PH 206 LECTURE 23

INTERACTION OF RADIATION WITH MATTER - MICROSCOPIC VIEW

OUR STUDIES OF ELECTROMAGNETIC RADIATION BEGAN WITH GIVEN WAVES INTERACTING WITH BULK MATTER - DIELECTRICS AND CONDUCTORS. WE WENT ON TO STUDY THE GENERATION OF WAVES BY MACROSCOPIC CURRENT DISTRIBUTIONS. THEN WITH THE HELP OF RELATIVITY WE INVESTIGATED THE GENERATION OF RADIATION ON THE MICROSCOPIC SCALE BY INDIVIDUAL ACCELERATING CHARGES. TO COMPLETE OUR PICTURE WE NOW CONSIDER THE INTERACTION OF RADIATION WITH MATTER FROM AN ATOMIC VIEWPOINT. [SOME RESULTS HAVE ALREADY BEEN PRESENTED IN LECTURE 12.]

THE CLASSICAL PICTURE OF AN ATOM ~~WHICH~~ HAS TAUGHT US THE MOST IS THAT OF AN ELECTRON ON THE END OF A SPRING WHICH IS TIED TO A HEAVY NUCLEUS. WRITING THE SPRING EQUATION AS

$$m\ddot{\bar{x}} = -K\bar{x}$$

WE HAVE CALLED $\omega_0 = \sqrt{K/m}$ THE NATURAL FREQUENCY OF THE ATOM. AN ATOM MIGHT HAVE MORE THAN ONE NATURAL FREQUENCIES IF IT CONTAINS MORE THAN ONE ELECTRON. BUT EXPERIMENTALLY EVEN THE HYDROGEN ATOM HAS MORE THAN ONE 'NATURAL FREQUENCY' (BALMER SERIES, LYMAN SERIES ....), WHICH INDICATED THE NEED FOR A MODIFICATION OF OUR CLASSICAL VIEW...

FOR ATOMS IN BULK MATTER WE CONSIDERED THE POSSIBILITY THAT AN UNSPECIFIED INTERACTION BETWEEN THE ELECTRONS AND/OR NEIGHBORING ATOMS MIGHT ACT LIKE A DAMPING FORCE ON THE MOTION OF AN ELECTRON:

$$m(\ddot{\bar{x}} + \gamma\dot{\bar{x}} + \omega_0^2\bar{x}) = \bar{F}_{EXT}$$

$1/\gamma$ ~ CHARACTERISTIC TIME OF THE DAMPING MECHANISM.

WE CAN NOW RECOGNIZE THAT EVEN AN ISOLATED ATOM IS SUBJECT TO DAMPING DUE TO THE RADIATION REACTION

$$m(\ddot{\bar{x}} + \omega_0^2\bar{x}) = \bar{F}_R = \frac{2}{3}\frac{e^2\dddot{\bar{x}}}{c^3}$$

OR $$\ddot{\bar{x}} - \frac{2}{3}\frac{e^2}{mc^3}\dddot{\bar{x}} + \omega_0^2\bar{x} = 0$$

$$\ddot{\bar{x}} - \frac{2}{3}\frac{r_0}{c}\dddot{\bar{x}} + \omega_0^2\bar{x} = 0$$

AGAIN WE INTRODUCE $r_0 = \frac{e^2}{mc^2} = 2.8 \times 10^{-13}$ cm = CLASSICAL ELECTRON RADIUS.

NOTE THAT $\tau_0 = \frac{r_0}{c} \sim 10^{-23}$ sec IS A TIME CHARACTERISTIC OF THE RADIATION REACTION. IT IS JUST THE TIME REQUIRED FOR RADIATION TO PROPAGATE FROM ONE END TO THE OTHER OF A 'CLASSICAL' ELECTRON.

SINCE τ_0 IS SO SMALL THE RADIATION DAMPING IS A TINY EFFECT, BUT IT IS ALWAYS PRESENT.

HENCE IF WE STRETCH THE ELECTRON IN AN ISOLATED ATOM TO x_0 AND THEN LET GO, WE EXPECT IT WILL OSCILLATE ALMOST ACCORDING TO $x \sim x_0 e^{-i\omega_0 t}$

THEN $\dddot{x} \sim -\omega_0^2 \dot{x}$, AND WE CAN APPROXIMATE THE EQUATION OF MOTION AS

$$\ddot{x} + \tfrac{2}{3}\omega_0^2 \tau_0 \dot{x} + \omega_0^2 x = 0$$

WE DEFINE $\gamma_0 = \frac{2}{3}\omega_0^2 \tau_0$ = RADIATION DAMPING CONSTANT

SO THAT $\ddot{x} + \gamma_0 \dot{x} + \omega_0^2 x = 0$

IN THE USUAL MANNER WE INTRODUCE THE TRIAL SOLUTION $x = x_0 e^{\alpha t}$ WHICH LEADS TO A QUADRATIC EQUATION IN α WITH ROOTS

$$\alpha \sim -\frac{\gamma_0}{2} \pm i\omega_0\left(1 - \frac{\gamma_0^2}{8\omega_0^2}\right) \qquad \text{ASSUMING } \frac{\gamma_0}{\omega_0} \ll 1$$

TO
$$x = \underbrace{x_0 e^{-\frac{\gamma_0 t}{2}}}_{\text{DAMPING}} \; e^{-i(\omega_0 - \underbrace{\Delta\omega}_{\text{FREQUENCY SHIFT}})t}$$

THE SHIFT IS $\Delta\omega = \frac{\gamma_0^2}{8\omega_0} = \frac{1}{18}\omega_0^3 \tau_0^2$

REMARK: IF WE HAD USED $\ddot{x} - \frac{2}{3}\tau_0 \dddot{x} + \omega_0^2 x = 0$ AND TRIED $x = x_0 e^{\alpha t}$ WE WOULD GET A CUBIC EQUATION FOR α. TWO OF THE ROOTS ARE ESSENTIALLY THOSE GIVEN. THE THIRD ROOT SUGGESTS $x = x_0 e^{\frac{t}{2\tau_0}}$, WHICH IS THE UNPHYSICAL 'RUNAWAY' SOLUTION MENTIONED IN LECTURE 22.

THE DAMPING FACTOR $e^{-\frac{\gamma_0 t}{2}}$ MEANS THAT AN ISOLATED ATOM, ONCE EXCITED, WILL NOT RADIATE FOREVER - TO MAINTAIN CONSERVATION OF ENERGY! THE OSCILLATION DIES OUT IN CHARACTERISTIC TIME $\tau \sim \frac{2}{\gamma_0} \sim \frac{3}{4\tau_0\omega_0^2}$

FOR EXAMPLE, A GREEN ATOM HAS $\lambda \sim 6\times10^{-5}$ cm $\Rightarrow \omega_0 \sim 3\times10^{15}$

$\Rightarrow \tau \sim 10^{-8}$ SEC WHICH IS BORNE OUT BY EXPERIMENT!

OF COURSE THIS ESTIMATE REPRESENTS A MAXIMUM TIME SCALE FOR THE RADIATION PROCESS. ANY OTHER DAMPING MECHANISM WILL ONLY INCREASE γ COMPARED TO $\gamma_0 \Rightarrow \tau$ BECOMES SHORTER.

THE FREQUENCY SHIFT EFFECT IS TINY.

$$\frac{\Delta\omega}{\omega_0} \sim (\omega_0\tau_0)^2 \sim 10^{-15} \quad \text{FOR GREEN LIGHT}$$

IT ONLY BECOMES SIGNIFICANT AT ULTRA HIGH FREQUENCIES $\omega_0\tau_0 \sim 1$ AT WHICH WE SUGGESTED IN LECTURE 22 OUR CLASSICAL THEORY WILL BREAK DOWN.

A 'PHOTON' OF FREQUENCY $\omega \sim 1/\tau_0$ HAS ENERGY

$$U = \hbar\omega = \frac{\hbar}{\tau_0} = \frac{\hbar c}{r_0} = \left(\frac{\hbar c}{e^2}\right) mc^2 = 137 \times \text{REST MASS OF ELECTRON}$$

FOR EXAMPLE, AT THESE HIGH FREQUENCIES A QUANTUM MECHANICAL EFFECT CALLED 'VACUUM POLARIZATION' BECOMES IMPORTANT. THE HIGH FREQUENCY FIELD CAN CREATE e^+-e^- PAIRS (ELECTRON-POSITRON) SEEMINGLY OUT OF THE VACUUM - BUT ACTUALLY OUT OF THE TREMENDOUS ENERGY DENSITY OF THE FIELD. IN SOME SENSE THIS REVIVES MAXWELL'S NOTION THAT THE VACUUM IS A COMPLICATED STRUCTURE CONTAINING + AND - CHARGES WHICH NORMALLY CANCEL, BUT CAN BE SEPARATED ON OCCASION ...

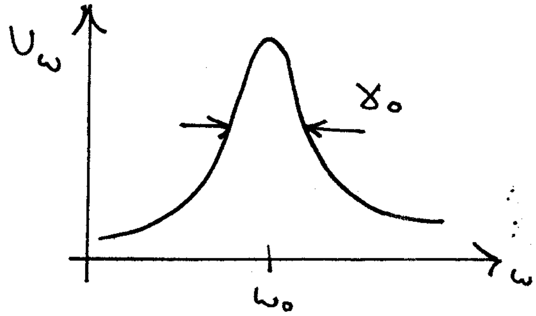
FREQUENCY SPECTRUM AS IN PROBLEM (5), SET 7, THE DAMPING TERM CAUSES A PULSE OF RADIATION TO BE EMITTED, WHOSE FINITE DURATION IMPLIES THE RADIATION CANNOT BE A PURE FREQUENCY.

THAT IS,
$$\bar{E}(t) \sim \begin{cases} \bar{E}_0\, e^{-\frac{\gamma_0 t}{2}}\, e^{-i\omega_0 t} & t>0 \\ 0 & t<0 \end{cases} \quad \text{(NEGLECTING } \Delta\omega)$$

SO
$$E_\omega = \frac{E_0}{2\pi}\int_0^\infty e^{-\frac{\gamma_0 t}{2}}\, e^{i(\omega-\omega_0)t}\,dt = \frac{E_0}{2\pi}\,\frac{i}{\omega-\omega_0+i\gamma_0/2}$$

THE PULSE ENERGY $U = \int \frac{dU}{dt}\,dt = \int U_\omega\, d\omega$ WITH $U_\omega \sim 4\pi |E_\omega|^2$

SO $U_\omega \sim \dfrac{1}{(\omega-\omega_0)^2 + \gamma_0^2/4}$



THE SPECTRUM PEAKS AT ω_0, BUT IT HAS A SPREAD AT HALF MAXIMUM OF

$$\Delta\omega \approx \gamma_0$$

OF COURSE THIS IS JUST WHAT WE WOULD EXPECT FROM OUR 'UNCERTAINTY RELATION' $\Delta\omega\Delta t \sim 1$: $\Delta t \sim \frac{2}{\gamma_0} \Rightarrow \Delta\omega \sim \frac{\gamma_0}{2}$

NATURE ABHORS PURE FREQUENCIES! A SPECTRAL 'LINE' ALWAYS HAS A FINITE SPREAD OF FREQUENCIES (OR WAVELENGTHS) EVEN IF FOR NO OTHER REASON THAN THE RADIATION REACTION.
[SEE A.L. SCHAWLOW IN PHYSICS TODAY, DEC. 1982 REGARDING NARROW SPECTRAL LINES.]

INTERACTION OF ELECTRONS WITH EXTERNAL RADIATION

ABOVE WE CONSIDERED AN ATOM THAT WAS INSTANTANEOUSLY 'STRETCHED' AT $t=0$ AND THEN LET GO. SUCH STRETCHING MIGHT BE DUE TO A COLLISION, OR PERHAPS A RADIOACTIVE DECAY..

WE NOW TURN TO THE VERY COMMON CASE IN WHICH THE ATOM IS STRETCHED BY AN EXTERNAL ELECTROMAGNETIC WAVE WHICH PERSISTS IN TIME. SO WE WILL HAVE A FORCED OSCILLATOR PROBLEM. THE OSCILLATING ATOM THEN RADIATES. ON THE MACROSCOPIC SCALE THIS LEADS TO REFLECTION AND REFRACTION.

ON THE MICROSCOPIC SCALE WE REPORT THIS 'RERADIATION' IN TERMS OF A CROSS-SECTION

$$\frac{d\sigma}{d\Omega} \equiv \frac{\left.\frac{dU}{dt\,d\Omega}\right|_{\text{RADIATED}}}{|S|_{\text{INCIDENT}}} \qquad \text{AS IN LECTURE 16.}$$

LATER IN THIS LECTURE WE INDICATE HOW SUMMING THIS MICROSCOPIC RE-RADIATION OVER ALL ATOMS LEADS TO THE INDEX OF REFRACTION - THE APPARENT REDUCTION OF THE SPEED OF LIGHT IN A DIELECTRIC.

THOMSON SCATTERING

WE BEGIN WITH THE SIMPLE CASE IN WHICH THE ELECTRON IS FREE - NOT TIED TO A NUCLEUS.

WE SUPPOSE THE INDUCED MOTION OF THE ELECTRON IS NON RELATIVISTIC, SO THAT WE MAY USE THE LARMOR FORMULA

$$\frac{dU}{dt\,d\Omega} = \frac{e^2 a^2 \sin^2\alpha}{4\pi c^3}$$

FOR A FREE ELECTRON $m\bar{a} = e\bar{E}$ WITH $\bar{E} = E_0 e^{i(kz-\omega t)}$

FOR A PLANE WAVE INCIDENT IN THE $+z$ DIRECTION.

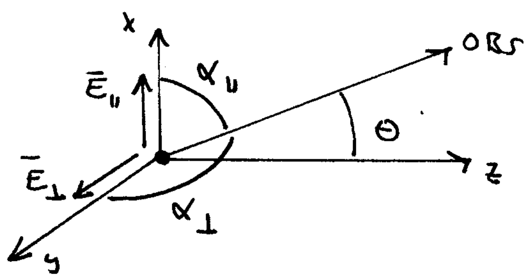
$$\text{SO} \quad \left\langle \frac{dU}{dt\,d\Omega} \right\rangle = \frac{e^4 E_0^2 \sin^2\alpha}{8\pi m^2 c^3}$$

$$\text{WHILE} \quad \langle \bar{S} \rangle_{IN} = \frac{c}{8\pi} \mathrm{Re}(\bar{E}\times\bar{B}^*) = \frac{c}{8\pi} E_0^2$$

$$\text{THUS} \quad \frac{d\sigma}{d\Omega} = \frac{e^4}{m^2 c^4} \sin^2\alpha = r_0^2 \sin^2\alpha$$

WHERE $r_0 = e^2/mc^2$ = CLASSICAL ELECTRON RADIUS

FOR AN OBSERVER IN THE X-Z PLANE, WE CAN DISTINGUISH 2 CASES:



$$\bar{E} \parallel \hat{x} : \quad \frac{d\sigma_{\parallel}}{d\Omega} = r_0^2 \cos^2\Theta$$

$$\bar{E} \perp \hat{x} \quad \frac{d\sigma_{\perp}}{d\Omega} = r_0^2$$

FOR UNPOLARIZED INCIDENT WAVES, (OR, AVERAGING OVER OBSERVERS WITH ARBITRARY AZIMUTHAL ANGLE)

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{d\sigma_{\parallel}}{d\Omega} + \frac{d\sigma_{\perp}}{d\Omega}\right) = r_0^2 \, \frac{1+\cos^2\Theta}{2}$$

$$\text{AND} \quad \sigma = \sigma_{\text{THOMSON}} = \frac{8\pi}{3} r_0^2 \sim 10^{-24}\ \text{cm}^2$$

OF COURSE πr_0^2 IS THE GEOMETRICAL CROSS SECTION OF A 'CLASSICAL ELECTRON'.

THE AREA $10^{-24}\ cm^2$ IS CALLED A BARN (FROM 'AS BIG AS A ....') AND SETS THE SCALE OF CROSS SECTIONS OF SCATTERING OFF INDIVIDUAL ELECTRONS.

REMARK: THE TOTAL CROSS SECTION $\sigma_{THOMSON}$ CAN BE GOTTEN MORE QUICKLY FROM THE OTHER LARMOR FORMULA

$$\frac{dU}{dt} = \frac{2}{3}\frac{e^2a^2}{c^3} \Rightarrow \left\langle \frac{dU}{dt}\right\rangle = \frac{e^4 E_0^2}{3m^2c^3}$$

$$\sigma_{THOMSON} = \frac{\langle dU/dt\rangle}{|\bar{S}|_{IN}} = \frac{8\pi}{3}\frac{e^4}{m^2c^4} = \frac{8\pi}{3}r_0^2$$

OUR USE OF THE LARMOR FORMULA IS JUSTIFIED IF THE VELOCITY OF THE ELECTRON REMAINS SMALL DURING THE OSCILLATIONS. FOR INTENSE INCIDENT WAVES THIS MAY NOT BE TRUE, LEADING TO CORRECTIONS TO THOMSON'S ANALYSIS. SEE THE HOMEWORK SET.

FOR VERY HIGH FREQUENCY WAVES THE PHOTON CONCEPT BECOMES MORE RELEVANT. IN PROBLEM 6, SET 9 WE FOUND THAT ONE CONSEQUENCE IS THE COMPTON EFFECT: THE SCATTERED RADIATION HAS LOWER FREQUENCY THAN THE INCIDENT RADIATION

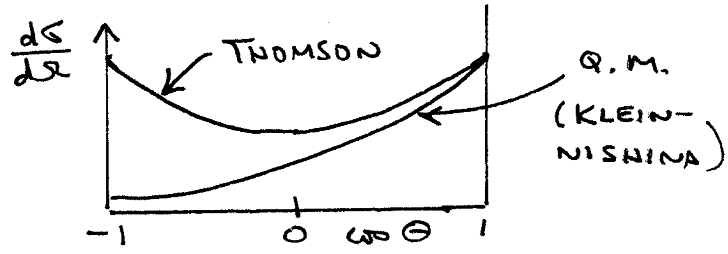
$$\frac{\omega_{SCAT}}{\omega_{IN}} = \frac{1}{1+\frac{\hbar\omega}{mc^2}(1-\cos\theta)} \leq 1$$

THIS OCCURS BECAUSE SOME OF THE INCIDENT PHOTON'S ENERGY AND MOMENTUM IS GIVEN UP TO THE RECOIL OF THE ELECTRON. = A KIND OF "RADIATION REACTION".

ALONG WITH THE FREQUENCY SHIFT THERE IS A MODIFICATION TO THE SCATTERING CROSS SECTION, BUT THIS TURNS OUT TO BE SURPRISINGLY SIMPLE. IN PROBLEM 6 SET 7 ON THE RAYLEIGH-JEANS LAW YOU SHOWED THAT THE NUMBER OF MODES FOR WAVES IN A BOX (=THE UNIVERSE) IS PROPORTIONAL TO ω^2.
AS SUCH THERE AREN'T AS MANY WAYS TO SCATTER AND PRODUCE LOW FREQUENCY PHOTONS

$$\left.\frac{d\sigma}{d\Omega}\right|_{Q.M.} = r_0^2\frac{(1+\cos^2\theta)}{2}\left|\frac{1}{1+\frac{\hbar\omega}{mc^2}(1-\cos\theta)}\right|^2$$ FOR UNPOLARIZED INCIDENT PHOTONS

THIS HAS THE EFFECT OF REDUCING THE BACKWARD SCATTERING.



Ph 206 Lecture 23

Scattering off Atomic Electrons

For scattering of light off an electron bound in an atom we can follow the analysis for Thomson scattering if we use the acceleration from

$$m(\ddot{\bar{x}} + \gamma \dot{\bar{x}} + \omega_0^2 \bar{x}) = \bar{F}_{EXT} = e\bar{E}_0 e^{-i\omega t} \qquad (z=0)$$

Here we define $\gamma = \gamma_0 +$ any other damping term present.

The oscillatory trial solution $\bar{x} = \bar{x}_0 e^{-i\omega t}$ leads to

$$\bar{x} = \frac{-e\bar{E}_0/m}{\omega^2 - \omega_0^2 + i\gamma\omega} \qquad \bar{a} = \ddot{\bar{x}} = \frac{e\bar{E}}{m} \frac{\omega^2}{\omega^2 - \omega_0^2 + i\gamma\omega}$$

Thus $$\left.\frac{d\sigma}{d\Omega}\right|_{UNPOL.} = r_0^2 \left(\frac{1 + \cos^2\theta}{2}\right) \frac{\omega^4}{(\omega^2-\omega_0^2)^2 + \gamma^2\omega^2}$$

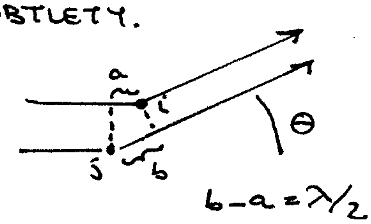
and hence $$\sigma_{SCATTER} = \sigma_{THOMSON} \frac{\omega^4}{(\omega^2-\omega_0^2)^2 + \gamma^2\omega^2}$$

We consider this expression in 3 frequency regions

1. Low frequency $\quad \omega \ll \omega_0 \qquad \sigma_{SCAT} \rightarrow \sigma_{THOMSON} \dfrac{\omega^4}{\omega_0^4}$

The behavior $\sigma \sim \omega^4$ is exactly like that we found in lecture 16 for scattering of waves of a small dielectric sphere. This frequency region is relevant to the scattering of sunlight off air molecules. Our result confirms that the sky should be blue ($\omega_{BLUE} > \omega_{RED}$).

Experts (= Einstein, 1910) will note a subtlety. Given scattering off molecule i, we can always find another molecule j such that the path difference for the scattered light is $\lambda/2$ (except for $\Theta = 0$). So if $\lambda \gg$ distance between molecules, and the density of molecules were uniform we would have complete destructive interference $\Rightarrow$ no blue sky!



Einstein saw that the sky is blue because the density of molecules varies from place to place.

IF WE DIVIDE THE VOLUME UP INTO LITTLE CELLS LABELLED BY i AND SUPPOSE THE NUMBER OF MOLECULES IN CELL i IS

$$N_i = N_{0i} + \Delta N_i \qquad (\text{CELL SIZE} \ll \lambda)$$

THEN $E_{SCAT} \sim \sum_i N_i e^{i\phi_i}$ WHERE ϕ_i = PHASE OF SCATTERED LIGHT

$$= \sum_i \Delta N e^{i\phi_i}$$

SINCE $E = 0$ IF DENSITY IS UNIFORM

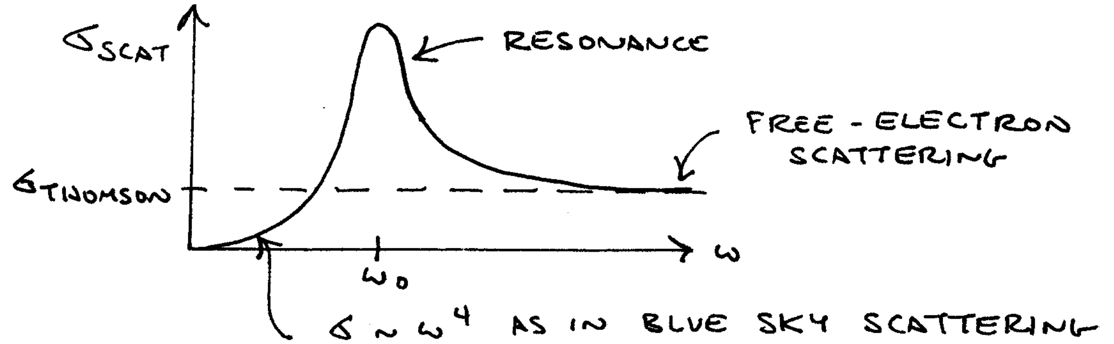
THE INTENSITY $I \sim E^2 \sim \left|\sum_i \Delta N_i e^{i\phi_i}\right|^2 = \sum_i \Delta N_i^2 + \sum_{i,j} \Delta N_i \Delta N_j e^{i(\phi_i - \phi_j)}$

FOR RANDOM DENSITY FLUCTUATIONS THE SECOND TERM VANISHES SO $I \sim \sum_i' \Delta N_i^2$. THE LAWS OF PROBALITY TELL US THAT FOR RANDOM FLUCTUATIONS $\Delta N_i^2 = N_i$ ON THE AVERAGE SO $I \sim \sum_i' N_i$ AFTER ALL — DUE ONLY BECAUSE OF THE EXISTENCE OF THE FLUCTUATIONS!

EINSTEIN EXTRACTED EVEN MORE PHYSICS FROM THIS PHENOMENON... [WE RECOMMEND THE RECENT BOOK BY PAIS FOR A BROAD APPRECIATION OF EINSTEIN'S WORK.]

2. RESONANT SCATTERING $\omega \sim \omega_0$. THE SCATTERING CROSS-SECTION GETS VERY BIG $\sigma \to \sigma_{TH} \frac{\omega_0^2}{\gamma^2}$. THE ATOMS LOVE TO RERADIATE AT THEIR NATURAL FREQUENCIES! THIS IS CALLED RESONANCE FLUORESCENCE, MORE COMMONLY KNOWN AS DAY-GLO PAINT! PROPAGATION OF WAVES IN A DIELECTRIC MEDIUM WHEN $\omega \sim \omega_0$ LEADS TO PECULIAR EFFECTS — 'ANOMALOUS DISPERSION' (LECTURE 12). IT IS POSSIBLE TO HAVE BOTH V_{PHASE} AND V_{GROUP} GREATER THAN C. IN THIS CASE A MORE POWERFUL ANALYSIS IS REQUIRED THAN THAT APPENDED TO LECTURE 12. SEE JACKSON SEC 7.11 FOR SOME OF THE DETAILS OF A VERY COMPLICATED SUBJECT. SEE ALSO L. BRILLOUIN "WAVE PROPAGATION & GROUP VELOCITY" - A BOOK.

3. HIGH FREQUENCY $\omega \gg \omega_0$. IN THIS REGIME THE ELECTRON IS ESSENTIALLY FREE, AND $\sigma_{SCAT} \to \sigma_{THOMSON}$.



SCATTERING AND ABSORPTION

IF THE ATOMS HAVE A DAMPING MECHANISM OTHER THAN THE RADIATION REACTION EFFECT, THEN SOME ENERGY IS 'LOST' TO THIS MECHANISM AND DOES NOT APPEAR IN THE SCATTERED RADIATION.

WE ENCOUNTERED THIS EFFECT IN LECTURE 17 ON DIFFRACTION, WHERE WE WROTE

$$\sigma_{TOT} = \sigma_{SCAT} + \sigma_{ABSORPTION}$$

IN OUR MICROSCOPIC PICTURE IT IS HARD TO GET AT σ_{ABS} DIRECTLY, SO WE PLAY A TRICK: FIND σ_{TOT}, THEN

$$\sigma_{ABS} = \sigma_{TOT} - \sigma_{SCAT}$$

WE CAN FIND σ_{TOT} BY NOTING THAT IT IS PROPORTIONAL TO THE TOTAL ENERGY LOST BY THE INCIDENT WAVE

$$\sigma_{TOT} = \frac{\left\langle \frac{dU}{dt}\Big|_{LOST} \right\rangle}{\langle |\bar{S}_{IN}| \rangle}$$

BUT WE CAN RELATE $\qquad \frac{dU}{dt}\Big|_{LOST} = \bar{F}_{EXT} \cdot \bar{v}$

WHERE $\bar{F}_{EXT}$ IS DUE TO THE INCIDENT WAVE ONLY, NOT THE DAMPING FORCES. $\qquad \bar{F}_{EXT} = e\,\bar{E}$

MEANWHILE $\qquad \bar{v} = \dot{\bar{x}} = \dfrac{i\omega e \bar{E}/m}{\omega^2 - \omega_0^2 + i\gamma\omega} \qquad$ FROM p 278

WE ARE USING COMPLEX NOTATION, SO $\left\langle \frac{dU}{dt}\Big|_{LOST} \right\rangle = \frac{1}{2} \mathrm{Re}\left(\bar{F}_{EXT} \cdot \bar{v}^*\right)$

OR $\left\langle \frac{dU}{dt}\Big|_{LOST} \right\rangle = \frac{1}{2} \mathrm{Re}\, \frac{e^2 E_0^2}{m} \frac{i\omega(\omega^2 - \omega_0^2 - i\gamma\omega)}{(\omega^2-\omega_0^2)^2 + \gamma^2\omega^2} = \frac{e^2 E^2}{2m} \frac{\gamma\omega^2}{(\omega^2-\omega_0^2)^2 + \gamma^2\omega^2}$

OF COURSE $\langle \bar{S}_{IN} \rangle = \frac{1}{2}\frac{c}{4\pi} \,\mathrm{Re}\, \bar{E}\times\bar{B}^* = \frac{c E_0^2}{8\pi}\hat{n}$

SO $$\sigma_{TOT} = 4\pi \frac{e^2}{Mc}\frac{\gamma\,\omega^2}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2} = 4\pi r_0 c \frac{\gamma\omega^2}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}$$

ON COMPARING WITH P. 273 WE NOTE THAT THE CONTRIBUTION TO THE DAMPING FROM THE RADIATION REACTION IS

$$\tfrac{2}{3}\omega^2\tau_0 = \gamma_0\frac{\omega^2}{\omega_0^2} \qquad (\gamma_0 \equiv \tfrac{2}{3}\omega_0^2\tau_0) = \frac{2}{3}\frac{\omega_0^2 r_0}{c}$$

WHEN THE INCIDENT WAVE HAS FREQUECY ω. THUS $\gamma = \gamma_{OTHER} + \frac{\omega^2}{\omega_0^2}\gamma_0$

THEN WITH $\sigma_{THOMSON} = \frac{8\pi}{3}r_0^2$ WE MAY WRITE

$$4\pi r_0 c\frac{\gamma_0}{\gamma_0} = \frac{8\pi}{3}r_0^2\frac{\omega_0^2}{\gamma_0} = \sigma_{THOM}\frac{\omega_0^2}{\gamma_0}$$

$$\sigma_{TOTAL} = \sigma_{THOMSON}\frac{\frac{\gamma}{\gamma_0}\omega^2\omega_0^2}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}$$

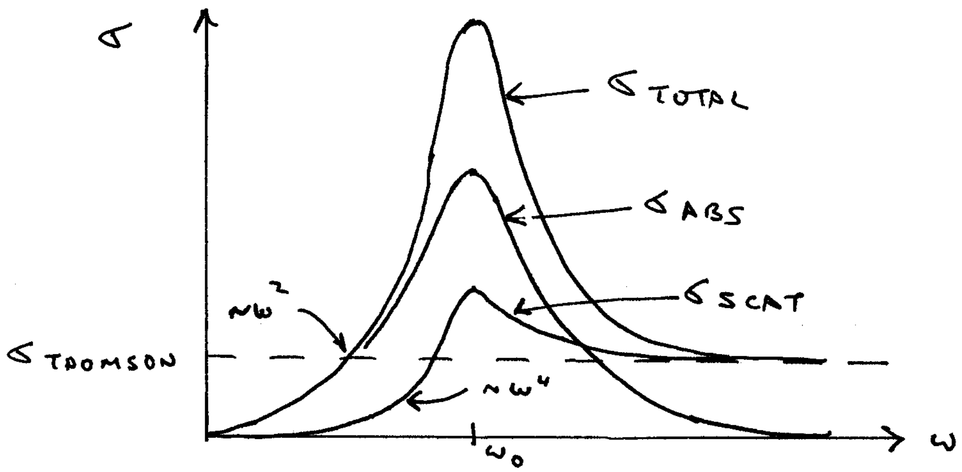
$$\sigma_{SCAT} = \sigma_{THOMSON}\frac{\omega^4}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}$$

$$\frac{\omega^2\omega_0^2}{\gamma_0}\left(\frac{\gamma}{\cancel{\gamma_0}} - \frac{\gamma_0\omega^2}{\omega_0^2}\right)$$

$$\Rightarrow \sigma_{ABS} = \sigma_{THOMSON}\frac{\frac{\gamma_{OTHER}}{\gamma_0}\omega^2\omega_0^2}{(\omega^2-\omega_0^2)+\gamma^2\omega^2} = \sigma_{TOTAL} - \sigma_{SCAT}$$

FOR VERY LARGE FREQUENCIES, $\omega \gg \frac{1}{\gamma_{OTHER}}$, THE ELECTRONS APPEAR TO BE FREE AND $\sigma_{TOT} \to \sigma_{SCAT}$ $\quad (\gamma \to \omega^2\gamma_0/\omega_0^2)$.

BUT AT VERY LOW FREQUENCIES, THE ABSORPTION OF THE WAVE DOMINATES OVER SCATTERING, SO LONG AS $\gamma_{OTHER} \neq 0$.



IN THE PROBLEM SET WE ENCOURAGE YOU TO SHOW THAT σ_{TOT} CAN ALSO BE CALCULATED BY MEANS OF THE 'OPTICAL THEOREM' DISCUSSED IN LECTURE 17

A "SUM RULE"

WE CAN VERIFY AN AMUSING RESULT OF A TYPE WHICH APPEARS OFTEN IN SCATTERING.

THE TOTAL CROSS SECTION FOUND ABOVE VARIES WITH THE FREQUENCY ω, SO WE MIGHT ACTUALLY LABEL IT $\sigma_{TOT}(\omega)$.

THEN $$\Sigma_1 = \int_0^\infty \sigma_{TOT}(\omega)\, d\omega = \text{CONSTANT}$$

INDEPENDENT OF ω_0 OR γ. THIS TYPE OF RELATION IS CALLED A SUM RULE

NOW $$\Sigma_1 = \sigma_{THOMSON} \int_0^\infty \frac{\frac{\gamma}{\gamma_0}\,\omega^2 \omega_0^2}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}\, d\omega$$

BECAUSE OF THE RESONANCE ALL OF THE INTEGRAL COMES FROM THE REGION NEAR ω_0. SO WE MAY SET ALL TERMS EXCEPT $\omega - \omega_0$ TO THEIR VALUES WHEN $\omega = \omega_0$, WITHOUT GREAT ERROR.

$$\Sigma_1 \approx \frac{\gamma\,\omega_0^4}{\gamma_0}\sigma_{TH} \int_0^\infty \frac{d\omega}{[2\omega_0(\omega-\omega_0)]^2 + \gamma^2\omega_0^2}$$

LET $x = \omega - \omega_0$, AND EXTEND THE LIMITS OF INTEGRATION..

$$\underline{\Sigma_1} = \frac{\gamma\,\omega_0^4}{\gamma_0}\frac{\sigma_{TH}}{4\omega_0^2}\int_{-\infty}^\infty \frac{dx}{x^2+\gamma^2/4} = \frac{\pi}{2}\frac{\omega_0^2}{\gamma_0}\sigma_{TH} = \underline{2\pi^2 r_0 c} = 2\pi^2\frac{e^2}{mc}$$

FOR WHAT IT'S WORTH.

ORIGIN OF THE REFRACTIVE INDEX

WE RECALL BRIEFLY OUR EXPLANATION OF THE REFRACTIVE INDEX GIVEN IN LECTURE 1

THE WAVE EQUATION IN A DIELECTRIC MEDIUM TELLS US THAT THE WAVE VELOCITY IS $v = \frac{c}{\sqrt{\epsilon}}$

SO $n = \sqrt{\epsilon}$ = INDEX OF REFRACTION.

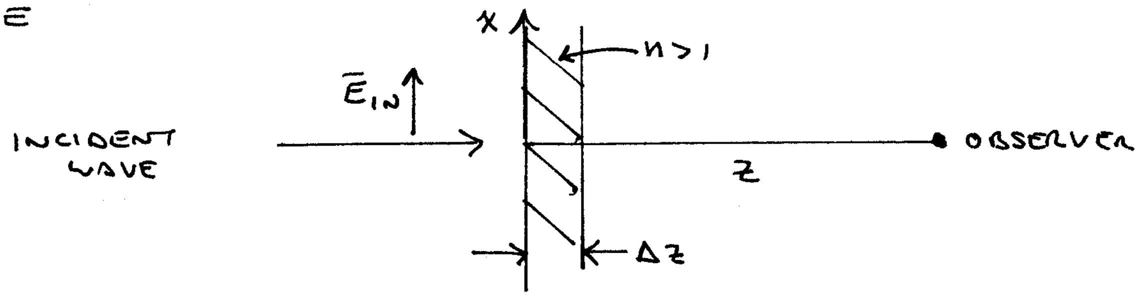
THE DIELECTRIC CONSTANT IS RELATED BY $\bar{P} = \frac{\epsilon - 1}{4\pi} \bar{E}$

WHERE $\bar{P} = Ne\bar{x} = \dfrac{Ne^2 \bar{E}/m}{\omega_0^2 - \omega^2 - i\gamma\omega}$ AS ON p 278

SO $n^2 = \epsilon = 1 + \dfrac{4\pi Ne^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}$

WE NOW WISH TO LOOK AT THIS ANOTHER WAY. THE TRANSMITTED WAVE IN A DIELECTRIC IS THE RESULT OF THE INTERFERENCE BETWEEN THE INCIDENT WAVE AND THE RADIATION OF THE OSCILLATING DIPOLES IN THE MEDIUM. CAN WE SHOW THAT THE RESULTING WAVE APPEARS TO MOVE WITH VELOCITY $v = c/n$?

FOR SIMPLICITY WE ONLY CONSIDER A THIN SLAB OF DIELECTRIC AS VIEWED BY AN OBSERVER ON THE FAR SIDE



WHAT SHOULD THE OBSERVER SEE AS A CONSEQUENCE OF VELOCITY $v = c/n$ INSIDE THE SLAB? CERTAINLY THE WAVE IS MOVING WITH $v = c$ AS IT PASSES THE OBSERVER. HOWEVER IT WILL APPEAR TO ARRIVE SOMEWHAT LATE, DUE TO BEING SLOWED DOWN WHILE TRAVERSING THE DIELECTRIC SLAB.

THAT IS, IF THERE WERE NO SLAB, THE OBSERVER WOULD SEE

$$\bar{E}(z,t) = \bar{E}_0 \, e^{i(kz - \omega t)} \qquad (k = \omega/c)$$

BUT WITH THE SLAB IN PLACE, THE WAVE AT $z = \Delta z$ IS

$$\bar{E}(\Delta z, t) = \bar{E}_0 \, e^{i\left(\frac{\omega n}{c}\Delta z - \omega t\right)}$$

AND SO $E(z,t) = \bar{E}(\Delta z, t)\, e^{i(k(z - \Delta z) - \omega t)} = \bar{E}_0 \, e^{i\left(\frac{\omega}{c} z - \omega t\right)} \underbrace{e^{i\frac{\omega}{c}(n-1)\Delta z}}_{\text{PHASE CHANGE}}$

THAT IS, THE DELAYED WAVE LOOKS LIKE THE INCIDENT WAVE BUT WITH A <u>PHASE CHANGE</u>. FOR SMALL Δz, WE CAN ALSO WRITE

$$\bar{E}(z,t) \approx \underbrace{\bar{E}_0 \, e^{i\left(\frac{\omega}{c} z - \omega t\right)}}_{\text{ORIGINAL WAVE}} + \underbrace{i\frac{\omega}{c}(n-1)\Delta z \, \bar{E}_0 \, e^{i\left(\frac{\omega}{c} z - \omega t\right)}}_{\text{RADIATED WAVE}}$$

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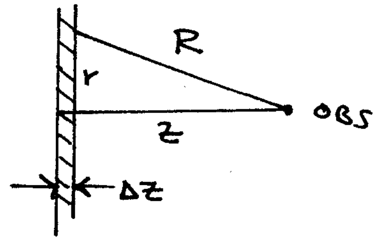
WE WILL NOW TRY TO SHOW THAT THE TOTAL SCATTERED WAVE AS SEEN AT THE OBSERVER HAS THE ABOVE FORM.

FOR THE RADIATION FIELDS, WE CAN DERIVE $\bar{E}$ FROM THE VECTOR POTENTIAL: $\bar{E}_{SCAT} = -\frac{1}{c}\dot{\bar{A}}$ SINCE $\phi = 0$ IN THIS PROBLEM.

WHERE $\bar{A}$ IS THE <u>RETARDED</u> POTENTIAL DUE TO THE OSCILLATING ATOMS IN THE DIELECTRIC SLAB.

$$\bar{A} = \frac{1}{c}\int_{SLAB} \frac{[J]}{R} = \frac{1}{c}\int_{SLAB} \frac{[\dot{\bar{P}}]}{R}$$



RECALLING HERTZ' METHOD.

NOW $\bar{P} = Ne\bar{x}$ IS THE POLARIZATION DENSITY

SO $\dot{\bar{P}} = Ne\dot{\bar{x}}$.

WITH $\bar{E}(0,t) = E_0\,\hat{x}\,e^{-i\omega t}$ THEN $\bar{x} = \frac{eE_0}{m}\hat{x}\,\frac{e^{-i\omega t}}{\omega_0^2-\omega^2-i\gamma\omega}$

AND $\dot{\bar{P}} = -\frac{i\omega Ne^2}{m}\,\frac{E_0\,\hat{x}\,e^{-i\omega t}}{\omega_0^2-\omega^2-i\gamma\omega}$

WE EVALUATE THIS AT THE RETARDED TIME $t' = t - R/c$

AND INTEGRATE OVER $d\,vol' = \Delta z\, r\,dr\,d\varphi$

SINCE $\dot{\bar{P}}$ HAS ONLY AN $\hat{x}$ COMPONENT, WE NEED ONLY A_x

$$A_x = \frac{-i\omega Ne^2 E_0 e^{-i\omega t}}{mc\,(\omega_0^2-\omega^2-i\gamma\omega)}\,\Delta z\int_0^{2\pi} d\varphi \int_0^{\infty} \frac{r\,dr\, e^{i\omega R/c}}{R}$$

NOW $R = \sqrt{r^2+z^2}$ SO $dR = r\,dr/R$ AND $R = z$ WHEN $r = 0$

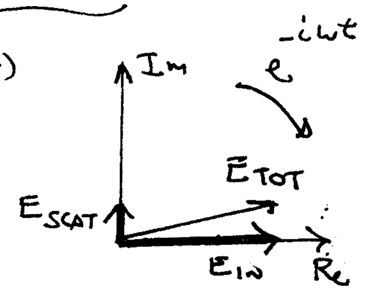
HENCE $$\iint = \frac{2\pi c}{i\omega}\left[e^{\frac{i\omega\infty}{c}} - e^{\frac{i\omega z}{c}}\right] \rightarrow \frac{2\pi i c}{\omega}\,e^{i\frac{\omega z}{c}}$$

IGNORING THE OSCILLATORY TERM AT $r\to\infty$ AS USUAL.

THUS $$A_x = 2\pi\,\frac{Ne^2}{m}\,\frac{E_0\,e^{i(\frac{\omega}{c}z-\omega t)}}{\omega_0^2-\omega^2-i\gamma\omega}\,\Delta z$$

THEN $\bar{E}_{SCAT} = -\frac{\dot{\bar{A}}}{c} = i\frac{\omega}{c}\frac{\Delta z}{2}\frac{4\pi N e^2}{m(\omega_0^2-\omega^2-i\gamma\omega)} \underbrace{\bar{E}_0 e^{i(\frac{\omega}{c}z-\omega t)}}_{\bar{E}_{IN}(z,t)}$

AND SO $E_{TOT} = E_{IN}\left(1 + i\frac{\omega}{c}\frac{\Delta z}{2}\frac{4\pi N e^2}{m(\omega_0^2-\omega^2-i\gamma\omega)}\right)$



COMPARING WITH WHAT WE EXPECTED TO FIND, AS ON P 283, WE IDENTIFY

$$n - 1 = \frac{1}{2}\frac{4\pi N e^2}{m}\frac{1}{\omega_0^2-\omega^2-i\gamma\omega}$$

$$\text{SO} \quad n^2 \sim 1 + \frac{4\pi N e^2}{m}\frac{1}{\omega_0^2-\omega^2-i\gamma\omega} \qquad \text{AS BEFORE!}$$

NOTE THAT THE WAVE REFLECTED TO AN OBSERVER AT $-z$ IS THE SAME AS THAT SCATTERED TO $+z$

SO $\bar{E}_{REFLECTED} = i\frac{\omega}{c}\frac{\Delta z}{2}\frac{4\pi N e^2}{m(\quad)} E_0 e^{-i(\frac{\omega}{c}z+\omega t)}$

OR IN BRIEF $\bar{E}_{TRANS} = \bar{E}_0(1+i\alpha)$
$\bar{E}_{REF.} = \bar{E}_0(i\alpha)$ } 90° PHASE DIFFERENCE

THESE ARE THE RELATIONS REFERRED TO IN PROBLEM 4, SET 6.

REMARK: WE MIGHT HAVE ARGUED THAT THE RADIATION FROM EACH ELECTRON IS $\bar{E}_{RAD} = -\frac{\dot{\bar{A}}_\perp}{c}$ (ATOM) WHICH IS $\perp$ TO THE LINE JOINING THE ATOM TO THE OBSERVER

THEN $E_{SCAT}|_x = \int E_{RAD}|_x = -\frac{1}{c}\int_{SLAB} \ddot{P}\cdot\frac{z^2}{R^2}\,dvol$

WHICH DIFFERS FROM THE ABOVE BY THE FACTOR z^2/R^2. AGAIN WE MUST ARGUE THAT ALL OF THE INTEGRAL COMES FROM THE CENTRAL REGION IN WHICH $z \sim R$, SO THAT THE RESULT IS UNCHANGED.