVOKE GREEN'S IDENTITY [BECKER (10-15)].

$$\int_V \bar{\nabla}\cdot(\psi\bar{\nabla}\phi)\, dvol = \int_V (\psi\nabla^2\phi + \bar{\nabla}\psi\cdot\bar{\nabla}\phi)\, dvol = \int_S \psi \frac{\partial \phi}{\partial n}\, ds$$

WITH $\phi = \psi = U$, $\displaystyle\int_V (U\nabla^2 U + |\bar{\nabla}U|^2)\, dvol = \int_S U\frac{\partial U}{\partial n}\, ds$

BUT ON THE SURFACE EITHER $U = 0$ OR $\frac{\partial U}{\partial n} = 0$ BY HYPOTHESIS, SO $\displaystyle\int_V |\bar{\nabla}U|^2\, dvol = 0 \Rightarrow \bar{\nabla}U = 0 \Rightarrow U = \text{CONST.}$

SO AT WORST $\phi_1 = \phi_2 + \text{CONST} \Rightarrow \bar{E}_1 = \bar{E}_2$ ALWAYS.

Ph 206 Lecture 4

IN SUMMARY, TO SOLVE FOR ϕ AND $\vec{E}$ WITHIN A VOLUME WE NEED:

ρ WITHIN THE VOLUME

AND ϕ ON THE SURFACE OR $\frac{\partial \phi}{\partial n} = E_n = 4\pi\sigma$ ON THE SURFACE.

(INDEED, ϕ & $\frac{\partial \phi}{\partial n}$ CANNOT BOTH BE SPECIFIED ARBITRARILY ON A CLOSED BOUNDARY.)

WE NOW DISCUSS SOME SPECIFIC METHODS OF PROBLEM SOLVING

1. THE METHOD OF IMAGES
2. SEPARATION OF VARIABLES AND EXPANSION IN ORTHOGONAL FUNCTIONS

OTHER CLASSIC METHODS WHICH WE WILL NOT CONSIDER MUCH INCLUDE

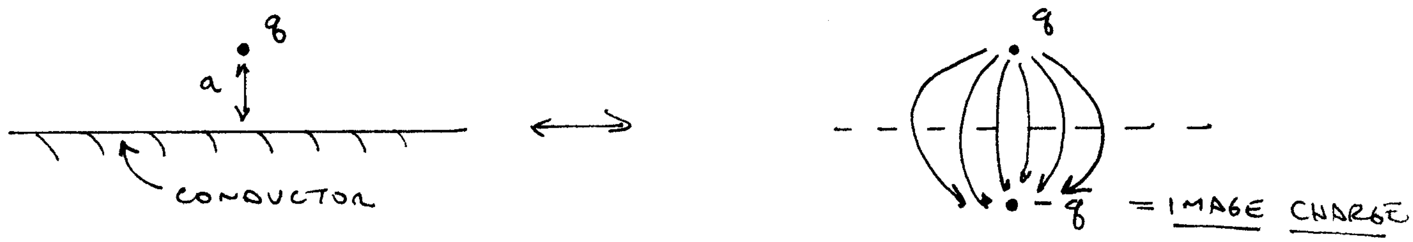
3. GREEN'S FUNCTIONS. SOLVE THE PROBLEM FOR A SINGLE POINT CHARGE IN THE VOLUME - BUT SAME BOUNDARY CONDITIONS. THEN BUILD THE GENERAL SOLUTION FOR A GIVEN ρ BY SUPERPOSITION. (SEE JACKSON FOR MANY EXAMPLES.)
4. INVERSION. SOLUTIONS IN ONE VOLUME CAN SOMETIMES BE TRANSFORMED INTO SOLUTIONS FOR ANOTHER VOLUME.
5. COMPLEX VARIABLES - ANY COMPLEX, ANALYTIC FUNCTION $f(z) = u + iv$ OBEYS LAPLACE EQUATION: $(z = x + iy)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$ AS A CONSEQUENCE OF

THE CAUCHY-RIEMANN CONDITIONS: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

THE METHOD OF IMAGES (BECKER SEC 23)

THE FIELDS AND POTENTIAL DUE TO SINGLE POINT CHARGES NEAR CONDUCTORS OF SIMPLE GEOMETRY CAN BE "GUESSED". THE CLASSIC EXAMPLE IS A POINT CHARGE ABOVE A CONDUCTING PLANE



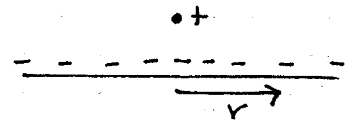
THE FIELD SET UP BY q AND IT'S IMAGE CHARGE $-q$

IN THE ABSENCE OF THE CONDUCTOR CLEARLY SATISFY THE BOUNDARY CONDITIONS FOR THE CASE WITH THE CONDUCTOR: ϕ = CONST ON BOUNDARY. $\therefore$ THIS IS THE UNIQUE SOLUTION!

THE FORCE ON THE CHARGE IS $F = \frac{q^2}{(2a)^2}$ ATTRACTIVE

CHARGE q INDUCES A SURFACE CHARGE ON THE CONDUCTOR

$$\sigma = \frac{1}{4\pi} E_\perp = -\frac{2q}{4\pi} \frac{a}{(r^2+a^2)^{3/2}}$$

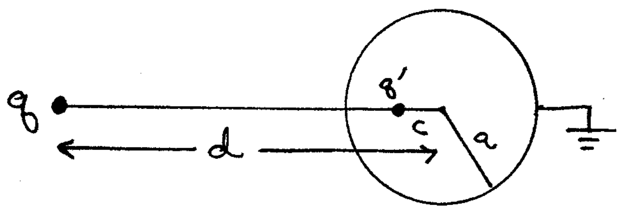


THE TOTAL INDUCED CHARGE IS

$$Q = \int_0^\infty 2\pi r \sigma \, dr = -q.$$

THE IMAGE CHARGE IDEA CAN ALSO BE USED IN ENERGY CONSIDERATIONS. SEE THE PROBLEM SET.

EXAMPLE POINT CHARGE AND A CONDUCTING SPHERE

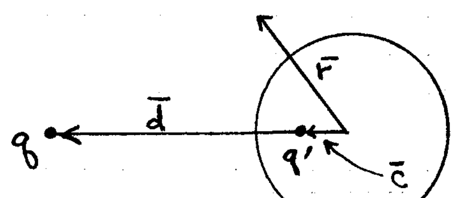


CONSIDER POINT CHARGE q AT DISTANCE d FROM A GROUNDED CONDUCTING SPHERE OF RADIUS a.

BY "GROUNDED" WE MEAN THAT IT IS HELD AT POTENTIAL $\phi = 0$ BY EXTERNAL MEANS (WHICH ARE SUPPOSED NOT TO DISTORT THE FIELD OUTSIDE THE SPHERE OTHERWISE).

CAN WE LOCATE AN IMAGE CHARGE q' SOMEWHERE INSIDE THE SPHERE SO THAT THE BOUNDARY CONDITION $\phi = 0$ IS SATISFIED? AMAZINGLY, THE ANSWER IS YES. CLEARLY q' WILL LIE ON THE LINE JOINING q TO THE CENTER OF THE SPHERE, SAY AT DISTANCE c.

THE POTENTIAL AT AN ARBITRARY POINT $\bar{r}$ WOULD BE



$$\phi(\bar{r}) = \frac{q}{|\bar{r} - \bar{d}|} + \frac{q'}{|\bar{r} - \bar{c}|}$$

$$= \frac{q}{r\,|\hat{r} - \frac{d}{r}\hat{d}|} + \frac{q'}{c\,|\frac{r}{c}\hat{r} - \hat{d}|} \qquad \text{SINCE } \bar{c} = c\hat{d}$$

TRICK: C NOT r

FOR $r = a$ WE WANT $\phi = 0$ $\Rightarrow$ $\dfrac{q/a}{|\hat{r} - \frac{d}{a}\hat{d}|} = \dfrac{-q'/c}{|\frac{a}{c}\hat{r} - \hat{d}|}$

THIS CAN BE ARRANGED BY SETTING $\dfrac{d}{a} = \dfrac{a}{c}$ AND $\dfrac{q}{a} = -\dfrac{q'}{c}$

SINCE $|\hat{r} - \frac{d}{a}\hat{d}| = 1 + \left(\frac{d}{a}\right)^2 - 2\frac{d}{a}\hat{r}\cdot\hat{d}$ AND $|\frac{a}{c}\hat{r} - \hat{d}| = \left(\frac{a}{c}\right)^2 + 1 - 2\frac{a}{c}\hat{r}\cdot\hat{d}$

HENCE $\underline{c = \dfrac{a^2}{d}}$ AND $\underline{q' = -\dfrac{a}{d}q}$

SUPPOSE INSTEAD THAT THE SPHERE WERE UNCHARGED AND ISOLATED SO THAT $Q_{\text{SPHERE}} = 0$ ALWAYS. OF COURSE THE SURFACE OF A CONDUCTING SPHERE MUST ALWAYS BE AN EQUIPOTENTIAL, BUT NOW THIS NEED NOT BE $\phi = 0$.

WE CAN SIMPLY EXTEND OUR SOLUTION ABOVE BY ADDING A SECOND IMAGE CHARGE $q'' = -q'$ TO RESTORE NEUTRALITY OF THE SPHERE. TO KEEP THE SPHERE'S SURFACE AS AN EQUIPOTENTIAL, WE PLACE q'' AT THE CENTER OF THE SPHERE!

THEN $\phi_{\text{SURFACE}} = \dfrac{q''}{a} = \dfrac{q}{d}$

WE CAN NOW WRITE DOWN SIMPLE EXPRESSIONS FOR ϕ AND $\vec{E}$ EVERYWHERE, AND DEDUCE THE FORCE ON q, THE CHARGE DENSITY σ ON THE SPHERE, ETC. ETC. [RELATED TOPICS: IMAGE METHOD IN CYLINDRICAL GEOMETRIES; IMAGE METHOD WITH DIELECTRICS..

EXPANSION IN A SERIES OF ORTHOGONAL FUNCTIONS

WE SHALL SOON FIND METHODS IN WHICH THE SOLUTION FOR THE POTENTIAL IS NOT GIVEN AS A SINGLE FUNCTION, BUT AS A SERIES EXPANSION:

$$\phi(x) = \sum_n a_n f_n(x)$$

WHERE $f_n(x)$ IS A FAMILY OF FUNCTIONS AND a_n ARE CONSTANTS.

{THE $f_n(x)$ MAY BE COMPLEX. THEN WE REALLY MEAN

$$\phi(x) = \mathrm{Re}\left[\sum_n a_n f_n(x)\right]$$
}

IF THE SERIES IS TO HOLD ON SOME INTERVAL $x_1 < x < x_2$, WE DETERMINE THE COEFFICIENTS a_n BY MINIMIZING THE "SQUARED ERROR"

$$\Delta^2 = \int_{x_1}^{x_2} \left[\phi(x) - \sum_n a_n f_n(x)\right]^2 dx \equiv \text{SQUARED ERROR}$$

TO MINIMIZE Δ^2, WE REQUIRE $\frac{\partial \Delta^2}{\partial a_n} = 0$ FOR EACH n:

$$\frac{\partial \Delta^2}{\partial a_n} = -2\int_{x_1}^{x_2} f_n^*(x) \left(\phi(x) - \sum_m a_m f_m(x)\right) dx = 0$$

OR $$\sum_m a_m \int_{x_1}^{x_2} f_n^*(x) f_m(x)\, dx = \int_{x_1}^{x_2} \phi(x) f_n^*(x)\, dx$$

A MESS IN GENERAL ($f^* \equiv$ COMPLEX CONJUGATE OF f)

THINGS BECOME MORE SENSIBLE IF WE RESTRICT OURSELVES TO A FAMILY OF ORTHOGONAL FUNCTIONS,

i.e., $$\int_{x_1}^{x_2} f_n^* f_m\, dx = \delta_{mn} .$$

NOTE THAT WE HAVE CHOSEN THAT $\int_{x_1}^{x_2} |f_n(x)|^2 dx = 1$ FOR EACH n.

WITH THIS EXTRA CONDITION, WE SPEAK OF ORTHONORMAL FUNCTIONS [WE SOMETIMES WILL ALLOW THE NORMALIZATION TO BE CONSTANTS $N_n \neq 1$.]

WE NOW HAVE $a_n = \int_{x_1}^{x_2} \phi(x) f_n^*(x)\, dx$ FOR THE COEFFICIENTS.

WE HOPE THAT IF WE KEEP ENOUGH TERMS IN THE SERIES, THE ERROR $\Delta^2 \to 0$. IF SO, WE SAY THE FAMILY OF FUNCTIONS IS COMPLETE.

SUPPOSE WE PLUG OUR SOLUTION FOR a_n BACK INTO THE ORIGINAL EXPANSION:

$$\phi(x) = \sum_n \int_{x_1}^{x_2} \phi(x') f_n^*(x')\, dx' \, f_n(x) = \int_{x_1} \left(\sum_n f_n^*(x') f_n(x)\right) \phi(x')\, dx'$$

FOR THIS TO BE TRUE, WE MUST HAVE

$$\sum_n f_n^*(x')\, f_n(x) = \delta(x'-x) = \text{DIRAC'S DELTA FUNCTION.}$$

THE SIMPLEST FAMILY OF SUCH "COMPLETE, ORTHONORMAL FUNCTIONS" ARE JUST SINES AND COSINES.

IF THE INTERVAL IS $-a/2 < x < a/2$, THE FAMILY IS

$$\sqrt{\frac{2}{a}} \sin \frac{2n\pi x}{a} \quad , \quad \sqrt{\frac{2}{a}} \cos \frac{2n\pi x}{a}$$

THEN $$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{2n\pi x}{a} + B_n \sin\frac{2n\pi x}{a}\right)$$

WITH $$A_n = \frac{2}{a}\int_{-a/2}^{a/2} \phi(x) \cos\frac{2n\pi x}{a} \quad ; \quad B_n = \frac{2}{a}\int_{-a/2}^{a/2} \phi(x) \sin\frac{2n\pi x}{a}$$

WHICH IS THE FAMILIAR FOURIER SERIES EXPANSION.

NOTE THAT $$\delta(x'-x) = 1 + \sum_n \cos\frac{2n\pi x'}{a}\cos\frac{2n\pi x}{a} + \sin\frac{2n\pi x'}{a}\sin\frac{2n\pi x}{a}$$

$$= \sum_n \cos\frac{2n\pi(x'-x)}{a}$$

IF $a \to \infty$, WE OBTAIN THE FOURIER INTEGRAL EXPANSION

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dx$$

WITH $$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(x)\, e^{-ikx}\, dx$$

NOTING THAT $$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-k')x}\, dx = \delta(k-k')$$

AND $$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\, dk = \delta(x-x')$$

SOLUTION OF LAPLACE'S EQUATION BY SEPARATION OF VARIABLES

WE WISH TO USE THE SERIES-EXPANSION METHOD FOR POTENTIAL PROBLEMS IN 3 DIMENSIONS. WE LOOK FOR THE SEPARATION OF VARIABLES SOLUTION:

$$\phi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

IF THE BOUNDARY CONDITIONS ARE KNOWN FOR PLANES OF X, y AND z = CONSTANT, WE MAY BE ABLE TO REDUCE THE 3 DIMENSIONAL PROBLEM TO 3 1-DIMENSIONAL PROBLEMS.

THIS METHOD WORKS FOR SITUATIONS WHERE LAPLACE'S EQUATION HOLDS:

$$\nabla^2\phi = 0 \iff \text{CHARGE FREE VOLUME}$$

NOW $$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

SO WITH $\phi = XYZ$, $\quad X''YZ + XY''Z + XYZ'' = 0$

DIVIDING THRU BY ϕ :

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

FUNCTION OF X ONLY FUNCTION OF y ONLY FUNCTION OF z ONLY

HENCE WE CAN SAY $\frac{Z''}{Z} = k_1 = \text{CONST.} \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = -k_1$

AND FURTHER WE CAN SET $\frac{X''}{X} = k_2 \Rightarrow \frac{Y''}{Y} = -k_1 - k_2$

AND ALL THE DIFFERENTIAL EQUATIONS ARE 'SPRING-LIKE'.

THE ABOVE PROCEDURE CAN'T BE COMPLETED IF THE BOUNDARIES ARE NOT OF RECTANGULAR GEOMETRY. BUT IF THE BOUNDARIES ARE SURFACES OF q = CONST IN SOME OTHER COORD SYSTEM (q_1, q_2, q_3), THE SEPARATION OF VARIABLES METHOD MAY STILL WORK. i.e., IN SPHERICAL COORDS, WE TRY

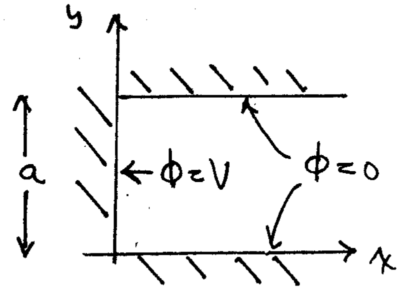
$$\phi(r,\theta,\varphi) = R(r)\,\Theta(\theta)\,\Phi(\varphi)$$

APPARENTLY THERE ARE 11 COORD. SYSTEMS IN WHICH $\nabla^2\phi = 0$ MAY BE SEPARATED. WE WILL CONSIDER ONLY 3 SYSTEMS HERE. (PLUS A PEEK AT A 4RTH IN LECTURE 6)

Ph 206 Lecture 4

Two-Dimensional Problems with Rectangular Geometry

EXAMPLE Find the potential everywhere in a 2-dimensional slot of width a. The walls are conductors held at fixed potentials 0 and V as shown.



$$\phi = \phi(x,y) = X(x)\,Y(y)$$

$$\nabla^2 \phi = 0 \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0$$

AND $\quad \dfrac{X''}{X} = K \qquad \dfrac{Y''}{Y} = -K \qquad$ K can be pos. or neg.

It is more clever to choose K positive, so as to have oscillatory solutions in y. You can, however, also obtain a solution for $K<0 \Rightarrow$ oscillatory functions in x.

LET $K = k^2 \Rightarrow X'' = k^2 X \qquad Y'' = -k^2 Y$

$$X = e^{\pm kx} \qquad Y = e^{\pm iky} \quad \text{(or linear combo's...)}$$

Then $\quad \phi = \sum_{k_n} A_n e^{\pm k_n x} e^{\pm i k_n y}$

The boundary condition $\phi(x, y=0) = 0$ suggests we consider functions $\sin ky$ rather than $e^{\pm iky}$

$\phi(x, y=a) = 0 \Rightarrow$ only $\sin\left(\dfrac{n\pi y}{a}\right)$ need be considered.

i.e. $\quad k_n = \dfrac{n\pi}{a} \qquad n = 1, 2, 3 \dots$

Also as $x \to +\infty$ we expect that $\phi \to 0$. Hence we need only consider the functions $e^{-kx} = e^{-\frac{n\pi x}{a}}$ in x.

Thus $\quad \phi(x,y) = \sum_n A_n e^{-\frac{n\pi x}{a}} \sin \dfrac{n\pi y}{a}$

To find the A_n we use the boundary condition at $x=0$

$$\phi(0,y) = V = \sum_n A_n \sin\frac{n\pi y}{a}$$

$$A_n = \frac{2}{a}\int_0^a V \sin\frac{n\pi y}{a}\,dy = \frac{-2V}{n\pi}\cos\frac{n\pi y}{a}\Big|_0^a = \begin{cases} 0 & n \text{ even} \\ \dfrac{4V}{n\pi} & n \text{ odd} \end{cases}$$

THUS $\phi(x,y) = \frac{4V}{\pi} \sum_{n\,odd} \frac{1}{n} e^{-\frac{n\pi x}{a}} \sin\frac{n\pi y}{a}$

WHICH IS A 'COMPLETE' SOLUTION, IN PRINCIPLE.

NEAR $x=0$, ALL THE $e^{-\frac{n\pi x}{a}} \sim 1$ AND MANY TERMS MUST BE KEPT FOR GOOD ACCURACY. BUT AS $x \to \infty$, CLEARLY ONLY THE FIRST TERM IN THE SERIES MATTERS. IN PRACTICE "∞" SETS IN QUICKLY. FOR $x > a/2$ THE 1ST TERM ALONE IS ACCURATE TO 5% EVERYWHERE!

HOWEVER, IT TURNS OUT THAT WE CAN ACTUALLY SUM THE SERIES TO YIELD A SINGLE FUNCTION:

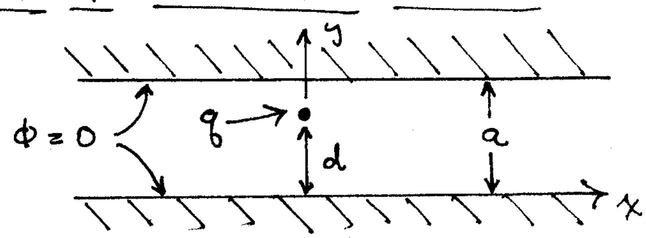
$$\phi(x,y) = \frac{2V}{\pi} \tan^{-1}\left(\frac{\sin\frac{\pi y}{a}}{\sinh\frac{\pi x}{a}}\right).$$

THIS CAN BE DONE BY A TRICK SKETCHED BELOW, AND THE FACT THAT

$$\sum_{n\,odd} \frac{1}{n} z^n = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$$

EXAMPLE LINE CHARGE BETWEEN 2 CONDUCTING PLANES

SUPPOSE THE PLANES ARE GROUNDED, AND THAT THE LINEAR CHARGE DENSITY IS q PER UNIT LENGTH (IN z DIR.) THE LINE CHARGE IS AT HEIGHT $y=d$.



BECAUSE OF THE LINE CHARGE, LAPLACE'S EQUATION DOES NOT HOLD BETWEEN THE PLANES.

WE COULD SOLVE THIS BY THE IMAGE METHOD — WHICH WOULD REQUIRE AN INFINITE SERIES OF IMAGE LINE CHARGES TO MAINTAIN BOTH SURFACES AT $\phi = 0$.

IF WE SPLIT THE PROBLEM INTO 2 PARTS, $x>0$ AND $x<0$, THEN LAPLACE'S EQUATION HOLDS IN EACH REGION, AND WE MAY USE THE APPROACH OF THE PREVIOUS EXAMPLE.

BUT WHAT IS THE "BOUNDARY CONDITION" AT $x=0$?

WE DO NOT EXPECT $\phi =$ CONSTANT!

INSTEAD, WE KNOW THE CHARGE DISTRIBUTION ON THE PLANE $x=0$, SO WE EXPECT TO USE A CONDITION ON $\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x} = E_x$

WE MAY WRITE THE "SURFACE CHARGE" DENSITY AT $x=0$
AS $\sigma(y) = q\,\delta(y-d)$ (PER UNIT LENGTH ALONG z)

SINCE $\sigma(y)=0$ WHEN $y \neq d$, AND $q = \int \sigma(y)\,dy$.

GAUSS' LAW APPLIED TO A 'PILLBOX' SURROUNDING A REGION OF THE PLANE $x=0$, THEN TELLS US

$$\oint \vec{E}\cdot d\vec{s} = 4\pi Q_{\text{INSIDE}} \quad \text{or} \quad E_x\big|_{x>0} - E_x\big|_{x<0} = 4\pi\,\sigma(y),$$

AND HENCE,
$$-\frac{\partial \phi}{\partial x}\bigg|_{x>0} + \frac{\partial \phi}{\partial x}\bigg|_{x<0} = 4\pi q\,\delta(y-d).$$

THIS IS THE NEEDED BOUNDARY CONDITION.

FOR $x>0$, WE EXPECT $\phi(x,y) = \sum_n A_n e^{-\frac{n\pi x}{a}} \sin\frac{n\pi y}{a}$

WHILE FOR $x<0$ $\qquad \phi(x,y) = \sum_n B_n e^{+\frac{n\pi x}{a}} \sin\frac{n\pi y}{a}$

FROM THE PREVIOUS EXAMPLE. TO MATCH SOLUTIONS AT $x=0$, $A_n = B_n$

THEN $\frac{\partial \phi}{\partial x} = \sum_n \mp\frac{n\pi}{a} A_n e^{\mp\frac{n\pi x}{a}} \sin\frac{n\pi y}{a}$, SO $\frac{\partial \phi}{\partial x}\Big|_{x>0} = \sum_n -\frac{n\pi}{a} A_n \sin\frac{n\pi y}{a}$

LIKEWISE $\frac{\partial \phi}{\partial x}\Big|_{x<0} = \sum_n \frac{n\pi}{a} A_n \sin\frac{n\pi y}{a}$

THE B.C. SAYS: $\frac{2\pi}{a}\sum_n n A_n \sin\frac{n\pi y}{a} = 4\pi q\,\delta(y-d)$

HENCE $\frac{2\pi}{a} n A_n = \frac{2}{a}\int_0^a 4\pi q\,\delta(y-d)\sin\frac{n\pi y}{a}\,dy = \frac{8\pi q}{a}\sin\frac{n\pi d}{a}$

$$A_n = \frac{4q}{n}\sin\frac{n\pi d}{a}$$

AND $\phi(x,y) = 4q\sum_n \frac{1}{n}\sin\frac{n\pi d}{a}\sin\frac{n\pi y}{a}\,e^{\mp\frac{n\pi x}{a}}$ FOR $x \gtrless 0$

[WE COULD REGARD THIS SOLUTION AS THE BEGINNING OF A GREEN'S FUNCTION METHOD FOR AN ARBITRARY TWO DIMENSIONAL CHARGE DISTRIBUTION BETWEEN TWO CONDUCTING PLANES.]

AN INTERESTING PHYSICAL QUESTION REGARDING THIS EXAMPLE IS HOW MUCH CHARGE IS INDUCED ON EACH OF THE TWO PLANES AS A FUNCTION OF d? (SEE PURCELL FOR AN ELEMENTARY 'TRICK' SOLUTION)

GAUSS' LAW TELLS US THAT $q_{\text{induced}} = -q$

IF $d = a/2$ WE EXPECT $q = -q/2$ BY SYMMETRY

IF $d = 0$, WE EXPECT $-q$ ON THE LOWER PLANE, 0 ON THE UPPER.

NOW WE KNOW THAT THE SURFACE CHARGE DENSITY ON THE PLANE $y=0$ OBEYS

$$\sigma(x) = \frac{1}{4\pi} E_y(y=0) = -\frac{1}{4\pi}\frac{\partial \phi}{\partial y}(y=0)$$

SO $$\sigma(x) = -\frac{q}{a}\sum_n \sin\frac{n\pi d}{a}\, e^{\mp\frac{n\pi x}{a}}$$

WE SKETCH A TRICK FOR SUMMING THIS SERIES:

$$e^{\mp\frac{n\pi x}{a}}\sin\frac{n\pi d}{a} = \operatorname{Im} e^{\frac{in\pi d}{a}} e^{\mp\frac{n\pi x}{a}} = \operatorname{Im}\left[e^{\frac{i\pi d}{a}\mp\frac{\pi x}{a}}\right]^n$$

SO LET $z = e^{\frac{i\pi d}{a}\mp\frac{\pi x}{a}}$

THEN WE NEED $\operatorname{Im}\sum_n z^n = \operatorname{Im}\left(\frac{1}{1-z}\right)$

NOW $$\frac{1}{1-z} = \frac{1}{1-e^{\frac{i\pi d}{a}}e^{\mp\frac{\pi x}{a}}} = \frac{1-e^{-i\frac{\pi d}{a}}e^{\mp\frac{\pi x}{a}}}{1-2\cos\frac{\pi d}{a}\,e^{\mp\frac{\pi x}{a}}+e^{\mp\frac{2\pi x}{a}}}$$

SO $$\operatorname{Im}\frac{1}{1-z} = \frac{\sin\frac{\pi d}{a}}{2\left(\cosh\frac{\pi x}{a}-\cos\frac{\pi d}{a}\right)}$$

AND $$\sigma(x) = -\frac{q}{2a}\,\frac{\sin\frac{\pi d}{a}}{\cosh\frac{\pi x}{a}-\cos\frac{\pi d}{a}}$$

THIS MAY BE INTEGRATED (R. & G 2.444.2) TO YIELD

$$q_{\text{induced}} = -q\left(1-\frac{d}{a}\right)$$ — A SIMPLE LINEAR RELATION!

(OR INTEGRATE THE SERIES FOR $\sigma(x)$ TERM BY TERM, AND USE A VARIATION OF THE ABOVE TRICK TO SUM THE SERIES)

IT IS EASY TO CONVINCE YOURSELF THAT OUR EXPRESSION FOR q_{induced} ALSO HOLDS FOR A POINT CHARGE HEIGHT d BETWEEN TWO GROUNDED CONDUCTING PLANES.