Ph 206 Lecture 3

Energy in Electrostatics (Becker Chap. B III)

The electrostatic force between two point charges is

$$\bar{F} = \frac{q_1 q_2}{r^2}\hat{r}$$

The work done in bringing the charges from separation $r = \infty$ to separation r is

$$W = -\int_{\infty}^{r} \bar{F}\cdot d\bar{r} = \frac{q_1 q_2}{r}$$

[Move the charges slowly so the law of electrostatics holds]

We may say that this amount of energy is stored in the system as the potential energy. (Where is it stored?)

We write $U = \dfrac{q_1 q_2}{r}$ (The symbol V usually refers to the electric potential)

$$= q_1 \, \phi_1 \text{ (due to 2)}$$

In a configuration of many point charges, we have

$$U = \frac{1}{2}\sum_{ij} \frac{q_i q_j}{r_{ij}} \qquad \left(\tfrac{1}{2} \text{ since } \sum_{ij} \text{ double counts}\right)$$

$$= \frac{1}{2}\sum_{i} q_i \, \phi_i \text{ (due to all other charges)}$$

In the limit of continuous charge distributions, we have

$$\underline{U = \frac{1}{2}\int \rho\, \phi \, d\text{vol}}$$

Localised Charge Distribution in an External Field

As a special example we consider the energy of a localised charge distribution which is in an external field. The potential energy has 3 parts:

U_1. The energy of the local charge distribution by itself

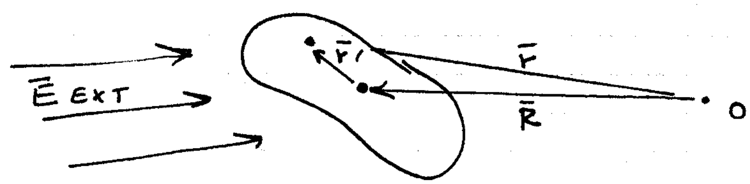
U_2. The energy of the system of charges which produces the external field.

U_3. The energy of interaction between the local charges and the external field.

We suppose U_1 and U_2 are constant, and consider only U_3.

THEN $U_{INTERACTION} = \int_{LOCAL} \rho\, \phi_{EXT}\, d\,vol$

THERE IS NO FACTOR ½ SINCE NO DOUBLE COUNTING EXISTS HERE.



WE AGAIN DO A TAYLOR EXPANSION. THIS TIME WE EXPAND THE EXTERNAL POTENTIAL $\phi_E(\bar{r})$ ABOUT THE VECTOR $\bar{R}$ TO AN APPROPRIATE POINT INSIDE THE LOCALISED CHARGE DISTRIBUTION. (WE CAN DEFINE THE ORIGIN O TO BE INSIDE THE DISTRIBUTION, AND SET $\bar{R}=0$.

$$\phi_E(\bar{r}) = \phi_E(\bar{R}) + r_i' \frac{\partial}{\partial r_i'} \phi_E(\bar{R}) + \frac{1}{2} r_i' r_j' \frac{\partial^2}{\partial r_i' \partial r_j'} \phi_E(\bar{R}) + \cdots$$

$$\text{so } U_{INT} = \phi_E(\bar{R}) \int \rho\, d\,vol + \left(\frac{\partial}{\partial r_i'} \phi_E(\bar{R})\right) \int r_i' \rho\, d\,vol + \frac{1}{2}\left(\frac{\partial^2}{\partial r_i' \partial r_j'} \phi_E(\bar{R})\right) \int r_i' r_j' \rho\, d\,vol + \cdots$$

$$= Q\, \phi_E(\bar{R}) - \bar{P} \cdot \bar{E}_E(\bar{R}) - \frac{1}{2} P_{ij} \frac{\partial E_j(\bar{R})}{\partial r_i'} + \cdots$$

NOTING $\bar{E}_E = -\bar{\nabla} \phi_E$ AND DEFINING $P_{ij} = \int r_i' r_j' \rho\, d\,vol$

TO RELATE THE THIRD TERM TO OUR DEFINITION OF Q_{ij} IN LECTURE 1, WE NOTE THAT $\bar{\nabla} \cdot \bar{E}_E = 0$ INSIDE THE LOCAL DISTRIBUTION, SINCE $\rho_E = 0$ THERE BY DEFINITION. HENCE

$$0 = \frac{\partial E_i}{\partial r_i'} = \delta_{ij} \frac{\partial E_j}{\partial r_i'}$$

AND WE CAN ADD $\frac{1}{6} \frac{\partial E_j}{\partial r_i'} \int r'^2 \delta_{ij} \rho\, d\,vol = 0$ TO U_{INT}

$$U_{INT}(\bar{R}) = Q\, \phi_E(\bar{R}) - \bar{P} \cdot \bar{E}_E(\bar{R}) - \frac{1}{6} Q_{ij} \frac{\partial E_{Ej}(\bar{R})}{\partial r_i} + \cdots$$

WHERE $Q_{ij} = \int (3 r_i' r_j' - \delta_{ij} r'^2) \rho\, d\,vol$ AS BEFORE.

WE IDENTIFY THE TERMS IN THIS SERIES AS THE ENERGY OF INTERACTION OF THE VARIOUS MULTIPOLES WITH THE EXTERNAL FIELD.

For example $U_{\text{dipole}} = -\bar{p}\cdot\bar{E}$ as was previously noted on p 22. A dipole aligned with the external field minimizes the potential energy.

Forces and Torques

From our original definition that $U = -\int \bar{F}\cdot d\bar{r}$

we have that $\underline{\bar{F} = -\bar{\nabla} U}$ is the force associated with motion in coordinate $\bar{r}$ (where $\bar{\nabla} = \frac{\partial}{\partial \bar{r}}$). Suppose we hold the external charges fixed ($\Rightarrow$ external field fixed) while considering motion of our localised charge distribution as a whole. Then we may use $\bar{R}$ as the appropriate coordinate.

$$\text{Then} \quad \bar{F} = -Q\bar{\nabla}\phi_E + \bar{\nabla}(\bar{p}\cdot\bar{E}_E) + \frac{1}{6}\bar{\nabla}\left(Q_{ij}\frac{\partial E_{Ej}}{\partial r_i'}\right) + \ldots$$

$$= Q\bar{E}_e(\bar{R}) + (\bar{p}\cdot\bar{\nabla})\bar{E}_E + \ldots$$

[NOTE: $\bar{F} = (\bar{p}\cdot\bar{\nabla})\bar{E}$ true even if $\bar{\nabla}\times\bar{E} \neq 0$ while $\bar{F} \neq \nabla(\bar{p}\cdot\bar{E})$ in general]

using $\bar{\nabla}(\bar{p}\cdot\bar{E}) = (\bar{p}\cdot\bar{\nabla})\bar{E} + (\bar{E}\cdot\bar{\nabla})\bar{p} + \bar{p}\times(\bar{\nabla}\times\bar{E}) + \bar{E}\times(\bar{\nabla}\times\bar{p})$

and noting that $\bar{p}$ is constant with respect to $\bar{\nabla}$.

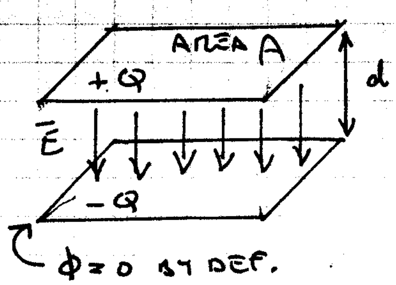
Thus a dipole feels a force if it is in a non-uniform electric field. It is pulled into regions of stronger field. (Assuming it was already lined up with the field to minimize U).

From Ph 205 we know that we can calculate the generalised force associated with any coordinate χ by means of $-\frac{\partial U}{\partial \chi}$.

Suppose $\chi = \theta$ = angle of the dipole axis. The generalised force is, of course, the torque $\tau = -\frac{\partial U}{\partial \theta} = -\frac{\partial}{\partial \theta}\, pE\cos\theta$

$\tau = pE\sin\theta$ or $\bar{\tau} = \bar{p}\times\bar{E}$ as on p 22.

EXAMPLE FORCE ON A CAPACITOR PLATE (FIXED CHARGES)



THE SURFACE CHARGE IS $\sigma = Q/A$

BY GAUSS' LAW $E = 4\pi\sigma$

THE POTENTIAL AT THE UPPER PLATE IS

$$\phi_{up} = -\int_0^d \bar{E}\cdot d\bar{l} = Ed$$

ACCORDING TO P 24, $U_{TOTAL} = \frac{1}{2}\int_{ALL} \rho\phi = \frac{1}{2}\sigma A \phi_{UPPER} = \frac{1}{2}\sigma A E d$

$$F = -\frac{\partial U}{\partial d} = -\frac{1}{2}\sigma A E \quad \longrightarrow \text{ PLATES ATTRACT}$$

NOTE $|F/A| = \frac{1}{2}\sigma E$ NOT σE. (COMPARE P. 15)

WE MIGHT ALSO REGARD THE UPPER PLATE AS OUR LOCALISED SYSTEM, AND THE LOWER PLATE THE SOURCE OF THE <u>EXTERNAL</u> FIELD. BUT THEN THE EXTERNAL FIELD ON THE UPPER PLATE IS ONLY $E_E = \frac{1}{2} 4\pi\sigma = \frac{E}{2}$

WHERE E = FIELD BETWEEN PLATES. $\Rightarrow \phi_{E\,UPPER} = \frac{1}{2}Ed$

$$U_{INT} = \int_{UPPER} \rho\phi = \frac{1}{2}\sigma A E d$$

AGAIN $F = -\frac{\partial U_{INT}}{\partial d} = -\frac{1}{2}\sigma A E$!

EXTERNAL FIELDS MAINTAINED BY BATTERIES

A VERY INTERESTING SITUATION IS WHEN THE EXTERNAL CHARGES ARE NOT FIXED, BUT RATHER THE EXTERNAL CHARGES RESIDE ON CONDUCTORS MAINTAINED AT FIXED POTENTIALS BY BATTERIES.

THIS WILL CHANGE OUR CALCULATION OF THE FORCE ON OUR LOCALISED CHARGE DISTRIBUTION. BECAUSE WHEN THE LOCAL DISTRIBUTION MOVES, THE CHARGE DISTRIBUTIONS ON THE CONDUCTORS MUST CHANGE TO REMAIN AT THEIR FIXED POTENTIALS. THE BATTERIES MUST DO WORK, AND SO THE EXPRESSION $U_{EL.} = \frac{1}{2}\int \rho\phi\, d\text{vol}$ DOES NOT INCLUDE ALL THE ENERGY TERMS!

LET U_{BAT} = ENERGY STORED IN THE BATTERIES.

NOW SUPPOSE SOME MOTION TAKES PLACE (SLOWLY, SO THAT NO KINETIC ENERGY DEVELOPS). TOTAL ENERGY MUST BE CONSERVED:

$$\delta U_{TOT} = 0 = \delta U_{EL} + \delta U_{BAT} + \delta(\text{MECHANICAL WORK})$$

$$\text{MECHANICAL WORK} = F_x\,\delta x$$

(COMPARE THE PRINCIPLE OF VIRTUAL WORK ...).

TO MAINTAIN A CONDUCTOR AT POTENTIAL ϕ, THE BATTERY SUPPLIES CHARGE $\delta\rho$ AND IN DOING SO LOSES ENERGY $(\delta\rho)\phi$

i.e., $\delta U_{BAT} = -\int \phi\,\delta\rho$.

MEANWHILE THE CHANGE IN ELECTRICAL POTENTIAL ENERGY IS

$$\delta U_{EL} = \frac{1}{2}\int \phi\,\delta\rho .$$

AMAZINGLY, $\delta U_{BAT} = -2\,\delta U_{EL}$.

HENCE, $\underline{F_x} = -\dfrac{\partial U_{EL}}{\partial x} - \dfrac{\partial U_{BAT}}{\partial x} = \underline{+\dfrac{\partial U_{EL}}{\partial x}}$, <u>FIXED POTENTIALS</u>,

COMPARED TO $\underline{F_x = -\dfrac{\partial U_{EL}}{\partial x}}$, <u>FIXED CHARGES</u>.

<u>EXAMPLE</u>: CAPACITOR PLATES MAINTAINED AT FIXED VOLTAGE.

DOES OUR NEW RESULT IMPLY THAT THE CAPACITOR PLATES WILL FLY <u>APART</u> IF THEY ARE CONNECTED TO A BATTERY?

NO! WE MUST BE CAREFUL. $U_{EL} = \frac{1}{2}\sigma A E d$ AS BEFORE

BUT σ AND E DO NOT REMAIN CONSTANT AS d VARIES!

INSTEAD, NOTE $E = \phi/d$ AND $\sigma = \dfrac{E}{4\pi} = \dfrac{\phi}{4\pi d}$

SO $U_{EL} = \dfrac{1}{2}\dfrac{\phi^2 A}{4\pi d}$ WITH ϕ A CONSTANT

THEN $F = +\dfrac{\partial U_{EL}}{\partial d} = -\dfrac{1}{2}\dfrac{\phi^2 A}{4\pi d^2} = -\dfrac{1}{2}\sigma E A$ AS BEFORE!

ELECTROSTATIC ENERGY AS A FIELD INTEGRAL

OUR EXPRESSION $U = \frac{1}{2}\int \rho\, \phi\, d\mathrm{vol}$

EMPHASIZES THE CHARGES AND THEIR RELATIVE CONFIGURATION.

WE NOW INTRODUCE AN ALTERNATIVE EXPRESSION WHICH INVOLVES ONLY THE ELECTRIC FIELD.

$$U = \frac{1}{8\pi}\int E^2\, d\mathrm{vol}$$

THIS EXPRESSION SUGGESTS THAT WE CONSIDER THE ENERGY TO BE STORED IN THE FIELD - AS IF THE FIELD HAS A DEFINITE PHYSICAL REALITY, RATHER THAN MERELY MATHEMATICAL SIGNIFICANCE IN $\bar{E} = \bar{F}/q$.

[MAXWELL FELT THAT $\bar{E}$ REPRESENTED A POLARIZATION OF THE ETHER, SO U WOULD BE THE ENERGY STORED IN THE DISTORTED ETHER.]

DERIVATION: $\rho = \frac{\bar{\nabla}\cdot\bar{E}}{4\pi} = -\frac{\nabla^2\phi}{4\pi}$

SO $U = \frac{1}{2}\int \rho\phi\, d\mathrm{vol} = \frac{-1}{8\pi}\int \phi\,\nabla^2\phi\, d\mathrm{vol}$

NOTE THAT $\bar{\nabla}\cdot(\phi\bar{\nabla}\phi) = (\bar{\nabla}\phi)^2 + \phi\,\nabla^2\phi$

SO $$U = \frac{1}{8\pi}\int (\bar{\nabla}\phi)^2\, d\mathrm{vol} - \frac{1}{8\pi}\int \bar{\nabla}\cdot(\phi\bar{\nabla}\phi)\, d\mathrm{vol}$$

$$= \frac{1}{8\pi}\int E^2\, d\mathrm{vol} - \frac{1}{8\pi}\int_{\mathrm{SURFACE}} \phi\bar{\nabla}\phi\cdot d\bar{S}$$

FOR A BOUNDED CHARGE DISTRIBUTION, VIEWED FROM ∞

$\phi \sim \frac{1}{r}$ $\nabla\phi \sim \frac{1}{r^2}$ WHILE $\int dS \sim r^2$

SO THE SURFACE INTEGRAL MAY BE NEGLECTED, AND THE RESULT HOLDS.

SELF ENERGY OF POINT CHARGES

OUR DEMONSTRATION OF $U = \frac{1}{8\pi}\int E^2\, d\mathrm{vol}$ STARTED FROM AN ASSUMPTION OF CONTINUOUS CHARGE DISTRIBUTIONS. WE RUN INTO TROUBLE IF WE APPLY IT TO POINT CHARGES.

EXAMPLE $-Q \bullet \xleftrightarrow{\quad r \quad} \bullet +Q$

$$U_{INT.} = \frac{Q_1 Q_2}{r} = -\frac{Q^2}{r} < 0$$ (THE FORCE IS ATTRACTIVE)

BUT $\int E^2 \, dvol > 0$ ALWAYS ???

CONSIDER A SPHERICAL SHELL OF CHARGE OF RADIUS a

THEN $E = \frac{q}{r^2}$, $r > a$; $E = 0$, $r < a$

$$U = \frac{1}{8\pi}\int E^2 \, dvol = \frac{q^2}{8\pi}\int_a^\infty \frac{4\pi r^2 \, dr}{r^4} = \frac{q^2}{2a} \to \infty \text{ AS } a \to 0$$

(YOU GET THE SAME RESULT USING $U = \frac{1}{2}\int \rho \phi \, dvol$)

CAN A 'POINT' CHARGE HAVE $a = 0 \Rightarrow U = \infty$? IS THERE ANY PHYSICAL RELEVANCE TO THIS CONCEPT?

NEITHER CLASSICAL NOR QUANTUM ELECTRODYNAMICS GIVE A DEFINITIVE ANSWER...

THE POINT-PARTICLE IDEA IS VERY CONVENIENT AND IS USED DESPITE THESE DIFFICULTIES.

AFTER EINSTEIN SAID $E = mc^2$, AN INTERESTING SUGGESTION AROSE: PERHAPS ALL THE MASS OF AN ELECTRON IS DUE TO ITS ELECTROMAGNETIC FIELD ENERGY.*

THEN (IGNORING THE FACTOR ½) PEOPLE SET $\frac{q^2}{a} = mc^2$

SO $a = \frac{q^2}{mc^2} \sim 2.8 \times 10^{-13}$ cm FOR AN ELECTRON

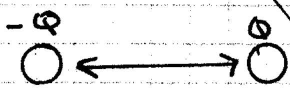
WE WILL CONSIDER THE POSSIBLE SIGNIFICANCE OF THIS EXPRESSION LATER IN THE COURSE. WE MAY NOTE THAT THE RUTHERFORD SCATTERING CROSS-SECTION DERIVED IN Ph 205 BECOMES

$$\frac{d\sigma}{d\Omega} = \left(\frac{q^2}{2mv^2}\right)^2 \frac{1}{\sin^4 \theta/2}$$ FOR ELECTRONS

SO AS $v \to c$ $\frac{d\sigma}{d\Omega} \to \frac{1}{4}\frac{a^2}{\sin^4 \theta/2}$ ....

* ACTUALLY, AN ELECTRODYNAMIC ARGUMENT LED J.J. THOMSON TO A SIMILAR CONCLUSION AS EARLY AS 1881. [A NUMERICAL VALUE FOR a COULD NOT BE GIVEN THEN.]

RETURNING TO OUR EXAMPLE OF A PAIR OF CHARGES



FOR SPHERE OF RADIUS a, WE FIND

$$U = \frac{1}{8\pi}\int E^2\, d\text{vol} \sim 2\cdot\left(\frac{Q^2}{2a}\right) - \frac{Q^2}{r} > 0 \quad \text{FOR } r > a$$

SELF ENERGY ↗ ↖ INTERACTION ENERGY

IN SUMMARY, $U = \frac{1}{8\pi}\int E^2\, d\text{vol}$ INCLUDES SELF ENERGY OF (POINT) CHARGES,

$U = \frac{1}{2}\sum_{\substack{i,j\\ i\neq j}} \frac{Q_i Q_j}{r_{ij}}$ DOES NOT.

FOR CONTINUOUS CHARGE DISTRIBUTIONS NO CONFLICT ARISES.

ENERGY AND DIELECTRICS

FOR ISOTROPIC DIELECTRICS WHERE $\bar{D} = \epsilon \bar{E}$

WE WILL FIND $U = \frac{1}{8\pi}\int \bar{E}\cdot\bar{D}\, d\text{vol}$.

TO DEMONSTRATE THIS WE IMAGINE ESTABISHING FIELDS $\bar{E}$ AND $\bar{D}$ BY ADDING INCREMENTS OF FREE (OR EXTERNAL) CHARGE $\delta\rho$ UNTIL THE DESIRED CONFIGURATION IS REACHED.

IF AT SOME MOMENT $\phi(\bar{r})$ IS THE EXISTING POTENTIAL, THEN TO ADD CHARGE $\delta\rho$ WE MUST DO WORK

$$\delta\,\text{WORK} = \int \delta\rho\, \phi\, d\text{vol}.$$

BUT SINCE $\bar{\nabla}\cdot\bar{D} = 4\pi\rho$, $\delta\rho = \frac{1}{4\pi}\delta(\bar{\nabla}\cdot\bar{D}) = \frac{1}{4\pi}\bar{\nabla}\cdot\delta\bar{D}$,

WHERE $\delta\bar{D}$ = CHANGE IN $\bar{D}$ DUE TO ADDING CHARGE $\delta\rho$.

SO

$$\begin{aligned}
\delta\,\text{WORK} &= \frac{1}{4\pi}\int \phi(\bar{\nabla}\cdot\delta\bar{D})\, d\text{vol} \\
&= \frac{-1}{4\pi}\int \bar{\nabla}\phi\cdot\delta\bar{D}\, d\text{vol} + \frac{1}{4\pi}\int \bar{\nabla}\cdot\phi\,\delta\bar{D}\, d\text{vol} \\
&= \frac{1}{4\pi}\int \bar{E}\cdot\delta\bar{D}\, d\text{vol} + \frac{1}{4\pi}\int \phi\,\delta\bar{D}\cdot d\bar{S} \\
&= \frac{1}{4\pi}\int \epsilon\,\bar{E}\cdot\delta\bar{E}\, d\text{vol}
\end{aligned}$$

$\to 0$ FOR SURFACE AT ∞

(THE LAST STEP ASSUMES ϵ IS INDEPENDENT OF E & D; ϵ CAN BE A FUNCTION OF POSITION, HOWEVER.)

$$\delta \text{work} = \frac{1}{8\pi}\int \epsilon\, \delta E^2\, dvol$$

$$\text{AND}\quad U = \int \delta \text{work} = \frac{1}{8\pi}\int \epsilon E^2\, dvol = \frac{1}{8\pi}\int \bar{E}\cdot\bar{D}\, dvol$$

THIS IS DIFFERENT THAN THE WORK DONE IN MERELY ASSEMBLING THE FREE CHARGES INTO THEIR FINAL CONFIGURATION — IT IS OFTEN LESS, IN THAT $\bar{D}$ DEPENDS ONLY ON ρ_{FREE}, WHILE $\bar{E} = \bar{D}/\epsilon < \bar{D}$ INSIDE DIELECTRICS. HOWEVER, IF WE REGARD $E^2/8\pi$ AS THE STORED ENERGY DENSITY OF THE ELECTRIC FIELD, THEN WE ATTRIBUTE THE EXTRA PIECE $(\epsilon-1)E^2/8\pi$ TO THE ENERGY STORED IN THE MOLECULAR DIPOLES CREATED BY THE APPLICATION OF $\bar{E}$ TO THE DIELECTRIC.

FOR AN ELECTRET, WITH PERMANENT POLARIZATION DENSITY $\bar{P}$, IT TURNS OUT THAT $\int \bar{E}\cdot\bar{D}\, dvol = 0$. (C.F. PROB 6, SET 1).

THUS OUR ENERGY EXPRESSION COMPLETELY FAILS FOR AN ELECTRET — WHICH MUST TAKE SOME ENERGY TO MAKE. THIS SERVES TO REMIND US THAT THE EXPRESSION $U = \frac{1}{8\pi}\int \bar{E}\cdot\bar{D}\, dvol$ HOLDS ONLY FOR MATERIALS IN WHICH $\bar{D} = \epsilon\bar{E}$ HOLDS.

FORCES ON DIELECTRICS (BECKER SEC. 35)

THE FORCE ASSOCIATED WITH A POSSIBLE MOTION OF A DIELECTRIC, WHICH (MOTION) CAN BE DESCRIBED BY COORDINATE q, IS

$$F = -\frac{\partial U}{\partial q} \qquad \text{WHERE} \qquad U = \frac{1}{8\pi}\int \bar{E}\cdot\bar{D}\, dvol.$$

HERE WE SEEK AN EXPRESSION FOR THE ELECTRICAL FORCE DENSITY $\bar{f}$ SUCH THAT THE TOTAL FORCE ON A VOLUME CAN BE OBTAINED BY INTEGRATING:

$$\bar{F} = \int \bar{f}\, dvol.$$

WE WILL DEDUCE $\bar{f}$ FROM AN ENERGY ARGUMENT. SUPPOSE THE VOLUME CONTAINS A DIELECTRIC, AND WE DISPLACE (DEFORM) IT BY A SMALL AMOUNT $\delta\bar{x}$. THEN THE WORK DONE BY THE ELECTRIC FORCE DENSITY $\bar{f}$ <u>LOWERS</u> THE STORED FIELD ENERGY:

$$\delta\text{work} = \int \delta\bar{x}\cdot f\, dvol = -\delta U = -\frac{1}{8\pi}\int \delta(\bar{E}\cdot\bar{D})\, dvol.$$

WE RESTRICT OUR ATTENTION TO ISOTROPIC, LINEAR DIELECTRICS, FOR WHICH $\bar{D} = \epsilon\bar{E}$. WE PERMIT ϵ TO VARY WITH POSITION (i.e., AT A BOUNDARY), AND WE NOTE THAT, IN GENERAL, IF THE DIELECTRIC DEFORMS THEN ϵ WILL CHANGE.

OUR METHOD IS TO TRANSFORM $\int \delta(\bar{E}\cdot\bar{D})\, dvol$ INTO $\int \delta\bar{x}\cdot(\dots)\, dvol$, SO WE CAN IDENTIFY $(\dots)$ WITH THE DESIRED FORCE DENSITY $\bar{f}$.

FIRST, $\delta U = \frac{1}{8\pi}\int \delta\left(\frac{D^2}{\epsilon}\right) dvol = \frac{1}{4\pi}\int \frac{\bar{D}\cdot\delta\bar{D}}{\epsilon}\, dvol - \frac{1}{8\pi}\int \frac{D^2}{\epsilon^2}\,\delta\epsilon\, dvol = \frac{1}{4\pi}\int \bar{E}\cdot\delta\bar{D}\, dvol - \frac{1}{8\pi}\int E^2\,\delta\epsilon\, dvol$

(IN THIS DERIVATION, $\delta\epsilon \neq 0$.)

COMPARING WITH P. 31 WE SEE THAT $\frac{1}{4\pi}\int \bar{E}\cdot\delta\bar{D}\, dvol$ CAN BE TRANSFORMED BACK TO $\int \phi\,\delta\rho\, dvol$ (EVEN IF ϵ CAN VARY).

CHARGE (AS WELL AS MASS) IS CONSERVED, AND OBEYS THE CONTINUITY EQUATION $\dot{\rho} = -\bar{\nabla}\cdot\bar{J}$. FOR CHANGES ASSOCIATED WITH DISPLACEMENT $\delta\bar{x}$ IN TIME δt, WE CAN WRITE $\dot{\rho} = \frac{\delta\rho}{\delta t}$ AND $\bar{J} = \rho\frac{\delta\bar{x}}{\delta t}$. HENCE $\delta\rho = -\bar{\nabla}\cdot(\rho\,\delta\bar{x})$

SIMILARLY, THE MASS DENSITY ρ_M OBEYS $\delta\rho_M = -\bar{\nabla}\cdot(\rho_M\,\delta\bar{x})$

HENCE $\int \phi\,\delta\rho\, dvol = -\int \phi\,\bar{\nabla}\cdot(\rho\,\delta\bar{x})\, dvol = -\int \bar{\nabla}\cdot(\phi\,\rho\,\delta\bar{x})\, dvol + \int \rho\,\delta\bar{x}\cdot\bar{\nabla}\phi\, dvol.$

VIA GAUSS' THEOREM WE TRANSFORM THE INTEGRAL ↑ TO A SURFACE INTEGRAL, WHICH VANISHES FOR A SURFACE LARGE ENOUGH TO CONTAIN ALL CHARGES INSIDE. THE REMAINING INTEGRAL TELLS US $\int \phi\,\delta\rho\, dvol = -\int \delta\bar{x}\cdot\rho\bar{E}\, dvol.$

THUS WE DEDUCE THAT PART OF THE FORCE DENSITY $\bar{f}$ IS JUST $\rho\bar{E}$, AS EXPECTED.

FOR THE OTHER PART WE NEED A MODEL OF HOW THE DIELECTRIC CONSTANT VARIES WITH DEFORMATION. FOR SOLIDS, SEE STRATTON, SEC. 2.22, OR LANDAU & LIFSHITZ, VOL 8, SEC. 16. FOR LIQUIDS OR GASES, ϵ DEPENDS ONLY ON THE MASS DENSITY (P. 21), SO

$\delta\epsilon = \frac{d\epsilon}{d\rho_M}\,\delta\rho_M = -\frac{d\epsilon}{d\rho_M}\,\bar{\nabla}\cdot(\rho_M\,\delta\bar{x})$ USING THE CONTINUITY EQUATION FOR MASS FLOW.

$$-\frac{1}{8\pi}\int E^2\,\delta\epsilon\, dvol = \frac{1}{8\pi}\int E^2\frac{d\epsilon}{d\rho_M}\,\bar{\nabla}\cdot(\rho_M\,\delta\bar{x})\, dvol = \overbrace{\frac{1}{8\pi}\int \bar{\nabla}\cdot\left(E^2\frac{d\epsilon}{d\rho_M}\,\rho_M\,\delta\bar{x}\right) dvol}^{0 \text{ VIA GAUSS}} -$$

$$-\frac{1}{8\pi}\int \rho_M\,\delta\bar{x}\cdot\left(\bar{\nabla}E^2\frac{d\epsilon}{d\rho_M}\right) dvol = -\frac{1}{8\pi}\int \delta\bar{x}\cdot\bar{\nabla}\left(E^2\rho_M\frac{d\epsilon}{d\rho_M}\right) dvol + \frac{1}{8\pi}\int \delta\bar{x}\cdot E^2\underbrace{\frac{d\epsilon}{d\rho_M}\bar{\nabla}\rho_M}_{\bar{\nabla}\epsilon}\, dvol.$$

ALTOGETHER,

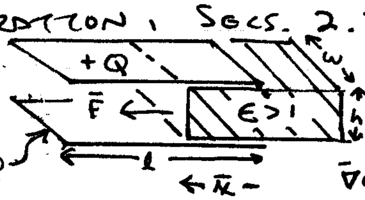
$$\int \delta\bar{x}\cdot\bar{f}\, dvol = -\delta U = \int \delta\bar{x}\cdot\left(\rho\bar{E} - \frac{1}{8\pi}E^2\bar{\nabla}\epsilon + \frac{1}{8\pi}\bar{\nabla}\left(E^2\rho_M\frac{d\epsilon}{d\rho_M}\right)\right) dvol.$$

HENCE, $\bar{f} = \rho\bar{E} - \frac{1}{8\pi}E^2\bar{\nabla}\epsilon + \frac{1}{8\pi}\bar{\nabla}\left(E^2\rho_M\frac{d\epsilon}{d\rho_M}\right)$ IS THE VOLUME FORCE DENSITY FOR LIQUID DIELECTRICS.

THE SECOND TERM, $-\frac{1}{8\pi} E^2 \vec{\nabla}\epsilon$, CORRESPONDS TO THE FORCE ON THE POLARIZATION CHARGE DENSITY AT A BOUNDARY WHERE THE DIELECTRIC CONSTANT CHANGES ABRUPTLY.

THE THIRD TERM, $\frac{1}{8\pi}\vec{\nabla}\left(E^2 \rho_M \frac{d\epsilon}{d\rho_M}\right)$, IS ASSOCIATED WITH THE PHENOMENON OF ELECTROSTRICTION, IN WHICH ELECTRIFIED DIELECTRICS EXPERIENCE A CHANGE IN THEIR INTERNAL PRESSURE — WHICH LEADS TO ADDITIONAL FORCES ON ANY CONSTRAINING BOUNDARIES. See Becker, sec. 37, Stratton, Secs. 2.25-26.

EXAMPLE:



$\vec{F} = -\frac{1}{8\pi}\int E^2 \vec{\nabla}\epsilon \, dvol$. $\vec{\nabla}\epsilon \neq 0$ ONLY AT BOUNDARY. EFFECTS AT TOP & BOTTOM CANCEL. AT LEFT FACE, $\vec{\nabla}\epsilon = -\frac{\epsilon - 1}{dx}\hat{x}$, $dvol = wh\,dx$, AND E IS INDEPENDENT OF x

SO $\vec{F} = \frac{E^2 wh(\epsilon-1)}{8\pi}\hat{x} \Rightarrow$ DIELECTRIC IS PULLED IN! CHALLENGE: VERIFY $\vec{F}$ VIA $F = -\frac{\partial U}{\partial x}$ & $U = \frac{1}{8\pi}\int \vec{E}\cdot\vec{D}\, dvol$

MAXWELL STRESS TENSOR (Becker sec. 36)

WHEN FARADAY TALKED OF 'LINES OF FORCE' HE INTENDED A VERY LITERAL INTERPRETATION. MAXWELL SHOWED HOW TO QUANTIFY THIS IDEA.

MAXWELL'S NOTION IS THAT ACROSS ANY SURFACE ELEMENT $d\vec{S}$, WHETHER IN A MATERIAL OR IN VACUUM, A FORCE IS TRANSMITTED IF $\vec{E} \neq 0$. THIS FORCE, $d\vec{F}$, NEED NOT BE PARALLEL TO $d\vec{S}$. IN GENERAL, WE HAVE A TENSOR RELATION — AS IN THE BEHAVIOR OF ELASTIC SOLIDS:

$$dF_i = T_{ij}\, dS_j \qquad T_{ij} \equiv \underline{\text{STRESS TENSOR}}$$

THE TOTAL FORCE ON A VOLUME WOULD THEN BE GIVEN BY

$$F_i = \int_{\text{SURFACE ENCLOSING THE VOLUME}} T_{ij}\, dS_j$$

BUT IF WE KNEW THE VOLUME FORCE DENSITY $\vec{f}$, WE COULD ALSO WRITE

$$F_i = \int_{\text{VOLUME}} f_i \, dvol$$

WE CAN RELATE THESE TWO EXPRESSIONS BY GAUSS' THEOREM:

$$\int_S T_{ij}\, dS_j = \int_S (\vec{T}_i)\cdot d\vec{S} = \int_V \vec{\nabla}\cdot(\vec{T}_i)\, dvol = \int_V \frac{\partial T_{ij}}{\partial x_j}\, dvol$$

HENCE WE IDENTIFY $f_i = \frac{\partial T_{ij}}{\partial x_j}$ OR $\vec{f} = \vec{\nabla}\cdot\overline{\overline{T}}$

WE APPLY THIS TO THE FORCE DENSITY $\bar{f} = \rho\bar{E} - \frac{E^2 \bar{\nabla}\epsilon}{8\pi} + \frac{1}{8\pi}\bar{\nabla}\left(E^2 \rho_M \frac{d\epsilon}{d\rho_M}\right)$

ONLY THE LAST TERM IS ALREADY THE DERIVATIVE OF SOMETHING. SO WE PLAY TRICKS:

$$\rho E_i = \frac{(\bar{\nabla}\cdot\bar{D})}{4\pi} E_i = \frac{E_i}{4\pi}\frac{\partial D_j}{\partial x_j} = \frac{1}{4\pi}\frac{\partial E_i D_j}{\partial x_j} - \frac{D_j}{4\pi}\frac{\partial E_i}{\partial x_j}$$

SINCE $\bar{\nabla}\times\bar{E} = 0$ WE HAVE $\frac{\partial E_i}{\partial x_j} = \frac{\partial E_j}{\partial x_i}$, AND THE LAST TERM BECOMES

$$-\frac{D_j}{4\pi}\frac{\partial E_j}{\partial x_i} = -\frac{\epsilon}{8\pi}\frac{\partial E^2}{\partial x_i}.$$

THIS COMBINES WITH THE MIDDLE TERM OF $\bar{f}$ TO GIVE A TOTAL DERIVATIVE

HENCE
$$f_i = \frac{1}{4\pi}\frac{\partial E_i D_j}{\partial x_j} - \frac{1}{8\pi}\left(\epsilon\frac{\partial E^2}{\partial x_i} + E^2\frac{\partial \epsilon}{\partial x_i}\right) + \frac{1}{8\pi}\frac{\partial}{\partial x_i}\left(E^2 \rho_M \frac{d\epsilon}{d\rho_M}\right)$$
$$= \frac{\partial}{\partial x_j}\left[\frac{E_i D_j}{4\pi} - \frac{\delta_{ij}}{8\pi}\left(\epsilon E^2 - E^2\rho_M\frac{d\epsilon}{d\rho_M}\right)\right]$$

THUS WE IDENTIFY THE STRESS TENSOR AS

$$T_{ij} = \frac{E_i D_j}{4\pi} - \frac{\delta_{ij}}{8\pi}\left(\epsilon E^2 - E^2\rho_M\frac{d\epsilon}{d\rho_M}\right) = \frac{1}{4\pi}\left[\epsilon E_i E_j - \frac{\delta_{ij}}{2}\left(\epsilon E^2 - E^2\rho_M\frac{d\epsilon}{d\rho_M}\right)\right]$$

T_{ij} IS A SYMMETRIC TENSOR. SO WE CAN ALWAYS CHOOSE (LOCAL) COORDINATE AXES SUCH THAT IT IS DIAGONAL. FOR EXAMPLE, CHOOSE THE X-AXIS ALONG $\bar{E}$. THEN

$$T_{ij} = \frac{1}{8\pi}\begin{pmatrix} E^2\left(\epsilon + \rho_M\frac{d\epsilon}{d\rho_M}\right) & & 0 \\ & -E^2\left(\epsilon - \rho_M\frac{d\epsilon}{d\rho_M}\right) & \\ 0 & & -E^2\left(\epsilon - \rho_M\frac{d\epsilon}{d\rho_M}\right)\end{pmatrix}$$

[LIQUID DIELECTRIC]

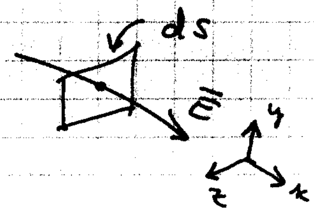
[$\bar{E}$ ALONG $\hat{x}$]

$$T_{ij} = \frac{1}{8\pi}\begin{pmatrix} E^2 & & 0 \\ & -E^2 & \\ 0 & & -E^2\end{pmatrix} \quad \text{(VACUUM)}$$

REMARK: IN A SYSTEM WITH NO DIELECTRICS WE COULD USE $\bar{f} = \rho\bar{E}$ TO ARRIVE QUICKLY AT THE STRESS TENSOR LABELLED 'VACUUM'.

WHAT DOES THIS MEAN?

CONSIDER A SURFACE ELEMENT $\perp$ TO $\bar{E}$



THEN $d\bar{S}$ IS $\parallel$ TO $\bar{E}$: $d\bar{S} = dS\,\hat{x}$

$$dF_x = T_{xx}\,dS_x + T_{xy}\,dS_y + T_{xz}\,dS_z$$

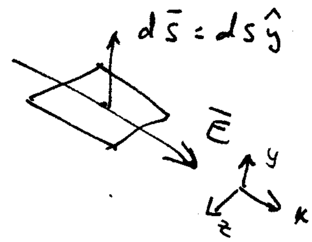
$$= +\frac{E^2}{8\pi}\,dS$$

THE + MEANS THE FORCE TENDS TO PULL THE SURFACE IN THE DIRECTION OF $\bar{E}$.

SINCE $\frac{dF_x}{dS}$ HAS DIMENSIONS OF PRESSURE, $\frac{E^2}{8\pi}$ IS SOMETIMES CALLED THE "ELECTROSTATIC PRESSURE". BUT STRICTLY SPEAKING, IT IS NOT A SCALAR - OBEYING ARCHIMEDES RULES, ETC.

FOR THE SURFACE ABOVE, $dF_y = dF_z = 0$

BUT CONSIDER A DIFFERENT SURFACE, WHICH IS PARALLEL TO $\bar{E}$. FOR EXAMPLE

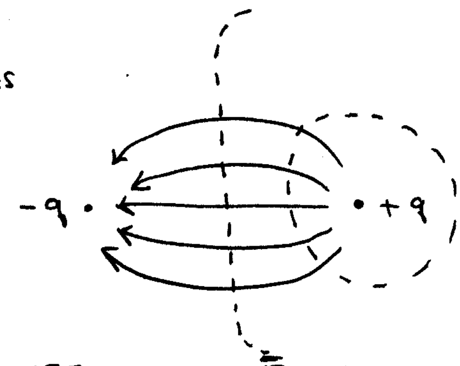


$$d\bar{S} = dS\,\hat{y}$$

THEN $dF_x = 0 = dF_z$

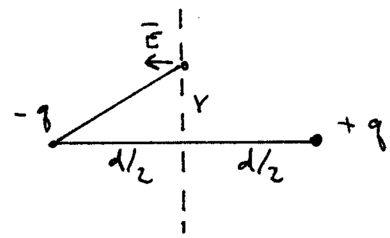
BUT $dF_y = -\frac{E^2}{8\pi}\,dS$ WHICH PUSHES IN THE $-y$ DIRECTION.

<u>EXAMPLE</u> TWO OPPOSITE POINT CHARGES



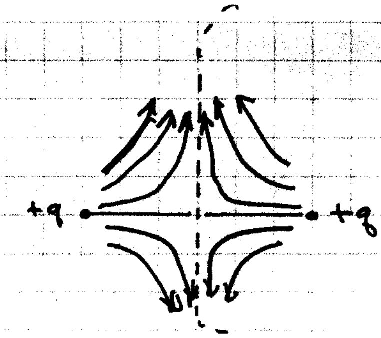
CONSIDER ANY SURFACE SURROUNDING THE + CHARGE. THE FORCE <u>ON THE CHARGE</u> IS THEN $F_i = \int_{\text{SURFACE}} T_{ij}\,dS_j$. QUALITATIVELY, WE SEE THAT THE FORCE MUST BE TO THE LEFT, SINCE $\bar{E}$ IS (ROUGHLY) PARALLEL TO THE OUTWARD NORMAL OF THE SURFACE ELEMENT TO LEFT OF q.

WE CAN BE QUANTITATIVE IF WE CONSIDER A SURFACE CONSISTING OF THE MIDPLANE AND A HEMISPHERE ON THE RIGHT AT $r = \infty$.



$$F = \frac{1}{8\pi}\int E^2\,dS = \frac{1}{8\pi}\int_0^\infty 2\pi r\,dr\left[\frac{2q\cdot d/2}{(r^2+(d/2)^2)^{3/2}}\right]^2$$

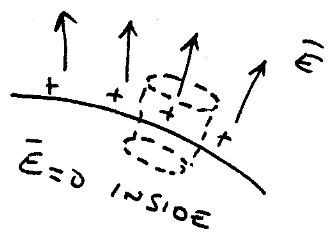
$$= \frac{q^2 d^2}{8}\int_0^\infty \frac{dr^2}{(r^2+(d/2)^2)^3} = \frac{q^2 d^2}{16}\left(\frac{2}{d}\right)^4 = +\frac{q^2}{d^2}$$



EXAMPLE TWO LIKE CHARGES **REPEL**

MAXWELL HAS THIS UNDER CONTROL. AT THE MIDPLANE $\bar{E}$ IS $\parallel$ TO THE SURFACE $\Rightarrow$ FORCE $\perp$ TO $\bar{E}$. FOR A SURFACE CLOSED AROUND THE CHARGE TO THE RIGHT, OUR SIGN CONVENTIONS TELL US THE FORCE IS TO THE RIGHT $\Rightarrow$ REPULSION!

$$F = -\frac{1}{8\pi}\int_0^\infty 2\pi r\,dr\left[\frac{2q\,r}{\left[r^2+\left(\frac{d}{2}\right)^2\right]^{3/2}}\right]^2 = -\frac{q^2}{d^2}$$



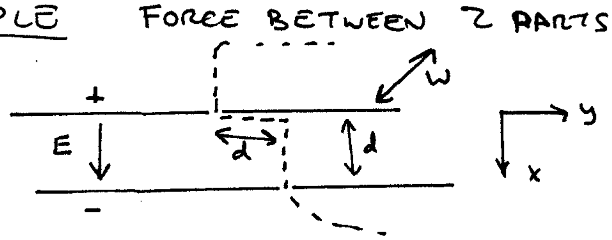
EXAMPLE FORCE AT THE SURFACE OF A CONDUCTOR

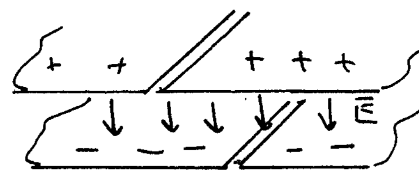
CONSIDER A PILLBOX WHICH SURROUNDS A PIECE dS ON THE SURFACE OF THE CONDUCTOR.
$\bar{E}$ IS $\perp$ TO THE SURFACE OUTSIDE, AND $E = 4\pi\sigma$

$$\therefore \quad F_\perp = \frac{1}{8\pi}E^2 dS = \frac{1}{2}\sigma E\,dS$$

AS DISCUSSED EARLIER

EXAMPLE FORCE BETWEEN 2 PARTS OF A SPLIT CAPACITOR



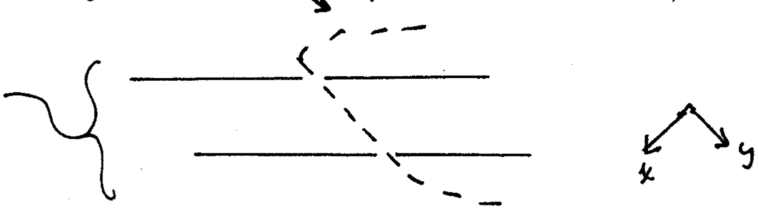


FOR THE SURFACE AS SHOWN, THE FORCE ON THE RIGHT HAND PIECE

IS $\bar{F} = \frac{E^2}{8\pi}wd\,\hat{x} + \frac{E^2}{8\pi}wd\,\hat{y}$ (DIRECTION OF $\bar{F}$)

$|\bar{F}| = \frac{E^2 wd\sqrt{2}}{8\pi}$

SUPPOSE WE CHOOSE INSTEAD



$$d\bar{S} = dS\,\hat{x} = \sqrt{2}\,wd\,\hat{x}$$

$$\bar{E} = \frac{E}{\sqrt{2}}\hat{x} + \frac{E}{\sqrt{2}}\hat{y}$$

$$T_{ij} = \frac{1}{4\pi}\begin{pmatrix} 0 & \frac{E^2}{2} & 0 \\ \frac{E^2}{2} & 0 & 0 \\ 0 & 0 & -E^2/2 \end{pmatrix}$$

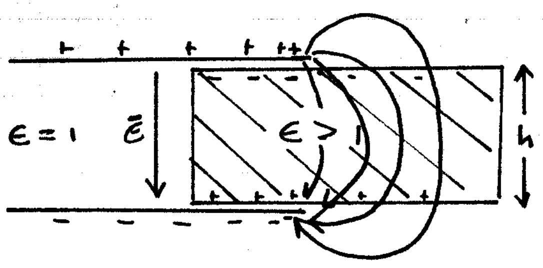
$F_x = T_{xx}\,dS_x = 0$

$F_y = T_{yx}\,dS_x = \frac{E^2}{8\pi}wd\sqrt{2}$

$F_z = T_{zx}\,dS_x = 0$

THE MAGNITUDE AND DIRECTION OF $\bar{F}$ ARE AS BEFORE!

EXAMPLE: FORCE ON DIELECTRIC SLAB PART WAY INSIDE A CAPACITOR (P.33).

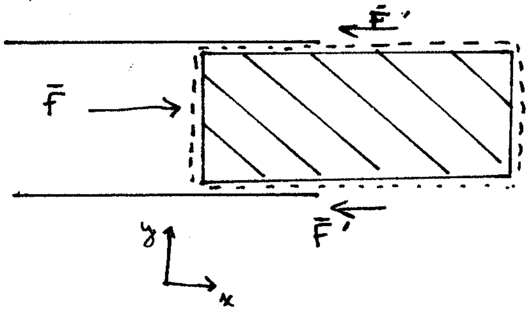


W = WIDTH ⊥ TO PAPER.

THIS PROBLEM IS COMPLICATED BY THE EFFECT OF THE FRINGE FIELD OF THE CAPACITOR!

THE SURFACE POLARIZATION CHARGE ON THE TOP & BOTTOM OF THE DIELECTRIC SLAB IS PULLED TO THE LEFT BY THE HORIZONTAL COMPONENT OF THE FRINGE FIELD.

FIRST TRY: SURFACE SURROUNDS DIELECTRIC

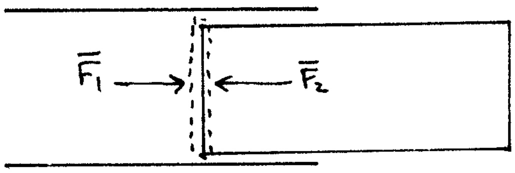


$$F = \frac{E^2}{8\pi} wh \quad \text{TO RIGHT}$$

$$F' = \frac{2w}{4\pi} \int E_x E_y \, dx \quad \text{TO LEFT}$$

BUT NOT EASY TO DO THE INTEGRAL!

SECOND TRY: SURFACE SURROUNDS ONLY THE LEFT FACE OF THE SLAB

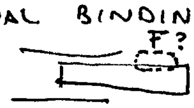


$$F_1 = \frac{E^2}{8\pi} wh \quad \text{TO RIGHT}$$

$$F_2 = \frac{\epsilon E^2}{8\pi} wh \quad \text{TO LEFT}$$

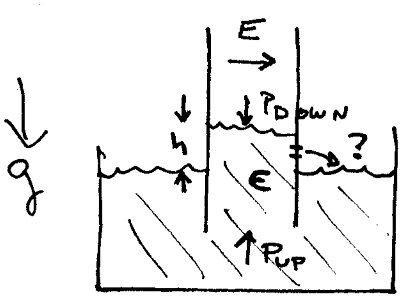
$$\bar{F}_{TOT} = \bar{F}_1 + \bar{F}_2 = (\epsilon - 1)\frac{E^2}{8\pi} wh \quad \text{TO } \underline{\text{LEFT}}$$

THIS FORCE IS TRANSMITTED TO THE ENTIRE SLAB BY ITS INTERNAL BINDING FORCES. CHALLENGE: SHOW $F_x = 0$ ON A PIECE OF THE TOP SURFACE SUCH AS



PARADOX: CAPACITOR PLATES IMMERSED IN A DIELECTRIC LIQUID

$$\Delta P = P_{UP} - P_{DOWN} = (\epsilon - 1)\frac{E^2}{8\pi} = \rho g h$$



$$\Rightarrow h = \frac{(\epsilon - 1) E^2}{8\pi \rho g}$$ = DISTANCE BY WHICH DIELECTRIC LIQUID IS DRAWN UP BETWEEN THE PLATES ABOVE THE NOMINAL LIQUID LEVEL.

? SUPPOSE A HOLE IS DRILLED IN ONE OF THE PLATES ABOVE THE NOMINAL LIQUID LEVEL, BUT BELOW HEIGHT h. WILL THERE BE A "PERPETUAL WATERFALL"?