RADIATION FROM RELATIVISTIC PARTICLES IN DIELECTRIC MEDIA

IN THIS LECTURE WE DISCUSS 2 INTERESTING EFFECTS IN WHICH CHARGED PARTICLES MOVING AT UNIFORM VELOCITY CAN EMIT RADIATION – IF THEY ARE INSIDE A DIELECTRIC MEDIUM.

FIRST IT IS USEFUL TO GIVE A GENERAL METHOD FOR CALCULATING

THE FREQUENCY SPECTRUM OF RADIATION SEEN BY A FIXED OBSERVER

AS WE REMARKED IN LECTURES 15 AND 20, THE TOTAL ENERGY IN A PULSE OF RADIATION CAN BE DESCRIBED BY A FREQUENCY SPECTRUM RELATED TO THAT OF THE ELECTRIC FIELD OF THE RADIATION. TO REPEAT:

$$\frac{dU}{d\Omega} = \int_{-\infty}^{\infty} \frac{dU}{dt\,d\Omega}\,dt = \frac{cr^2}{4\pi}\int_{-\infty}^{\infty} E^2\,dt = \frac{cr^2}{4\pi}\int_{-\infty}^{\infty} E\,dt \int_{-\infty}^{\infty} E_\omega e^{-i\omega t}\,d\omega = \frac{cr^2}{4\pi}\int_{-\infty}^{\infty} E_\omega\,d\omega \int_{-\infty}^{\infty} E\,e^{-i\omega t}\,dt$$

$$= \frac{cr^2}{2}\int_{-\infty}^{\infty} d\omega\, E_\omega E_\omega^* = cr^2\int_0^{\infty} d\omega\, E_\omega^2$$

SO IF WE DEFINE THE FREQUENCY SPECTRUM OF THE PULSE AS

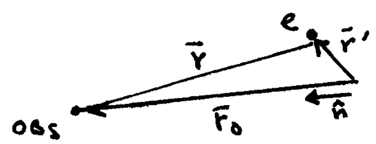
$$\frac{dU}{d\Omega} = \int_0^{\infty} \frac{dU_\omega}{d\Omega}\,d\omega \qquad \text{THEN} \qquad \frac{dU_\omega}{d\Omega} = c r^2 \bar{E}_\omega^2$$

WHERE
$$\bar{E}_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{E}(t)\, e^{+i\omega t}\,dt = \frac{e}{2\pi c r}\int_{-\infty}^{\infty} \left.\frac{\hat{n}\times\left[(\hat{n}-\bar{\beta})\times\dot{\bar{\beta}}\right]}{(1-\hat{n}\cdot\bar{\beta})^3}\right|_{\text{RETARDED}} e^{i\omega t}\,dt$$

USING THE RADIATION FIELD OF A MOVING CHARGE (p 235)

CLEARLY IT IS USEFUL TO CHANGE THE VARIABLE OF INTEGRATION TO BE THE RETARDED TIME $t' = t - r/c$ SO $dt = dt'(1-\hat{n}\cdot\bar{\beta})$ [p 239]

ALSO FOR AN OBSERVER FAR FROM THE CHARGE WE MAY APPROXIMATE



$r \approx r_0 - \bar{r}'\cdot\hat{n}$ WHERE r_0 IS FIXED

AND $\bar{r}'$ DESCRIBES THE MOVING SOURCE

THEN
$$\bar{E}_\omega = \frac{e}{2\pi c r}\, e^{+i\frac{\omega r_0}{c}} \int_{-\infty}^{\infty} \frac{\hat{n}\times\left[(\hat{n}-\bar{\beta})\times\dot{\bar{\beta}}\right]}{(1-\hat{n}\cdot\bar{\beta})^2}\, e^{i\omega\left(t' - \frac{\hat{n}\cdot\bar{r}'(t')}{c}\right)}\,dt'$$

WE WILL WRITE $\frac{\omega\hat{n}}{c}$ AS $\bar{k}$.

SOMEBODY NOTICED A CLEVER TRICK:

$$\left.\frac{\hat{n}\times\left[(\hat{n}-\bar{\beta})\times\dot{\bar{\beta}}\right]}{(1-\hat{n}\cdot\bar{\beta})^2}\right|_{RET} = \frac{d}{dt'}\left[\frac{\hat{n}\times(\hat{n}\times\bar{\beta})}{1-\hat{n}\cdot\bar{\beta}}\right]$$

SO WE CAN INTEGRATE BY PARTS:

$$E_\omega = \frac{e}{2\pi c r}\, e^{\frac{i\omega r_0}{c}} (i\omega) \int_{-\infty}^{\infty} \hat{n}\times(\hat{n}\times\bar{\beta})\, e^{i(\omega t' - \bar{k}\cdot\bar{r}')} dt'$$

IF $\bar{\beta} = 0$ AT $t' = \pm\infty$. THIS RESULT IS SOMEWHAT PECULIAR IN THAT IT WILL GIVE US THE RADIATION SPECTRUM IN TERMS OF AN INTEGRAL OF THE VELOCITY RATHER THAN ACCELERATION.

THEN
$$\frac{dU_\omega}{d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{n}\times(\hat{n}\times\bar{\beta})\, e^{i(\omega t' - \bar{k}\cdot\bar{r}')} dt'\right|^2$$

[COMPARE P.182]

THIS COULD BE USED TO GIVE DETAILED EXPRESSIONS FOR THE SPECTRA OF BREMSSTRAHLUNG AND SYNCHROTRON RADIATION, BUT THE EXPRESSIONS ARE COMPLICATED ... SEE THE HOMEWORK SET FOR SOME SIMPLE CASES.

ČERENKOV RADIATION (Č RADIATION)

WHEN A CHARGED PARTICLE MOVES WITH CONSTANT VELOCITY THRU A DIELECTRIC MEDIUM, THE ELECTRIC FIELD AT A FIXED POINT IN THE MEDIUM CHANGES RAPIDLY. THE MEDIUM RESPONDS TO THIS CHANGE AND IN SOME CIRCUMSTANCES EMITS RADIATION. THE ENERGY FOR THIS RADIATION IS OBTAINED FROM THE MOVING CHARGE, WHICH THEREFORE SLOWS DOWN. THE RADIATION IS NOT HOWEVER JUST ANOTHER FORM OF BREMSSTRAHLUNG. IF THE PARTICLE'S MASS IS LARGE, THEN a IS SMALL $\Rightarrow$ TINY BREMSSTRAHLUNG. WE WILL SHOW THAT THE CERENKOV EFFECT IS INDEPENDENT OF THE PARTICLE'S MASS.

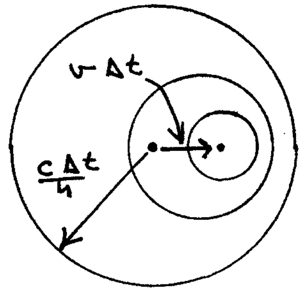
THE SPECIAL CONDITION REQUIRED FOR CERENKOV RADIATION IS THAT THE PARTICLE'S VELOCITY BE GREATER THAN THAT OF LIGHT IN THE MEDIUM:

$$v > \frac{c}{n} \qquad (\text{OF COURSE } v < c)$$

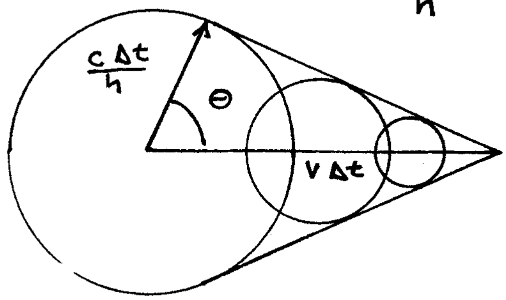
WHERE n = INDEX OF REFRACTION.

THE BASIC IDEA CAN BE SEEN IN A SORT OF HUYGHEN'S CONTRUCTION FOR THE EQUIPOTENTIALS.

$v < c/n$



$v > \frac{c}{n}$



THE EQUIPOTENTIALS ARE CENTERED ON THE POSITION OF THE CHARGE AT $t' = t - r/c$

$$\cos\theta = \frac{c}{nv} = \frac{1}{n\beta} < 1$$

WE SEE THAT FOR $v < \frac{c}{n}$ THE PART OF $\bar{E}$ DUE TO $-\bar{\nabla}\phi$ IS STRONGER FORWARDS THAN BACKWARDS. THE PIECE DUE TO $-\dot{\bar{A}}$ BEHAVES JUST THE OPPOSITE, LEADING TO A RADIAL $\bar{E}$ FIELD (ABOUT THE PRESENT POSITION OF THE CHARGE. BUT FOR $v > c/n$ WE SEE THAT THERE IS A WAVE FRONT AT WHICH ϕ AND $\bar{A}$ PILE UP, SO THAT THEIR DERIVATES ARE VERY LARGE HERE. THIS 'SHOCK WAVE' EFFECT IS THE ČERENKOV RADIATION.

THE RADIATED LIGHT RAYS MAKE ANGLE $\theta = \cos^{-1}\left(\frac{c}{nv}\right)$ TO THE VELOCITY $\bar{v}$

THUS THE ČERENKOV EFFECT GIVES A METHOD OF DETERMINING THE VELOCITY OF VERY FAST PARTICLES. JUST LOOK FOR VISIBLE LIGHT AS THE PARTICLE PASSES THRU A GAS OR TRANSPARENT LIQUID, AND MEASURE THE ANGLE OF THE LIGHT. THEN $v = \frac{c}{n\cos\theta}$!

THE RADIATION IS POLARISED IN THE PLANE OF $\bar{v}$ AND THE OBSERVER. SINCE $\bar{E} = -\bar{\nabla}\phi - \dot{\bar{A}}$ AND $\bar{A} \sim \bar{v}$, NO DERIVATIVES GIVE COMPONENTS $\perp$ TO THE PLANE.

WE NOW TURN TO THE QUESTION OF INTENSITY AND FREQUENCY OF THE RADIATION.

WE CAN READILY ADAPT OUR FANCY FORMULA FOR THE FREQUENCY SPECTRUM TO THE CASE OF A DIELECTRIC MEDIUM:

$c \to c/n$ AND $k \to \frac{n\omega}{c}$ (WE MUST NOT CONFUSE n AND $\hat{n}$)

THEN $$\frac{dU_\omega}{d\Omega} = \frac{n e^2 \omega^2}{4\pi^2 c}\left|\int \hat{n}\times(\hat{n}\times\bar{\beta})\, e^{i(\omega t' - \bar{k}\cdot\bar{r}')}\, dt'\right|^2$$

WE PUT $\bar{\beta} = \bar{v}/c$ AND $\bar{r}' = \bar{v}t'$

FINALLY, WE OBSERVE THE RADIATION AT ANGLE θ TO DIRECTION $\bar{v}$

$$\frac{dU_\omega}{d\Omega} = \frac{n e^2 \omega^2 v^2 \sin^2\theta}{4\pi^2 c^3} \left| \int e^{i(\omega - kv\cos\theta)t'} dt' \right|^2$$

INTEGRATING OVER ALL TIME LEADS TO A DIRAC DELTA FUNCTION $\sim \delta(\omega - kv\cos\theta)$. HENCE WE CAN ONLY HAVE RADIATION AT $\cos\theta = \frac{\omega}{kv} = \frac{c}{nv}$ AS BEFORE.

(IF $n = 1$, THIS REQUIRES $\cos\theta = \frac{c}{v} > 1$, SO OUR FANCY FORMULA DOES NOT PREDICT RADIATION BY A UNIFORMLY MOVING CHARGE IN A VACUUM.)

WE MAY AVOID THE APPEARANCE OF THE DELTA FUNCTION BY SUPPOSING WE OBSERVE THE RADIATION ONLY OVER A FINITE SEGMENT OF THE PARTICLE'S PATH, SAY $x' = (-L, L)$ SO t' RUNS FROM $-L/v$ TO L/v

THEN THE $\int \rightarrow \dfrac{e^{i(\omega - kv\cos\theta)L/v} - e^{-i(\omega - kv\cos\theta)L/v}}{i(\omega - kv\cos\theta)}$

$$= 2\frac{L}{v}\frac{\sin z}{z} \qquad \text{WHERE } z = \omega\left(1 - \frac{nv}{c}\cos\theta\right)\frac{L}{v}$$

WE CAN INTEGRATE THE INTENSITY OVER SOLID ANGLE:

$$U_\omega = \int \frac{dU_\omega}{d\Omega} d\Omega = \frac{2 n e^2 \omega^2 L^2}{\pi c^3} \int_{-1}^{1} \sin^2\theta \frac{\sin^2 z}{z^2} \, d\cos\theta$$

THE TRICK IS TO NOTE THAT ALL THE INTEGRAL COMES FROM THE REGION $z \sim 0 \longleftrightarrow \cos\theta \sim \frac{c}{nv}$

IN THAT REGION $\sin^2\theta = 1 - \cos^2\theta \sim 1 - \frac{c^2}{n^2 v^2}$ = CONSTANT

ALSO $dz = -\frac{\omega n L}{c} d\cos\theta$, SO

$$U_\omega \sim \frac{2 e^2 \omega L}{\pi c^2}\left(1 - \frac{c^2}{n^2 v^2}\right) \underbrace{\int_{-\infty}^{\infty} \frac{\sin^2 z}{z^2} dz}_{\pi}$$

WHERE WE EXTEND THE LIMITS OF INTEGRATION ON z WITHOUT GREAT ERROR

THE TOTAL PATH OBSERVED IS $2L$, SO

$$U_\omega \text{ PER UNIT PATH} = \frac{e^2}{c^2}\left(1 - \frac{c^2}{n^2 v^2}\right)\omega$$

REMEMBER THAT $U_\omega \, d\omega$ = TOTAL ENERGY IN FREQUENCY INTERVAL $d\omega$.

NOW e^2/c^2 IS A SMALL NUMBER, SO THE ČERENKOV EFFECT IS NOT VERY STRONG. THEREFORE THE PHOTON CONCEPT IS ESPECIALLY RELEVANT. WE WRITE $U_\omega = N_\omega(\hbar\omega)$ SO THAT

N_ω = NUMBER OF PHOTONS EMITTED IN FREQUENCY INTERVAL $d\omega$

THEN $$\frac{N_\omega}{\text{UNIT PATH}} = \frac{e^2}{\hbar c}\left(1 - \frac{c^2}{n^2 v^2}\right)\frac{1}{c} \qquad \text{INDEPENDENT OF } \omega!$$

$$= \frac{\sin^2\theta_c}{137\, c} \qquad \left(\cos\theta_c \equiv \frac{c}{nv}\right)$$

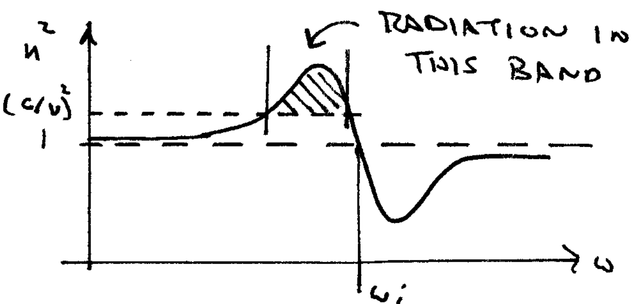
FOR EXAMPLE, A PARTICLE WITH $v \sim c$ PASSING THRU 1 METER OF WATER ($n \sim 1.33$) YIELDS ABOUT 30,000 PHOTONS IN THE OPTICAL RANGE 3.5×10^{-5} cm $< \lambda < 7\times10^{-5}$ cm.

IN YOUR EYE, THE PATH IS $\sim$ 1 cm, SO THERE MIGHT BE $\sim$ 300 PHOTONS. YOU CAN SEE AS FEW AS $\sim$ 20 PHOTONS IN A BURST. THUS YOU SHOULD BE ABLE TO SEE ČERENKOV LIGHT FROM COSMIC RAYS PASSING THRU YOU. THE RATE IS ABOUT 1 PER SECOND PER cm^2. JUST CLOSE YOUR EYES AND WAIT TILL YOU SEE STARS!

SINCE $U_\omega \sim \omega$ IF ALL FREQUENCIES WERE POSSIBLE, WE WOULD HAVE $U = \int U_\omega d\omega \to \infty$. HOWEVER WE GET ČERENKOV RADIATION ONLY IF $n > \frac{c}{v} > 1$.

ACCORDING TO OUR ATOMIC MODEL OF DIELECTRICS:

$$n^2 = \epsilon = 1 + \frac{4\pi N e^2}{m}\sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\gamma_i\omega}$$



THUS ČERENKOV RADIATION IS LIMITED TO NARROW BANDS OF FREQUENCIES JUST BELOW THE NATURAL FREQUENCIES.

U_{TOTAL} IS QUITE SMALL.

TRANSITION RADIATION

ANOTHER EXOTIC RADIATION EFFECT OCCURS WHEN A PARTICLE CROSSES A BOUNDARY BETWEEN TWO DIFFERENT DIELECTRIC MEDIA. SINCE THE DIELECTRIC CONSTANTS ARE DIFFERENT, THE FIELDS OF A UNIFORMLY MOVING CHARGE DO NOT HAVE QUITE THE SAME SHAPE IN THE TWO MEDIA. AT THE BOUNDARY AN EXTRA PIECE IS NEEDED TO MATCH THE FIELDS. THIS PIECE IS NOT A STATIC FIELD, BUT A PULSE GENERATED AS THE PARTICLE CROSSES THE BOUNDARY. AFTERWARDS THE PULSE PROPAGATES AT THE SPEED OF LIGHT c/n. THIS IS THE <u>TRANSITION RADIATION</u>.

IT IS A VERY WEAK EFFECT. IT WAS DISCOVERED IN 1946, BUT 'PRACTICAL' APPLICATIONS OF IT ARE ONLY NOW BEING MADE. IT IS STRONGER FOR HIGH VELOCITIES $v \to c$, BUT IT DOES NOT DEPEND ON THE CONDITION $v > c/n$.

TO ANALYSE THE TRANSITION RADIATION WE WILL USE OUR FANCY EXPRESSION FOR THE FREQUENCY SPECTRUM. WE NOW HAVE TWO MEDIA WITH INDICES OF REFRACTION n_1 AND n_2, WITH THE BOUNDARY AT $x = 0$.

WE OBSERVE THE PARTICLE OVER THE INTERVAL $-L < x < L$, OR $-L/v < t < L/v$.

THE FREQUENCY INTEGRAL MUST BE MODIFIED SLIGHTLY:

$$\frac{dU_\omega}{d\Omega} = \frac{n_2 e^2 \omega^2 v^2 \sin^2\theta}{4\pi^2 c^3} \left| \int_{-L/v}^{0} dt'\, e^{i(\omega - k_1 v \cos\theta)t'} + \int_{0}^{L/v} dt'\, e^{i(\omega - k_2 v\cos\theta)t'} \right|^2$$

WHERE $k_i = \frac{\omega}{c} n_i$, AND THE OBSERVER IS IN MEDIUM 2.

$$= \frac{n_2 e^2 \omega^2 v^2 \sin^2\theta}{4\pi^2 c^3} \left| \frac{1 - e^{i\omega[1]L/v}}{i\omega[1]} + \frac{e^{i\omega[2]L/v} - 1}{i\omega[2]} \right|^2$$

WHERE $[1] = \frac{\omega - k_1 v\cos\theta}{\omega} = 1 - \frac{v}{c} n_1 \cos\theta$ ETC.

WE NOTE TWO ESSENTIAL FEATURES AT ONCE

a) THE EFFECT IS BIG ONLY WHEN $[1]$ AND $[2]$ ARE SMALL.
THIS MEANS $v \to c$, $\theta \to 0$ AND $n_i \sim 1$

[WE SUPPOSE $[1]$ AND $[2]$ CANNOT VANISH - IN WHICH CASE WE WOULD HAVE THE CERENKOV EFFECT JUST DISCUSSED i.e. WE SUPPOSE $\frac{vn}{c} < 1$]

LIKE THE CERENKOV EFFECT, THE TRANSITION RADIATION DEPENDS ON THE RESPONSE OF THE MEDIUM TO THE MOVING CHARGE, WHICH IS MARVELOUSLY SUMMARIZED IN OUR FREQUENCY EXPRESSION.

FROM THE ATOMIC MODEL OF THE INDEX OF REFRACTION,

$$n^2 = \epsilon = 1 + \frac{4\pi N e^2}{m} \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\gamma_i\omega}$$

THE REQUIREMENT THAT $n \sim 1$ TELLS US THAT THE IMPORTANT FREQUENCY REGION WILL BE <u>LARGE ω</u>

$$\text{THEN} \quad n^2 \sim 1 - \frac{4\pi N e^2}{m\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

WHERE ω_p IS THE <u>PLASMA FREQUENCY</u> OF THE MEDIUM. (LECTURE 12).

$$\text{SO FOR} \quad \omega > \omega_p \qquad n \sim 1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}$$

$$\text{AND} \quad [1] = 1 - \frac{v}{c} n \cos\theta \sim 1 - \beta\left(1 - \frac{\omega_p^2}{2\omega^2}\right)\left(1 - \frac{\theta^2}{2}\right)$$

$$\sim 1 - \beta + \frac{\omega_p^2}{2\omega^2} + \frac{\theta^2}{2} \sim \frac{1}{2}\left(\frac{1}{\gamma^2} + \frac{\omega_p^2}{\omega^2} + \theta^2\right) \qquad [\beta \sim 1]$$

b) THE EFFECT IS SIGNIFICANT ONLY IF THE OSCILLATING TERMS IN THE NUMERATOR OF $\frac{dU_\omega}{d\Omega}$ CAN BE NEGLECTED: $e^{i\omega[1]L/v}$

PHYSICALLY THIS MEANS THAT L MUST BE LARGE ENOUGH SO THAT MANY OSCILLATIONS HAVE OCCURED AND THE AVERAGE EFFECT IS ZERO - SO WE JUST DROP THE OSCILLATORY TERM. (WE INVOKED THIS SOMEWHAT DEVIOUS ARGUMENT PREVIOUSLY IN OUR STUDY OF DIFFRACTION.)

THAT IS, WE NEED $\frac{\omega L}{v}[1] \gg 1$ BEFORE THE EFFECT IS IMPORTANT. IF MEDIUM 2 IS A THIN SLAB (SEE THE HOMEWORK SET), THE EFFECT INDEED VANISHES AS THE THICKNESS $\to 0$.

AT $\theta = 0$, THE CONDITION IS $\frac{\omega L}{2v}\left(\frac{1}{\gamma^2} + \frac{\omega_p^2}{\omega^2}\right) \gg 1$

$$\text{NOW} \quad \frac{1}{\gamma^2} + \frac{\omega_p^2}{\omega^2} = \frac{\omega_p}{\gamma\omega}\left(\frac{\omega}{\gamma\omega_p} + \frac{\gamma\omega_p}{\omega}\right)$$

SO AS $v \to c$, WE REQUIRE $L \gg \frac{2c\gamma}{\omega_p} \dfrac{1}{\frac{\omega}{\gamma\omega_p} + \frac{\gamma\omega_p}{\omega}}$

THE FUNCTION $f = \dfrac{1}{x + \frac{1}{x}}$ HAS A MAXIMUM AT $x = 1$

SO WE NEED $L \gg \dfrac{c\gamma}{\omega_p}$. ALSO WE SEE THAT THE

FREQUENCY $\omega = \gamma\omega_p$ WILL HAVE SPECIAL SIGNIFICANCE. IT WILL BE THE CHARACTERISTIC FREQUENCY OF TRANSITION RADIATION — WHICH IS AN X-RAY FREQUENCY IF γ IS LARGE. EVEN METALS ARE 'TRANSPARENT' TO SUCH HIGH FREQUENCIES. THE MOST SUCCESSFUL TRANSITION RADIATION DEVICE HAS IN FACT BEEN MADE OUT OF LITHIUM METAL.

THE LENGTH $D = \dfrac{c\gamma}{\omega_p}$ IS CALLED THE <u>FORMATION LENGTH</u>. THE RADIATION BUILDS UP AS L IS INCREASED UNTIL $L \sim D$, AFTER WHICH IT IS ROUGHLY INDEPENDENT OF L (EXCEPT FOR POSSIBLE REABSORPTION OF THE RADIATION BY THE MEDIUM.)

RETURNING TO $\dfrac{dU_\omega}{d\Omega}$, WE SUPPOSE $L \gg D$ SO WE CAN IGNORE THE OSCILLATORY TERMS $e^{i\omega[\;]L/v}$ ....

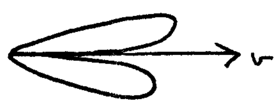
WE APPROXIMATE $n_2 \sim 1$ IN THE NUMERATOR, SET $\sin\theta \sim \theta$ AND SUPPOSE $v \sim c$. THEN

$$\frac{dU_\omega}{d\Omega} = \frac{e^2\theta^2}{\pi^2 c}\left\{\frac{1}{\left(\frac{1}{\gamma^2} + \frac{\omega_{p1}^2}{\omega^2} + \theta^2\right)} - \frac{1}{\left(\frac{1}{\gamma^2} + \frac{\omega_{p2}^2}{\omega^2} + \theta^2\right)}\right\}^2$$

OF COURSE, IF $\omega_{p1} = \omega_{p2}$ THIS VANISHES.

THE ANGULAR DISTRIBUTION PEAKS IN THE REGION

$$\frac{1}{\gamma^2} + \frac{\omega_{p1}^2}{\omega^2} < \theta^2 < \frac{1}{\gamma^2} + \frac{\omega_{p2}^2}{\omega^2}$$



(BREMSSTRAHLUNG-LIKE)

THE ANGULAR DISTRIBUTION CAN BE INTEGRATED (I DID IT) TO GIVE:

$$U_\omega = \frac{e^2}{\pi^2 c}\left[\frac{\omega_1^2 + \omega_2^2 + 2\omega^2/\gamma^2}{\omega_1^2 - \omega_2^2}\ln\left(\frac{\omega_1^2 + \omega^2/\gamma^2}{\omega_2^2 + \omega^2/\gamma^2}\right) - 2\right]$$

WHERE $\omega_1 \equiv \omega_{p1}$ ETC.

IN THE HIGH FREQUENCY LIMIT (WE MUST CALCULATE TO 3RD ORDER!)

$$U_\omega \to \frac{2e^2}{3\pi c} \frac{\gamma^4}{\omega^4} \left(\frac{\omega_1^2 - \omega_2^2}{2}\right)^2 = \frac{1}{6\pi} \frac{e^2}{c} \left(\frac{\gamma \omega_2}{\omega}\right)^4$$

IF MEDIUM 1 = VACUUM

IF WE DIVIDE BY $\hbar$ WE OBTAIN THE NUMBER OF PHOTONS PER UNIT FREQUENCY INTERVAL

$$N_\omega = \frac{1}{6\pi} \frac{e^2}{\hbar c} \left(\frac{\gamma \omega_2}{\omega}\right)^4 \sim \frac{1}{6\pi} \frac{1}{137} \left(\frac{\gamma \omega_2}{\omega}\right)^4$$

THUS THE EFFECT IS INDEED VERY WEAK!

PRACTICALLY IT DOES NOT PAY TO CONSIDER THE LOW FREQUENCY REGIME, AS MOST MATERIALS ARE NOT VERY TRANSPARENT THERE....

THE INTEREST IN TRANSITION RADIATION LIES IN THE γ^4 DEPENDENCE OF THE RADIATION. THUS THE STRENGTH OF THE TRANSITION RADIATION CAN BE USED TO MEASURE THE VELOCITY OF PARTICLES IN THE EXTREME RELATIVISTIC LIMIT, $\gamma \gg 1$.

IN THIS LIMIT THE CERENKOV EFFECT CERTAINLY TAKES PLACE ALSO, BUT IS NOT VERY USEFUL FOR VELOCITY MEASUREMENT.

$\cos\theta_{\check{c}} = \frac{c}{v n} \to \frac{1}{n}$, AND THE ČERENKOV RADIATION INTENSITY BECOMES INDEPENDENT OF γ.

BECAUSE TRANSITION RADIATION IS SO WEAK, SUCCESSFUL DETECTORS MUST INCORPORATE ~ 1000 BOUNDARIES TO INCREASE THE SIGNAL STRENGTH. IN THIS CASE ONE MUST TAKE INTO ACCOUNT POSSIBLE INTERFERENCE EFFECTS BETWEEN RADIATION FROM DIFFERENT BOUNDARIES...

SEE: M.L. CHERRY, ET AL. PHYS. REV <u>D10</u>, 3594 (1974)

J. COBB ET AL, NUC. INSTR. & METH. <u>140</u>, 413 (1977)

A LOT OF THE OLDER WORK WAS DONE BY RUSSIANS. YOU COULD TRACE THIS STARTING WITH

G. M. GARIBIAN, SOV. PHYS. JETP <u>33</u>, 23 (1971)

AN IMPORTANT EXAMPLE OF TRANSITION RADIATION IS IMPLICIT IN THE USUAL DERIVATION OF ČERENKOV RADIATION, AS NOTED BY I. TAMM J. PHYS (USSR) 1, 439 (1939).

CONSIDER A PARTICLE OF CHARGE e MOVING WITH VELOCITY v IN A MEDIUM OF INDEX OF REFRACTION n THAT IS BOUNDED BY TWO PERFECTLY CONDUCTING REGIONS AT $-L$ AND $+L$. THEN WE FIND, AS ON p 251 OF MY Ph 206 NOTES THAT $U_\omega \equiv dU/d\omega$ = SPECTRAL ENERGY DENSITY OF THE RADIATION, IS

$$U_\omega = \frac{2ne^2\omega^2L^2}{\pi c^3}\sin^2\theta\,\frac{\sin^2 Z}{Z^2}\,d\cos\theta$$

WHERE $Z = \frac{\omega L}{v}\left(1 - \frac{nv}{c}\cos\theta\right)$

IF $n > 1$ WE COULD PROCEED TO DEDUCE ČERENKOV RADIATION.

BUT EVEN FOR $n = 1$ (VACUUM), U_ω IS NOT ZERO!

WE WILL ONLY CONSIDER $\beta = \frac{v}{c} \approx 1$ i.e. $\beta \sim 1 - \frac{1}{2\gamma^2}$ $\gamma \gg 1$

THEN (WITH $n = 1$), $Z = \frac{\omega L}{c}\left(\frac{1}{\beta} - \cos\theta\right) \approx \frac{\omega L}{c}\left(1 - \cos\theta + \frac{1}{2\gamma^2}\right)$

WE CONSIDER VARIOUS FREQUENCY REGIMES, STARTING WITH LARGE ω, WHICH PROVES TO BE THE MOST IMPORTANT.

1. $\frac{\omega L}{2c\gamma^2} > 1$ EQUIVALENTLY, $\frac{\pi L}{\gamma^2\lambda} > 1$ OR $L > \frac{\gamma^2\lambda}{\pi} \equiv L_0$

L_0 IS THE 'FORMATION LENGTH', THE DISTANCE OVER WHICH A PHOTON OVERRUNS THE CHARGED PARTICLE BY $\lambda/2\pi$. $\left[\frac{\lambda}{2\pi} = (c - v)\frac{L_0}{v} \approx (1-\beta)L_0 \approx \frac{L_0}{2\gamma^2}\right]$

IN THIS REGIME, $Z > 1$ AND $\sin^2 Z$ VARIES BETWEEN 0 AND 1 AS θ VARIES. TRICK: REPLACE $\sin^2 Z(\theta)$ BY $1/2$, ITS AVERAGE VALUE!

THEN $U_\omega \sim 1/Z^2$ WHICH IS BIG ONLY FOR SMALL θ. FOR $\theta \ll 1$

$Z \approx \frac{\omega L}{c}\left(\frac{\theta^2}{2} + 1/2\gamma^2\right)$ SO $U_\omega \approx \frac{4e^2}{\pi c}\frac{\theta^3\,d\theta}{\left(\theta^2 + 1/\gamma^2\right)^2}$

THE ANGULAR INTEGRAL IS $\int \theta^3 d\theta / (\theta^2 + \frac{1}{\gamma^2})^2 = \ln\gamma - \frac{1}{2}$

SO $U_\omega \sim \frac{4e^2}{\pi c}\left(\ln\gamma - \frac{1}{2}\right)$ WHICH HOLDS FOR $\omega > \frac{2c\gamma^2}{L}$

IF THE BOUNDARIES WERE REALLY PERFECT CONDUCTORS TO ARBITRARILY HIGH FREQUENCIES THE TOTAL RADIATED ENERGY WOULD BE QUITE SIGNIFICANT. OF COURSE, NO PHOTON CAN TAKE AWAY MORE THAN THE INITIAL ENERGY $U_0 = \gamma mc^2$, SO $\omega_{MAX} = \gamma mc^2/\hbar = U_0/\hbar$.

$$U = \int^{\omega_{MAX}} U_\omega = \frac{4}{\pi}\frac{e^2}{\hbar c} U_0 \ln(\gamma - \tfrac{1}{2}) = \frac{4\alpha}{\pi} U_0 \ln(\gamma - \tfrac{1}{2})$$

HOWEVER, REAL CONDUCTORS CEASE TO BEHAVE AS CONDUCTORS AT FREQUENCIES ABOVE THEIR PLASMA FREQUENCY ω_p.

SO A BETTER ESTIMATE IS $U_{TOT} = \frac{4\alpha}{\pi} U_p \ln(\gamma - \frac{1}{2})$ WHERE $U_p = \hbar\omega_p$.

WE CAN ALSO WRITE $U_\omega = N_\omega \hbar\omega = \frac{dN}{d\omega}\hbar\omega$ WHERE N = # OF PHOTONS.

THUS $dN = \frac{4}{\pi}\frac{e^2}{\hbar c}\frac{d\omega}{\omega}(\ln\gamma - \frac{1}{2}) = \frac{4}{\pi}\alpha \frac{d\omega}{\omega}(\ln\gamma - \frac{1}{2})$

[THIS FORMS DOES NOT CONTAIN THE LENGTH L. ⇒ RADIATION IS A <u>BOUNDARY</u> EFFECT. THE INDUCED SURFACE CHARGE ON THE BOUNDARIES IS ACCELERATIN!]

THIS (AND THE PREVIOUS RESULTS) AS THE SUM OF RADIATION FROM THE TWO SURFACES AT $-L$ & $+L$. FOR A SINGLE SURFACE,

$$dN = \frac{2}{\pi}\alpha\frac{d\omega}{\omega}(\ln\gamma - \tfrac{1}{2})$$

[THIS IS ESSENTIALLY THE WEIZSÄCKER-WILLIAMS SPECTRUM (p 277b) FOR $b_{MIN} = \frac{\lambda}{2\pi}$.]

THE ANGULAR DISTRIBUTION ALSO FOLLOWS FROM ABOVE:

$$dN = \frac{2}{\pi}\alpha\frac{d\omega}{\omega}\frac{\theta^3 d\theta}{(\theta^2 + 1/\gamma^2)^2}$$

2. $\frac{\omega L}{2c\gamma^2} < 1$ OR $L < L_0 = \frac{\gamma^2\lambda}{\pi}$

IN THIS REGIME, $z \lesssim 1$ AND $\frac{\sin^2 z}{z^2} \approx 1$

FOR $\frac{\omega L}{2c\gamma^2}$ APPROACHING 1, HOWEVER, $z < 1$ ONLY FOR ANGLES UP TO SOME Θ_{MAX}, AND THE RADIATION IS SMALL FOR $\Theta > \Theta_{MAX}$

FOR $\gamma \gg 1$, Θ_{MAX} IS LESS THAN $90°$ WHEN $\omega \frac{L}{c} > 1$

2a. $1 < \omega \frac{L}{c} \lesssim 2\gamma^2$

IN THIS REGIME WE DEFINE Θ_{MAX} BY

$$z = 1 = \frac{\omega L}{c}\left(1 + \frac{1}{2\gamma^2} - \cos\Theta_{MAX}\right) \Rightarrow \cos\Theta_{MAX} = 1 + \frac{1}{2\gamma^2} - \frac{c}{\omega L}$$

AND $U_\omega \approx \frac{2 e^2 \omega^2 L^2}{\pi c^3} \sin^2\theta \, d\cos\theta$ FOR $\theta \lesssim \Theta_{MAX}$

INTEGRATING OVER ANGLES, WE HAVE $\int_{\cos\Theta_{MAX}}^{1} \sin^2\theta \, d\cos\theta = \cos\theta - \frac{\cos^3\theta}{3}\Big|_{\cos\Theta_{MAX}}^{1}$

$$\approx \left(\frac{1}{2\gamma^2} - \frac{c}{\omega L}\right)^2 \quad \text{AFTER SOME ALGEBRA}$$

$$\approx \left(\frac{c}{\omega L}\right)^2 \quad \text{IN REGION 2a.}$$

SO $U_\omega \approx \frac{2 e^2}{\pi c}$ HERE

AND $U = \int_{c/L}^{2\gamma^2 c/L} U_\omega \, d\omega \approx \frac{4\gamma^2 e^2}{\pi L} = \frac{4\gamma^2}{\pi}\frac{r_0}{L} mc^2$ WHERE $r_0 = \frac{e^2}{mc^2}$

IS THE CLASSICAL ELECTRON RADIUS, $\sim 10^{-13}$ cm.

FOR MACROSCOPIC $L \sim 1$ cm, THE RADIATION FROM REGIME 2a IS SMALL UNLESS $\gamma \gtrsim 10^6$, i.e. $\gtrsim 500$ GeV FOR AN ELECTRON.

THUS, FOR FREQUENCIES SUCH THAT L IS LESS THAN THE FORMATION LENGTH, THE TOTAL RADIATION IS SMALL.

2b. $\frac{\omega L}{c} < 1$ THIS IS THE VERY LOW FREQUENCY REGIME.

HERE z IS < 1 FOR ALL θ UP TO $90°$

SO $U_\omega \sim \frac{2 e^2 \omega^2 L^2}{\pi c^3} \int_0^1 \sin^2\theta \, d\cos\theta = \frac{4 e^2 \omega^2 L^2}{3\pi c^3}$

$U = \int_0^{c/L} U_\omega \, d\omega = \frac{4}{9\pi}\frac{e^2}{L} = \frac{4}{9\pi}\frac{r_0}{L} mc^2$ WHICH IS VERY SMALL.