Multipole Expansion of the Radiation Fields

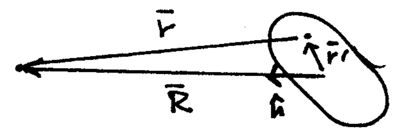
WE RESTRICT OURSELVES TO THE CASE WHERE THE OBSERVER IS FAR FROM A LOCALIZED CURRENT DISTRIBUTION. THE RADIATION FIELDS CAN BE FOUND FROM THE VECTOR POTENTIAL ALONE:

$$\bar{A}_{RAD} = \frac{1}{c}\int \frac{[\bar{J}]}{r}\, dvol' \approx \frac{1}{cr}\int [\bar{J}]\, dvol'$$

WE WILL USE (P.178) $\bar{B} = \frac{1}{c}\frac{\partial \bar{A}}{\partial t}\times\hat{n}$, $\bar{E} = \bar{B}\times\hat{n}$ NOT $E = -\frac{1}{c}\frac{\partial \bar{A}}{\partial t}$, $\bar{B} = \bar{\nabla}\times\bar{A}$. THIS TRICK DEPENDS ON $\bar{\nabla}\cdot\bar{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$

WE BRING THE FACTOR $1/r$ OUTSIDE THE INTEGRAL IN THE APPROXIMATION THAT THE SIZE OF THE SOURCE IS VERY MUCH SMALLER THAN THE DISTANCE TO THE OBSERVER, r. AS SUCH WE WILL ONLY OBTAIN THE RADIATION FIELDS WHICH VARY LIKE $1/r$. SEE THE HOMEWORK SET FOR A DERIVATION OF THE NEAR FIELDS IN AN INTERESTING CASE. WE NOW SHOW HOW $\int[\bar{J}]\,dvol$ MAY BE EXPANDED IN MULTIPOLES.

THE GEOMETRY IS SHOWN AT THE RIGHT.



$$r \approx R - \bar{r}'\cdot\hat{n} \qquad \text{WHERE } \hat{n} = \frac{\bar{R}}{R}$$

$\bar{R}$ = CONSTANT VECTOR FROM SOME CENTRAL POINT IN THE CHARGE DISTRIBUTION

IN EVALUATING $[\bar{J}]$ WE NEED THE RETARDED TIME

$$t' = t - r/c \approx t - \frac{R}{c} + \frac{\bar{r}'\cdot\hat{n}}{c} \equiv t_0' + \frac{\bar{r}'\cdot\hat{n}}{c} \qquad \left(t_0' \equiv t - \frac{R}{c}\right)$$

WE AVOID THE COMPLICATION OF THE VARIATION OF t' OVER THE SOURCE BY AN EXPANSION. IF f IS ANY FUNCTION THEN

$$[f] = f(t') \approx f(t_0') + \left.\frac{\partial f}{\partial t}\right|_{t_0'}\left(\frac{\bar{r}'\cdot\hat{n}}{c}\right) + \cdots$$

HENCE THE VECTOR POTENTIAL CAN BE EXPANDED

$$\bar{A}_{RAD} \approx \frac{1}{cR}\int \bar{J}(t_0')\,dvol' + \frac{1}{c^2R}\int \dot{\bar{J}}(t_0')(\hat{n}\cdot\bar{r}')\,dvol' + \cdots$$

USING R AS THE CHARACTERISTIC DISTANCE TO THE SOURCE.

IF THE CURRENTS ARE STEADY ($\dot{\bar{J}} = 0$) AND FORM CLOSED LOOPS THERE IS NO RADIATION! RECALL THAT FOR CURRENT IN LOOPS WE CAN WRITE

$$\int \bar{J}\,dvol = \sum_{\text{LOOPS}} I \oint d\bar{l} = 0.$$

ELECTRIC DIPOLE RADIATION

THE LEADING TERM IN OUR EXPANSION, $\bar{A}_1(t) \sim \frac{1}{cR}\int \bar{J}(t_0')\, dvol'$

CAN BE REARRANGED IN A USEFUL MANNER FOLLOWING HERTZ (1888), IN WHICH THE INTEGRAL OVER $\bar{J}$ IS REPLACED BY THE TIME DERIVATIVE OF THE TOTAL ELECTRIC DIPOLE MOMENT.

TRICK: $\bar{\nabla}\cdot(x\bar{J}) = x(\bar{\nabla}\cdot\bar{J}) + (\bar{J}\cdot\nabla)x = -x\dot{\rho} + J_x$, USING $\bar{\nabla}\cdot\bar{J} = -\dot{\rho}$.

THUS $\int J_x\, dvol = \int x\dot{\rho}\, dvol + \int \bar{\nabla}\cdot(x\bar{J})\, dvol$. [OR $\bar{J} = \sum e\bar{v} = \frac{d}{dt}\sum e\bar{x} = \dot{\bar{P}}$]

THE 2ND INTEGRAL IS TRANSFORMED BY GAUSS TO $\oint x\bar{J}\cdot d\bar{S} = 0$ ASSUMING THE CURRENTS ARE ALL IN SOME FINITE VOLUME.

THUS $\int \bar{J}\, dvol = \int \bar{x}\dot{\rho}\, dvol = \frac{d}{dt}\int \bar{x}\rho\, dvol = \dot{\bar{P}}$

WHERE $\bar{P} = \int \bar{x}\rho\, dvol$ = TOTAL ELECTRIC DIPOLE MOMENT.

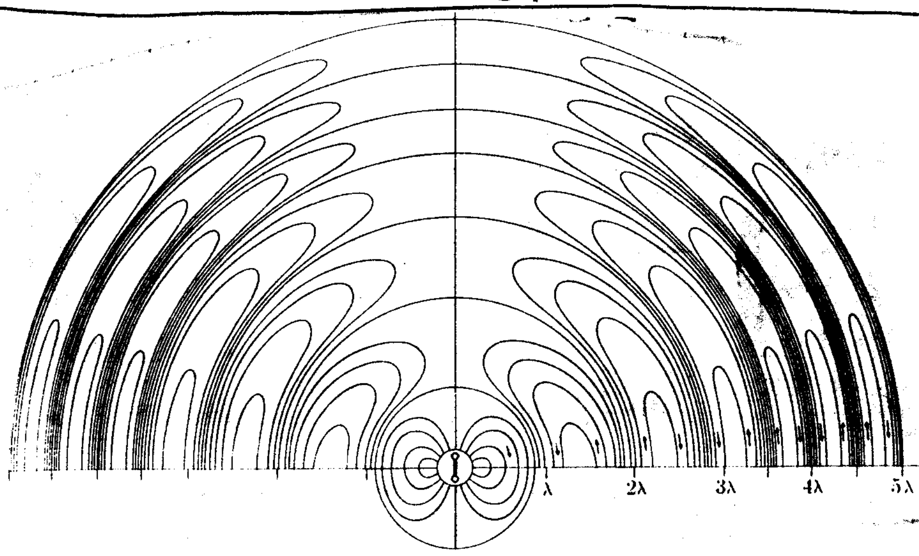
THUS $\bar{A}_1(t) = \frac{\dot{\bar{P}}(t_0')}{cr}$, WHERE WE EVALUATE THE DIPOLE MOMENT AT THE RETARDED TIME $t_0' = t - r/c$.

ON p 178 WE FOUND THE RADIATION FIELDS TO BE

$\bar{B} = \frac{1}{c}\frac{\partial \bar{A}}{\partial t}\times\hat{n}$, AND $\bar{E} = \bar{B}\times\hat{n}$.

NOW $\frac{\partial \bar{A}}{\partial t} = \frac{1}{cr}\frac{\partial \dot{\bar{P}}(t_0')}{\partial t} = \frac{\ddot{\bar{P}}(t_0')}{cr}$. WE WILL JUST WRITE THIS AS $\frac{[\ddot{\bar{P}}]}{cr}$.

$$\bar{B} = \frac{[\ddot{\bar{P}}]\times\hat{n}}{c^2 r}, \quad \bar{E} = \frac{([\ddot{\bar{P}}]\times\hat{n})\times\hat{n}}{c^2 r}, \quad \text{ELECTRIC DIPOLE RADIATION}$$

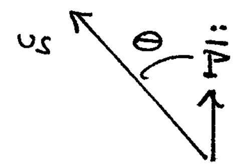


NODES = SURFACES OF ZERO FIELD, ARE SPHERES

THE RADIATED POWER IS GIVEN BY (SEE p 179)

$$\frac{dP}{d\Omega} = r^2\,\bar{S}\cdot\hat{n} = \frac{r^2 c}{4\pi}(\bar{E}\times\bar{B})\cdot\hat{n} = \frac{r^2 c}{4\pi}B^2 = \frac{1}{4\pi c^3}\left([\ddot{\bar{P}}]\times\hat{n}\right)^2$$

$$= \frac{\left[\ddot{P}^2 - (\ddot{\bar{P}}\cdot\hat{n})^2\right]}{4\pi c^3} = \frac{[\ddot{P}]^2\sin^2\theta}{4\pi c^3}.$$



IN ALL THESE EXPRESSIONS, $[\ddot{P}]$ IS TO BE EVALUATED AT THE RETARDED TIME $t_0' = t - R/c$

THE TOTAL POWER RADIATED IS OBTAINED ON INTEGRATION OVER $d\Omega$

$$\int \sin^2\theta\, d\Omega = 2\pi\int(1-\cos^2\theta)\,d\cos\theta = \frac{8\pi}{3}$$

$$\text{Power} = \frac{2}{3c^3}[\ddot{P}]^2$$

WE MAKE SEVERAL OBSERVATIONS

1) <u>LARMOR RADIATION.</u> IF THE RADIATION IS DUE TO A SINGLE MOVING CHARGE THEN $\bar{p} = e\,\bar{x}$ AND $\ddot{\bar{p}} = e\,\bar{a}$ ($\nwarrow$ CHARGE)
WHERE $\bar{a}$ = ACCELERATION.

HENCE
$$\boxed{\frac{dP}{d\Omega} = \frac{e^2 a^2 \sin^2\theta}{4\pi c^3}, \quad P_{TOTAL} = \frac{2e^2a^2}{3c^3}.}$$

THESE ARE THE <u>LARMOR RADIATION</u> FORMULAE WHICH HOLD FOR VELOCITIES $v \ll c$. THEY ARE WELL WORTH REMEMBERING AS THE BASIC RESULTS FOR RADIATION OF A MOVING CHARGE!

2) <u>OSCILLATING DIPOLE</u>. ANOTHER SIMPLE CASE IS THAT THE DIPOLE MOMENT OF THE CHARGE DISTRIBUTION OSCILLATES WITH FREQUENCY ω: $\bar{P} = \bar{P}_0 e^{-i\omega t}$.

THEN $\ddot{\bar{P}} = -\omega^2\bar{P}$.

BEFORE INSERTING THIS INTO THE EXPRESSIONS FOR $\bar{E}$ AND $\bar{B}$, WE MUST EVALUATE $[\ddot{\bar{P}}]$ AT THE RETARDED TIME $t_0' = t - R/c$.

i.e., $[\ddot{\bar{P}}] = -\omega^2\bar{P}_0\, e^{i\left(\frac{\omega R}{c} - \omega t\right)} = -\omega^2\bar{P}_0\, e^{i(kR-\omega t)}$

$$\bar{B} = -\underbrace{\frac{\omega^2}{c^2}}_{k^2}\,\frac{e^{i(kr-\omega t)}}{r}\left(\bar{P}_0\times\hat{n}\right), \qquad \bar{E} = -\underbrace{\frac{\omega^2}{c^2}}_{k^2}\,\frac{e^{i(kr-\omega t)}}{r}\,(\bar{P}_0\times\hat{n})\times\hat{n}.$$

SEE FIG. ON BOTTOM OF p 185

THE AVERAGE POWER RADIATED IS (FACTOR OF 2 LOST IN $\langle \cos^2 \omega t \rangle$)

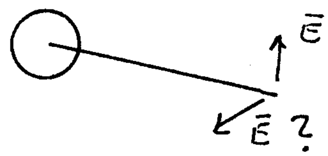
$$\frac{dP}{d\Omega} = \frac{c}{8\pi} r^2 |\bar{B}|^2 = \frac{\omega^4 p_0^2 \sin^2\theta}{8\pi c^3} \quad ; \quad P_{TOT} = \frac{\omega^4 p_0^2}{3c^3} = \frac{ck^4 p_0^2}{3}$$

THE DEPENDENCE ON THE 4RTH POWER OF THE FREQUENCY IS A MEMORABLE RESULT.

WE ENCOURAGE YOU TO CONSIDER MANY MORE DETAILS OF THE OSCILLATING DIPOLE FIELDS ON THE HOMEWORK SET.

3. THE LOWEST ORDER TERM IN THE MULTIPOLE EXPANSION, INVOLVES CHANGES IN THE DIPOLE MOMENT. WHY DON'T WE HAVE 'MONOPOLE' RADIATION — AS MIGHT BE PRODUCED BY A RADIALLY OSCILLATING SPHERE OF CHARGE?

THE RADIATION FIELDS MUST BE TRANSVERSE. BUT IF THE MOTION OF THE SOURCE IS SPHERICALLY SYMMETRIC AND RADIAL, WHAT DIRECTION COULD THE TRANSVERSE $\bar{E}$ TAKE?



IN VIEW OF THIS CONTRADICTION WE EXPECT NO MONOPOLE RADIATION TERM.

THIS IS QUITE UNLIKE SOUND WAVES WHICH INVOLVE LONGITUDINAL VIBRATIONS — AND CAN EASILY BE GENERATED BY A PULSING SPHERE.

4. SUPPOSE OUR SOURCE CONSISTS OF SEVERAL POINT CHARGES, ALL WITH THE SAME CHARGE TO MASS RATIO e/m.

THEN $\bar{p} = \sum_i e\bar{x}_i \sim \sum_i m\bar{x}_i = M_{TOT}\,\bar{X}_{\text{of C.M.}}$

IF THERE ARE NO OUTSIDE FORCES, THEN $\ddot{\bar{p}} = 0$, NO MATTER HOW COMPLICATED THE RELATIVE MOTION OF THE CHARGES MAY BE.

$\Rightarrow$ NO ELECTRIC DIPOLE RADIATION.

[TINY CORRECTION: IF QUADRUPOLE RADIATION EMITS NET MOMENTUM, THEN $\bar{P}_{MECH} \neq 0 \Rightarrow$ CAN HAVE TINY DIPOLE RADIATION (RADIATION REACTION.)]

THIS HAS A STRIKING APPLICATION IN THE CASE OF GRAVITY WAVES. THE SOURCE OF SUCH WAVES IS ACCELERATING MASSES.

BUT ACCORDING TO AN OUTSIDE OBSERVER, THE C.M. OF AN ISOLATED SYSTEM CANNOT ACCELERATE.

$\therefore$ THERE IS NO DIPOLE GRAVITY RADIATION!

EINSTEIN TELLS US THAT GRAVITY WAVES ARE TRANSVERSE.

$\Rightarrow$ NO MONOPOLE GRAVITY RADIATION.

HENCE THE SIMPLEST GRAVITY RADIATION IS QUADRUPOLE.

MAGNETIC DIPOLE AND ELECTRIC QUADRUPOLE RADIATION

WE CONSIDER THE 2ND TERM IN OUR EXPANSION,

$$\bar{A}_2 = \frac{1}{c^2 r} \int \dot{\bar{J}}(t_0') (\bar{r}' \cdot \hat{n}) \, dvol'.$$

WE RECALL PP. 84-85 - BUT FILL IN SOME DETAILS SKIPPED THERE.

WE FIRST LOOK AT $\bar{J}(\bar{r}' \cdot \hat{n})$, AND, AS BEFORE, LOOK FOR THE REST OF THE TRIPLE CROSS PRODUCT:

$$\bar{J}(\bar{r}' \cdot \hat{n}) = \frac{1}{2}\left[\bar{J}(\bar{r}' \cdot \hat{n}) - (\bar{J} \cdot \hat{n})\bar{r}'\right] + \frac{1}{2}\left[\bar{J}(\bar{r}' \cdot \hat{n}) + (\bar{J} \cdot \hat{n})\bar{r}'\right]$$

$$= \frac{1}{2} \hat{n} \times (\bar{J} \times \bar{r}') + \bar{V}$$

$$= \frac{1}{2} (\bar{r}' \times \bar{J}) \times \hat{n} + \bar{V}$$

REGARDING $\bar{V}$, TAKE COMPONENTS, AND IDENTIFY $\bar{J} = \rho \dot{\bar{r}}'$

$$V_i = \frac{1}{2} \rho (\dot{r}_i' r_j' n_j + r_i' \dot{r}_j' n_j) = \frac{1}{2} \frac{\partial}{\partial t} (\rho r_i' r_j' n_j)$$

ALTOGETHER

$$\bar{A}_2\Big|_i = \frac{1}{c^2 r} \left\{ \frac{\partial}{\partial t} \Bigg[\underbrace{\int \frac{\bar{r}' \times \bar{J}}{2} \, dvol'}_{c\bar{m} \,=\, \text{MAGNETIC DIPOLE MOMENT (p. 85)}} \Bigg] \times \hat{n} + \frac{1}{2} \frac{\partial^2}{\partial t^2} \left[\int \rho r_i' r_j' \, dvol' \right] n_j \right\}$$

FINALLY, SINCE $\bar{E}$ AND $\bar{B} \sim \bar{A} \times \hat{n}$, WE CAN ADD A PIECE PROPORTIONAL TO $\hat{n}$ TO $\bar{A}_2$ WITHOUT CHANGING ANY FIELDS. THIS ALLOWS US TO EXHIBIT THE QUADRUPOLE MOMENT TENSOR $Q_{ij} = \int \rho (3 r_i' r_j' - r'^2 \delta_{ij}) \, dvol'$

SO $$\bar{A}_2\Big|_i = \frac{\dot{\bar{m}} \times \hat{n}}{c r}\Big|_i + \frac{1}{6c^2 r} \dddot{Q}_{ij} n_j$$

IF WE DEFINE AN AUXILIARY VECTOR $\bar{Q}$ BY $Q_i = Q_{ij} n_j$

THEN $$\bar{A}_2 = \frac{\dot{\bar{m}} \times \hat{n}}{c r} + \frac{\ddot{\bar{Q}}}{6c^2 r}$$

ALTOGETHER $$\bar{A} \sim \frac{1}{cr}\left(\dot{\bar{p}} + \dot{\bar{m}} \times \hat{n} + \frac{\ddot{\bar{Q}}}{6c} + \dots\right),$$

$$\bar{B} = \frac{1}{c}\dot{\bar{A}} \times \hat{n} = \frac{1}{c^2 r}\left(\ddot{\bar{p}} \times \hat{n} + (\ddot{\bar{m}} \times \hat{n}) \times \hat{n} + \frac{\dddot{\bar{Q}} \times \hat{n}}{6c} + \dots\right),$$

$$\bar{E} = \bar{B} \times \hat{n} = \frac{1}{c^2 r}\left((\ddot{\bar{p}} \times \hat{n}) \times \hat{n} - \ddot{\bar{m}} \times \hat{n} + \frac{(\dddot{\bar{Q}} \times \hat{n}) \times \hat{n}}{6c} + \dots\right).$$

(ALL DERIVATIVES EVALUATED AT THE RETARDED TIME.)

NOTE THE 3RD DERIVATIVE ON THE QUADRUPOLE TERM!

WE CALCULATE THE ANGULAR DISTRIBUTION OF THE RADIATED POWER VIA $$\frac{dP}{d\Omega} = \frac{cr^2}{4\pi} \bar{E} \times \bar{B} \cdot \hat{n} = \frac{cr^2}{4\pi} B^2 = \frac{cr^2}{4\pi} E^2.$$

FOR PURE MAGNETIC DIPOLE RADIATION: $$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}|\ddot{\bar{m}} \times \hat{n}|^2 = \frac{\ddot{m}^2 \sin^2\theta}{4\pi c^3},$$

$$P_{TOT} = \frac{2\ddot{m}^2}{3c^3}.$$

FOR PURE ELECTRIC QUADRUPOLE RADIATION, $$\frac{dP}{d\Omega} = \frac{1}{144\pi c^5}|\dddot{\bar{Q}} \times \hat{n}|^2.$$

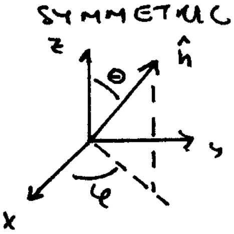
THIS MAY BE RE-EXPRESSED 2 WAYS. RECALLING $Q_i = Q_{ij} n_k$,

$$\frac{dP}{d\Omega} = \frac{\dddot{Q}_{ij}\dddot{Q}_{kl}}{144\pi c^5}\left(\delta_{ik} n_j n_l - n_i n_j n_k n_l\right),$$

WHICH YOU MAY WANT TO COMPARE TO RESULTS FROM GENERAL RELATIVITY SOME DAY.

MORE PRACTICALLY, SUPPOSE THE QUADRUPOLE IS ROTATIONALLY SYMMETRIC, SAY ABOUT THE Z-AXIS

$$Q_{ij} = \begin{pmatrix} -Q_0/2 & & 0 \\ & -Q_0/2 & \\ 0 & & Q_0 \end{pmatrix}$$



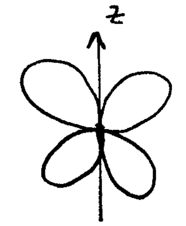
THEN $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

SO $$\bar{Q} = \left(-\frac{Q_0 \sin\theta\cos\phi}{2}, -\frac{Q_0 \sin\theta\sin\phi}{2}, Q_0\cos\theta\right)$$

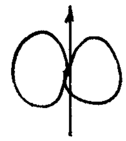
$$= -\frac{Q_0}{2}\hat{n} + \left(0, 0, \frac{3Q_0}{2}\cos\theta\right)$$

HENCE $|\bar{Q} \times \hat{n}| = \frac{3}{2} Q_0 \sin\theta\cos\theta$

HENCE $\frac{dP}{d\Omega} = \frac{\dddot{Q}_0^{\,2}}{64\pi c^5} \sin^2\theta \cos^2\theta$,



COMPARED TO



FOR A DIPOLE.

$$P_{TOT,\,QUADRUPOLE} = \frac{\dddot{Q}_0^{\,2}}{120\, c^5}$$

[FOR A GENERAL QUADRUPOLE, $P_{TOT} = \frac{\dddot{Q}_{ij}\,\dddot{Q}_{ij}}{180 c^5}$]

AGAIN SOME REMARKS:

1) POINT CHARGE MOVING IN A CIRCLE OF RADIUS r_0 AT VELOCITY $v \ll c$.
E1 AND E2 MULTIPOLES ARE GENERATED BY THIS MOTION.

THE ACCELERATION IS $a = v^2/r_0$

SO $P_{E\,DIPOLE} = \frac{2}{3}\frac{e^2}{c^3}\frac{v^4}{r_0^2} = \frac{2}{3}\frac{e^2 c}{r_0^2}\left(\frac{v}{c}\right)^4$

WHILE $P_{E,\,QUAD} \sim \left(\frac{v}{c}\right)^6$ (SEE HOMEWORK SET)

THIS SHOWS THE TYPICAL SUPPRESSION OF QUADRUPOLE RADIATION RELATIVE TO DIPOLE RADIATION.

2. <u>OSCILLATORY SOURCES</u>. IF

$$\bar{p} = \bar{p}_0\, e^{-i\omega t} \quad \Rightarrow \quad \ddot{\bar{p}} = -\omega^2 \bar{p}$$
$$\bar{m} = m_0\, e^{-i\omega t} \qquad\qquad \ddot{\bar{m}} = -\omega^2 \bar{m}$$
$$\bar{Q} = \bar{Q}_0\, e^{-i\omega t} \qquad\qquad \dddot{\bar{Q}} = i\omega^3 \bar{Q}$$

THEN $$\bar{A} = \frac{e^{i(kr-\omega t)}}{r}\left\{ -\frac{i\omega}{c}\bar{p}_0 - \frac{i\omega}{c}\bar{m}_0 \times \hat{n} - \frac{\omega^2}{6c^2}\bar{Q}_0 \right\},$$

$$\bar{B} = \frac{e^{i(kr-\omega t)}}{r}\left\{ \frac{\omega^2}{c^2}\bar{p}_0\times\hat{n} + \frac{\omega^2}{c^2}(\bar{m}_0\times\hat{n})\times\hat{n} + \frac{i\omega^3}{6c^3}\bar{Q}_0\times\hat{n} \right\},$$

$$\bar{E} = \frac{e^{i(kr-\omega t)}}{r}\left\{ \frac{\omega^2}{c^2}(\bar{p}_0\times\hat{n})\times\hat{n} - \frac{\omega^2}{c^2}(m_0\times\hat{n}) + \frac{i\omega^3}{6c^3}(\bar{Q}_0\times\hat{n})\times\hat{n} \right\},$$

$$\left.\frac{dP}{d\Omega}\right|_{MAG\ DIPOLE} = \frac{\omega^4 m_0^2 \sin^2\theta}{8\pi c^3}, \qquad \left.\frac{dP}{d\Omega}\right|_{\substack{SYM\ EL.\\ QUAD}} = \frac{\omega^6 Q_0^2 \sin^2\theta\cos^2\theta}{128\pi c^5},$$

$$P_{MAG\ DIPOLE} = \frac{\omega^4 m_0^2}{3c^3}, \qquad P_{SYM\ EL\ QUAD} = \frac{\omega^6 Q_0^2}{240\, c^5}.$$

CLEARLY $P \sim \omega^8$ FOR AN OCTUPOLE, ETC...

3. A COLLECTION OF LIKE CHARGES.

IF e/m = CONST FOR ALL CHARGES, THEN $\bar{m} = \frac{1}{2}\sum \bar{r}\times e\bar{v} \sim \sum \bar{r}\times m\bar{v} = \bar{L}$

FOR AN ISOLATED SYSTEM, THE ANGULAR MOMENTUM $\bar{L}$ IS A CONSTANT. HENCE NO MAGNETIC DIPOLE RADIATION IS POSSIBLE IN THESE CIRCUMSTANCES. WE ALSO CONCLUDE THAT THE GRAVITATIONAL ANALOGY OF MAGNETIC DIPOLE RADIATION DOES NOT EXIST.

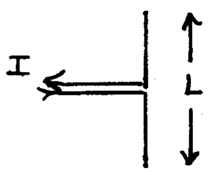
[IF E2 RADIATION EMITS ANGULAR MOMENTUM, THIS WOULD INDUCE A TINY AMOUNT OF M1 RADIATION...]

4. MORE ON MOTION IN A CIRCLE. IF A CHARGE MOVES IN A CIRCLE WITH ANGULAR FREQUENCY ω, THE TIME DEPENDENT PART OF THE QUADRUPOLE MOMENT HAS FREQUENCY 2ω. WE ENCOURAGE YOU TO EXPLORE THIS ON THE HOMEWORK SET.

RADIATION FROM ANTENNAS WITH KNOWN CURRENT DISTRIBUTIONS

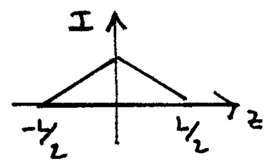
IT IS NOW RELATIVELY STRAIGHTFORWARD TO CALCULATE RADIATION PATTERNS IN VARIOUS SITUATIONS – IF THE DRIVING CURRENTS ARE KNOWN. WE START WITH 2 EXAMPLES, IN WHICH WE SUPPOSE THE SIZE OF THE ANTENNA IS MUCH LESS THAN THE WAVE-LENGTH OF THE BROADCAST WAVE.

EXAMPLE CENTER FED LINEAR ANTENNA



A VERY SIMPLE CURRENT DISTRIBUTION IS

$$I = I_0 e^{-i\omega t}\left(1 - \frac{2|z|}{L}\right),$$



WHICH VANISHES AT THE ENDS OF THE ANTENNA.

SINCE $L \ll \lambda$, $kL \ll 1$ AND IT IS SUFFICIENT TO USE THE DIPOLE APPROXIMATION.

WE NEED THE DIPOLE MOMENT $\quad P = \int_{-L/2}^{L/2} \rho z\, dz.$

NOW $\bar{\nabla}\cdot\bar{J} = -\dot{\rho} \Rightarrow \dot{\rho} = -\frac{\partial I}{\partial z} = -I_0 e^{-i\omega t}\left(\mp\frac{2}{L}\right),$

SO $\quad \rho = \frac{\pm 2 I_0 e^{-i\omega t}}{-i\omega L} = \pm\frac{2i I_0 e^{-i\omega t}}{\omega L}.$ INDEPENDENT OF z!

$\Rightarrow P = i\frac{I_0 L}{2\omega}e^{-i\omega t}$ UPON INTEGRATION OR $P_0 = i\frac{I_0 L}{2\omega}$,

$$\frac{d\,\text{Power}}{d\Omega} = \frac{\omega^4 P_0^2 \sin^2\theta}{8\pi c^3}, \quad P_{TOT} = \frac{\omega^4 P_0^2}{3c^3} = \frac{I_0^2 L^2 \omega^2}{12 c^3} = \frac{\pi^2}{3c} I_0^2\left(\frac{L}{\lambda}\right)^2.$$

THIS WAS THE FORM $\text{Power} = \frac{1}{2} I_0^2 R_{eff}$,

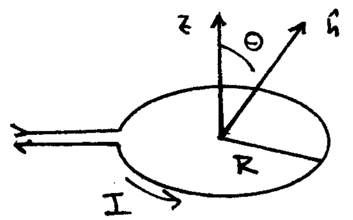
APPROPRIATE TO A.C. CIRCUITS CONTAINING A RESISTANCE:

$$R_{eff} = \frac{2\pi^2}{3c}\left(\frac{L}{\lambda}\right)^2 = \frac{\pi}{6}\left(\frac{4\pi}{c}\right)\left(\frac{L}{\lambda}\right)^2 .$$

RECALL $\frac{4\pi}{c} \sim 377$ OHMS $\equiv$ "RESISTANCE OF THE VACUUM".

IN ANY CASE, FROM THE CIRCUIT'S POINT OF VIEW, POWER IS DISAPPATED IN THE ANTENNA. WE KNOW THAT THIS IS RADIATED ENERGY, RATHER THAN JOULE HEATING, BUT THE ACCOUNTING IS IDENTICAL EITHER WAY.

<u>EXAMPLE</u> LOOP ANTENNA



IT IS EASY TO SUPPOSE THAT $I = I_0 e^{-i\omega t}$

THE MAGNETIC DIPOLE MOMENT IS

$$M = \frac{I}{c}\pi R^2$$

CLEARLY WE GET MAGNETIC DIPOLE RADIATION.

IF $R \ll \lambda$ WE CAN USE THE MAGNETIC DIPOLE FORMULA DIRECTLY

$$\frac{dP}{d\Omega} = \frac{\omega^4 M_0^2 \sin^2\theta}{8\pi c^3} \qquad P_{TOT} = \frac{\omega^4 M_0^2}{3c^3} = \frac{\omega^4 I_0^2 \pi^2 R^4}{3c^5}$$

IF WE REPLACE R BY THE TOTAL LENGTH OF THE LOOP, $L = 2\pi R$

AND USE $\omega = \frac{2\pi c}{\lambda}$, THEN $P_{TOT} = \frac{\pi^2}{3c} I_0^2 \left(\frac{L}{\lambda}\right)^4 = \frac{1}{2} I_0^2 R_{eff}$,

WITH $R_{eff} = \frac{\pi}{6}\left(\frac{4\pi}{c}\right)\left(\frac{L}{\lambda}\right)^4$.

COMPARING THE TWO DIPOLE ANTENNAS, BOTH OF TOTAL LENGTH L,

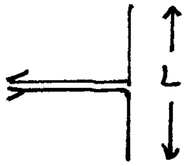
WE HAVE $\frac{P_{\text{MAG DIPOLE}}}{P_{\text{EL DIPOLE}}} = \left(\frac{L}{\lambda}\right)^2 \ll 1$

THIS SHOWS THE TYPICAL SUPPRESSION OF MAGNETIC DIPOLE RADIATION COMPARED TO ELECTRIC DIPOLE.

ALTHOUGH THE ANALOGY MAKE BE STRAINED IN DETAIL, THIS SUPPRESSION HOLDS TRUE IN ATOMS, WHERE L, THE SIZE OF THE ATOM, IS MUCH SMALLER THAN λ FOR OPTICAL RADIATION.

MAGNETIC DIPOLE RADIATION IS IMPORTANT IN ATOMS ONLY IF THE ELECTRIC DIPOLE TERM IS ABSENT FOR SOME REASON.

EXAMPLE CENTER FED LINEAR ANTENNA REVISITED



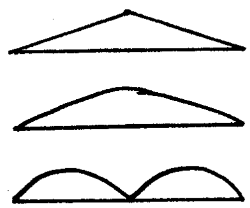
WE CONSIDER MORE GENERAL CURRENT DISTRIBUTIONS THAN BEFORE. BUT $I(z)$ REMAINS SYMMETRIC AND VANISHES AT THE ENDS. A POSSIBILITY BASED ON SINE FUNCTIONS IS

$$I(z) = I_0 \sin k\left(\frac{L}{2} - |z|\right) e^{-i\omega t}, \quad \text{WHERE } k = \frac{2\pi}{\lambda} = \frac{\omega}{c}.$$

$kL \ll 1 \longrightarrow$ AS IN OUR PREVIOUS EXAMPLE

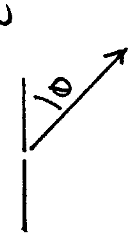
$kL = \pi \longrightarrow$ 'HALF-WAVE' ANTENNA

$kL = 2\pi \longrightarrow$ 'FULL-WAVE' ANTENNA



IN GENERAL, L AND λ ARE COMPARABLE, SO WE MUST TAKE INTO ACCOUNT THE VARIATION IN PHASE OF THE RADIATION OVER THE ANTENNA ($\Rightarrow$ 'INTERFERENCE')

INSTEAD OF USING THE MULTIPOLE EXPANSION, WE RETURN TO THE DETAILED EXPRESSION (p 181, LECTURE 15),



$$\frac{dP}{d\Omega} = \frac{1}{8\pi c}\left[\int (\bar{J}_\omega \times \hat{k})\, e^{-ik\cdot\bar{r}'}\, d\mathrm{vol}\right]^2.$$

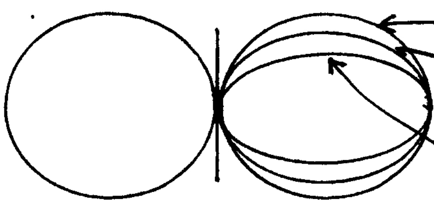
FOR US THE INTEGRAL IS $\displaystyle\int = I_0 k \sin\theta \int_{-L/2}^{L/2} dz \sin k\left(\frac{L}{2} - |z|\right)\cos(kz\cos\theta)$,

AFTER TAKING THE REAL PART.

WITH THE HELP OF A TRIG IDENTITY, $\sin\theta_1 \cos\theta_2 = \frac{1}{2}\sin(\theta_1+\theta_2) + \frac{1}{2}\sin(\theta_1-\theta_2)$.

$$\int = \frac{2I_0}{\sin\theta}\left(\cos\frac{kL\cos\theta}{2} - \cos\frac{kL}{2}\right),$$

SO $$\frac{dP}{d\Omega} = \frac{I_0^2}{2\pi c}\left[\frac{\cos\frac{kL\cos\theta}{2} - \cos\frac{kL}{2}}{\sin\theta}\right]^2.$$



$\sim \sin^2\theta$ IF $kL \ll 1$

$\sim \cos^2(\frac{\pi}{2}\cos\theta)/\sin^2\theta$ IF $kL = \pi$

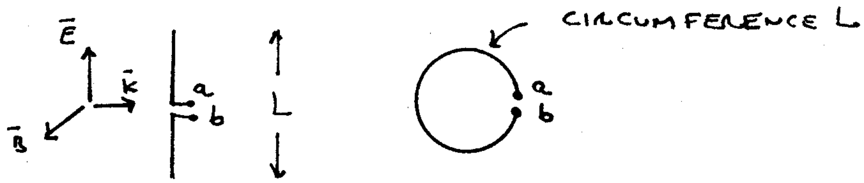
$\sim \cos^4(\frac{\pi}{2}\cos\theta)/\sin^2\theta$ IF $kL = 2\pi$

RECEIVING ANTENNAS

MOST OF US ARE MORE FAMILIAR WITH ANTENNAS USED FOR RECEIVING WAVES THAN FOR BROADCASTING.

JUST AS ANTENNAS RADIATE POWER AT DIFFERENT RATES IN DIFFERENT DIRECTIONS, SO DO THEY HAVE VARYING ABILITIES TO RECEIVE SIGNALS FROM DIFFERENT DIRECTIONS. IN FACT THE ANGULAR DEPENDENCE OF TRANSMISSION AND RECEPTION ARE THE SAME FOR A GIVEN ANTENNA. IT RECEIVES BEST FROM THE DIRECTION IT CAN BROADCAST INTO THE BEST, ETC. THIS IS A CONSEQUENCE OF TIME-REVERSAL INVARIANCE - AS DISCUSSED WHEN CONSIDERING THE EXCITATION OF WAVE GUIDE MODES BY ANTENNAS.

WHILE WE CONSIDERED THE CURRENT WHEN RADIATING THE WAVES, IT IS OFTEN RELEVANT TO CONSIDER THE VOLTAGE IN THE ANTENNA WHEN RECEIVING



WHAT IS V_{ab} INDUCED BY THE WAVES HITTING THESE ANTENNAS

FOR THE LINEAR ANTENNA THE BEST RECEPTION IS CLEARLY WITH THE ANTENNA ALIGNED ALONG $\bar{E}$. ON DIMENSIONAL GROUNDS,

$V_{ab} \sim E \cdot$ SOME DISTANCE.

FOR A SMALL GAP THE GAP DISTANCE IS IRRELEVANT.

THE WAVELENGTH λ WON'T ENTER, SINCE THE ANTENNA IS $\perp$ TO THE VARIATIONS IN THE WAVE WITH λ.

$\therefore V_{ab} \sim EL$ WHERE $L =$ LENGTH.

IF $L \ll \lambda$ IT TURNS OUT THAT $V_{ab} = \frac{1}{2} EL$.

[THE WAVE INDUCES $\Delta V = \int E\,dl = \frac{EL}{2}$ ON EACH HALF OF THE ANTENNA. MEASURING V_{ab} IS LIKE MEASURING ΔV ACROSS ONE PART OF THE ANTENNA]

FOR THE LOOP ANTENNA, WE CAN USE FARADAY'S LAW. FIRST, ORIENT THE ANTENNA SO ITS AXIS IS ALONG $\bar{B}$. IF $L \ll \lambda$, ELECTRIC EFFECTS CANCEL.

THEN, $V_{ab} = \oint \bar{E} \cdot d\bar{l} = -\frac{1}{c} \dot{\Phi}_{MAG} = i\omega B \cdot \text{AREA}$, IF $B = B_0 e^{-i\omega t}$.

FOR A PLANE WAVE, $E = B$; ALSO AREA $= \pi \left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi}$

SO $V_{ab} = \frac{\omega E L^2}{4\pi c} = \frac{1}{2} E L \left(\frac{L}{\lambda}\right)$

THUS $\frac{V_{ab}|_{LOOP}}{V_{ab}|_{LINEAR}} = L/\lambda \ll 1$!

[IF $R =$ RESISTANCE OF RECEIVING CIRCUIT, THEN $P = V_{ab}^2/R$ IS THE POWER TAKEN FROM THE WAVE. $\frac{P_{LOOP}}{P_{LINEAR}} = \left(L/\lambda\right)^2$. COMPARE P. 192]

Scattering of Waves by Small Dielectric Spheres

Suppose a plane wave is incident on a sphere of radius $a \ll \lambda$. The wave causes motions of charges in the sphere, which in turn cause radiation. We may describe this entire process as 'scattering' – even tho we have a more detailed viewpoint.

If $a \ll \lambda$ it will certainly be sufficient to consider only dipole radiation.

Suppose $\bar{p}$ and $\bar{m}$ are the electric and magnetic moments induced in the sphere by the incident wave. These, of course, oscillate at frequency ω of the wave.

Then $$\bar{E}_{scat} = k^2 \frac{e^{i(kr-\omega t)}}{r}\left((\bar{p}\times\hat{n})\times\hat{n} - \frac{\bar{m}\times\hat{n}}{c}\right),$$

$$\bar{B}_{scat} = \hat{n}\times\bar{E}_{scat},$$

while $$\bar{E}_{in} = \bar{E}_0\, e^{i(kz-\omega t)}, \quad \bar{B}_{in} = \hat{z}\times\bar{E}_{in}.$$

We define the scattering cross section to be

$$\frac{d\sigma}{d\Omega} = \frac{\text{power scattered into solid angle } d\Omega}{\text{incident power per unit area}}$$

The dimensions of $\frac{d\sigma}{d\Omega}$ is area.

This definition is similar to that used in Ph 205 for particle scattering, with the substitution of 'power' for 'number of particles'.

Relating power flow to the Poynting vector

$$\frac{d\sigma}{d\Omega} = \frac{\frac{c}{8\pi}\, \mathrm{Re}\left(\bar{E}_{scat}\times\bar{B}^*_{scat}\right)\cdot\hat{n}\, r^2}{\frac{c}{8\pi}\, \mathrm{Re}\left(\bar{E}_{in}\times\bar{B}^*_{in}\right)} = \frac{r^2|E_{scat}|^2}{|E_{in}|^2}$$

What are $\bar{p}$ and $\bar{m}$?

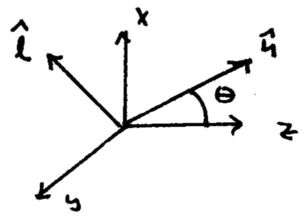
For a dielectric sphere, $\bar{m}=0$ – no current loops! [$J_{pol} = \dot{\bar{P}}$ gives only electric multipoles if $\lambda \gg a$]

With $a \ll \lambda$, the incident field is essentially constant over the entire sphere – so we have an electrostatics problem (p 57, Lecture 5)

THUS $\bar{P}_0 = -\left(\frac{\epsilon-1}{\epsilon+2}\right) a^3 \bar{E}_0$, ϵ = DIELECTRIC CONSTANT ,

SO $\bar{E}_{SCAT} = k^2 \frac{e^{i(kr-\omega t)}}{r} (\bar{p}_0 \times \hat{n}) \times \hat{n}$,

AND $r^2 |E_{SCAT}|^2 = k^4 a^6 \left(\frac{\epsilon-1}{\epsilon+2}\right)^2 \underbrace{\left[(E_0 \times \hat{n}) \times \hat{n}\right] \cdot \left[(E_0^* \times \hat{n}) \times \hat{n}\right]}_{(\bar{E}_0 \cdot \bar{E}_0^*) - (\bar{E}_0 \cdot \hat{n})(E_0^* \cdot \hat{n})}$.



CHOOSE A COORDINATE SYSTEM SUCH THAT THE OBSERVER LIES IN THE x-z PLANE, AT ANGLE Θ TO THE z AXIS.

THEN $\bar{E}_0 = E_0 \hat{\ell}$ FOR THE INCIDENT WAVE,

WHERE $\hat{\ell}$ MUST LIE IN THE x-y PLANE. $\hat{\ell}$ CAN BE COMPLEX

$\hat{\ell} = \hat{x} \pm i\hat{y}$ FOR CIRCULAR POLARIZATION.

THEN $\frac{d\sigma}{d\Omega} = k^4 a^6 \left(\frac{\epsilon-1}{\epsilon+2}\right)^2 \left(1 - (\hat{\ell}\cdot\hat{n})(\hat{\ell}^*\cdot\hat{n})\right)$.

WE DISTINGUISH 2 CASES OF LINEAR POLARIZATION

1. $\hat{\ell} = \hat{x} \Leftrightarrow \hat{\ell}$ PARALLEL TO THE SCATTERING PLANE

   THEN $\hat{\ell}\cdot\hat{n} = \sin\Theta$ SO THE FACTOR IS $1 - \sin^2\Theta = \cos^2\Theta$

2. $\hat{\ell} = \hat{y} \Leftrightarrow \hat{\ell}$ PERP TO THE SCATTERING PLANE

   AND THE FACTOR IS JUST 1

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left(\frac{\epsilon-1}{\epsilon+2}\right)^2 \begin{cases} \cos^2\Theta , & \parallel \text{ SCATTER} , \\ 1 , & \perp \text{ SCATTER} . \end{cases}$$

IF THE INCIDENT WAVE IS UNPOLARIZED, WE HAVE EACH TYPE OF SCATTER 50% OF THE TIME.

$$\left.\frac{d\sigma}{d\Omega}\right|_{UNPOL} = k^4 a^6 \left(\frac{\epsilon-1}{\epsilon+2}\right)^2 \frac{1+\cos^2\Theta}{2}$$

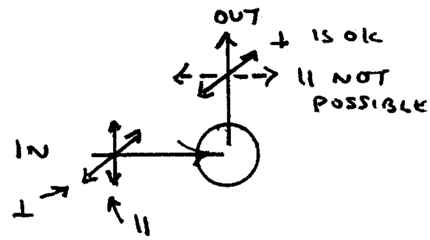
$$\sigma_{TOTAL} = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3} k^4 a^6 \left(\frac{\epsilon-1}{\epsilon+2}\right)^2$$

THE SCATTERED LIGHT RETAINS THE POLARIZATION OF THE INCIDENT LIGHT (I.E., IT REMAINS $\parallel$ OR $\perp$ TO THE SCATTERING PLANE).

WE CAN MEASURE THE DEGREE OF POLARIZATION OF THE SCATTERED LIGHT AS

$$\frac{\left.\frac{d\sigma}{d\Omega}\right|_{\perp} - \left.\frac{d\sigma}{d\Omega}\right|_{\parallel}}{\left.\frac{d\sigma}{d\Omega}\right|_{\perp} + \left.\frac{d\sigma}{d\Omega}\right|_{\parallel}} = \frac{1-\cos^2\theta}{1+\cos^2\theta} = \frac{\sin^2\theta}{1+\cos^2\theta}$$

AT $90°$ THE LIGHT IS 100% POLARIZED $\perp$ TO THE SCATTERING PLANE. SINCE THE INCOMING WAVE WAS ONLY TRANSVERSE POLARIZATION, IT CANNOT GENERATE ANY $\parallel$ POLARIZATION IN THE $90°$ OUTGOING WAVE.



AS A FINAL REMARK, NOTE THAT

$$\sigma \sim k^4 a^6 \sim \frac{a^6}{\lambda^4} \Rightarrow \text{STRONG SCATTERING AT SHORT } \lambda.$$

APPLYING THIS TO SCATTERING OF SUNLIGHT IN THE ATMOSPHERE, WE SEE THAT BLUE LIGHT IS SCATTERED MOST – IF ALL MOLECULES HAVE THE SAME SIZE a!

IF THE ATMOSPHERE IS DUSTY, a IS BIG IN SOME CASES, AND OUR ANALYSIS BREAKS DOWN. BUT WE CAN GET A ROUGH SENSE OF WHAT HAPPENS IF WE SUPPOSE THE SCATTERING OFF A BIG PARTICLE IS JUST THE RESULT OF THE INTERFERENCE OF SCATTERING OFF MANY SMALL PARTICLES. THIS INTERFERENCE IS DESTRUCTIVE AS a APPROACHES λ FROM BELOW. ALSO, IT WILL AFFECT THE BLUE LIGHT BEFORE RED LIGHT, AS BLUE HAS SMALLER λ.
$\therefore$ BLUE SCATTERING DIMINISHES AND RED SCATTER REMAINS.

AT SUNSET THE SUNLIGHT PASSES THRU A LOT OF DUSTY AIR. HENCE WE EXPECT REDDISH LIGHT NEAR THE SURFACE, YELLOW ABOVE THAT, AND BLUE LIGHT AT THE TOP...

Ph 206 LECTURE 16

FOOTNOTE ON ABSORPTION AND THE OPTICAL THEOREM

IN LECTURE 12 WE FOUND THAT THE DIELECTRIC 'CONSTANT' IS NOT REALLY CONSTANT IN AN ATOMIC MODEL OF THE DIELECTRIC MEDIUM. WE FOUND

$$\epsilon(\omega) = 1 + \frac{4\pi N e^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

WHERE ω_0 = NATURAL FREQUENCY OF THE ATOM

γ = DAMPING CONSTANT

THE PRESENCE OF DAMPING IN THE DIELECTRIC MEANS THAT ENERGY IS TAKEN OUT OF THE EXTERNAL FIELD AND CONVERTED TO HEAT.

FOR PLANE WAVES INCIDENT ON A DIELECTRIC SPHERE SOME OF THE INCIDENT WAVE ENERGY WILL DISAPPEAR INTO HEATING THE SPHERE, IF $\gamma \neq 0$. WE CALL THIS PROCESS ABSORPTION OF RADIATION, AND WE WISH TO CALCULATE THE ABSORPTION CROSS SECTION

$$\sigma_{ABS} = \frac{\text{POWER ABSORBED BY THE SPHERE}}{\text{INCIDENT POWER PER UNIT AREA}} .$$

BUT POWER ABSORBED $= \sum_{\text{ELECTRONS}} \bar{v}\cdot\bar{F} = \sum e\bar{v}\cdot\bar{E}_{IN}$.

NOTE THAT $\sum e\bar{v} = \frac{d}{dt}\sum e\bar{x} = \frac{d\bar{p}_0}{dt}$, WHERE $\bar{p}_0$ = DIPOLE MOMENT .

FROM p 196, $\bar{p}_0(t) = -\left(\frac{\epsilon-1}{\epsilon+2}\right) a^3 \bar{E}_{IN} e^{-i\omega t} \Rightarrow \dot{\bar{p}}_0 = i\omega\left(\frac{\epsilon-1}{\epsilon+2}\right) a^3 \bar{E}_{IN} e^{-i\omega t}$.

THE AVERAGE POWER ABSORBED IS THEN $\frac{1}{2}\mathrm{Re}\left(\dot{\bar{p}}_0\cdot\bar{E}_{IN}^*\right)$ BY AN EXTENSION OF OUR METHOD OF TIME AVERAGING

$$\text{POWER ABSORBED} = \frac{\omega}{2} a^3\, \mathrm{Im}\left(\frac{\epsilon-1}{\epsilon+2}\right) E_{IN}^2 .$$

IF $\gamma \neq 0$, ϵ HAS AN IMAGINARY PART, SO THIS IS NON-ZERO.

AGAIN, POWER INCIDENT PER UNIT AREA $= \frac{c}{8\pi} E_{IN}^2$

THUS $$\sigma_{ABS} = 4\pi k a^3\, \mathrm{Im}\left(\frac{\epsilon-1}{\epsilon+2}\right) = 4(ka)\,\mathrm{Im}\left(\frac{\epsilon-1}{\epsilon+2}\right)\cdot\pi a^2$$

SINCE $ka \ll 1$ THIS CAN BE SIGNIFICANT COMPARED TO THE SCATTERING CROSS SECTION FOUND ON p 196! $\sigma_{SCAT} = \frac{8}{3}(ka)^4\left(\frac{\epsilon-1}{\epsilon+2}\right)^2\cdot\pi a^2$

WE MAY USE THESE RELATIONS TO ILLUSTRATE (BUT NOT PROVE) THE OPTICAL THEOREM.

WE NOW DEFINE THE TOTAL CROSS SECTION AS

$$\sigma_{TOTAL} = \sigma_{ABS} + \sigma_{SCAT} \approx \sigma_{ABS} \quad \text{FOR } ka \ll 1$$

FROM P. 196 $$\frac{d\sigma_{SCAT}}{d\Omega} = k^4 a^6 \left(\frac{\epsilon - 1}{\epsilon + 2}\right)^2 \begin{cases} \cos^2\theta \\ 1 \end{cases} \text{DEPENDING ON THE POLARIZATION.}$$

SUPPOSE WE WRITE $$\frac{d\sigma_{SCAT}}{d\Omega} \equiv |f(\theta)|^2$$

SO $f(\theta) \equiv$ SCATTERING AMPLITUDE.

THEN CERTAINLY $f(0°) = k^2 a^3 \left(\frac{\epsilon-1}{\epsilon+2}\right)$ = FOWARD SCATTERING AMPLITUDE

THE RESULT THAT $\sigma_{TOTAL} = \frac{4\pi}{k} \mathrm{Im}[f(0)]$ IS CALLED THE OPTICAL THEOREM.

WE SHALL GIVE ANOTHER EXAMPLE OF THIS RESULT IN LECTURE 17. THERE WE CONSIDER THE 'OPTICAL LIMIT' IN WHICH THE WAVELENGTH OF LIGHT IS MUCH LESS THAN THE SIZE OF THE SCATTERER, i.e., $ka \gg 1$.

THEN $\sigma_{ABS} = \pi a^2$ SEEMS VERY NATURAL IN THIS CASE.

WE MAY NOTE THAT IN THE PRESENT CASE OF $ka \ll 1$ (THE LONG WAVELENGTH LIMIT) IT IS POSSIBLE THAT $\sigma_{ABS} > \pi a^2$ AT CERTAIN FREQUENCIES.

NAMELY, IF $\epsilon \to -2$ BOTH σ_{ABS} & σ_{SCAT} GET VERY LARGE!

RECALL THE DEFINITION $\omega_p^2 = \frac{4\pi N e^2}{m}$ = (PLASMA FREQUENCY)^2

THUS $$\epsilon = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 + i\gamma\omega}$$

IF $\omega \approx \frac{\omega_p}{\sqrt{3}}$ (AND $\omega \gg \omega_0$; $\omega \gg \gamma$) THEN $\epsilon \to -2$

IT TURNS OUT THAT FOR ALUMINUM SPHERES & FOR ULTRAVIOLET LIGHT OF ABOUT $\lambda = 1200$ Å THIS CONDITION IS SATISFIED, AND $\sigma_{ABS} \sim 18\pi a^2$. [SEE C.F. BOHREN, AM. J. PHYS. 51, 323 (1983)]

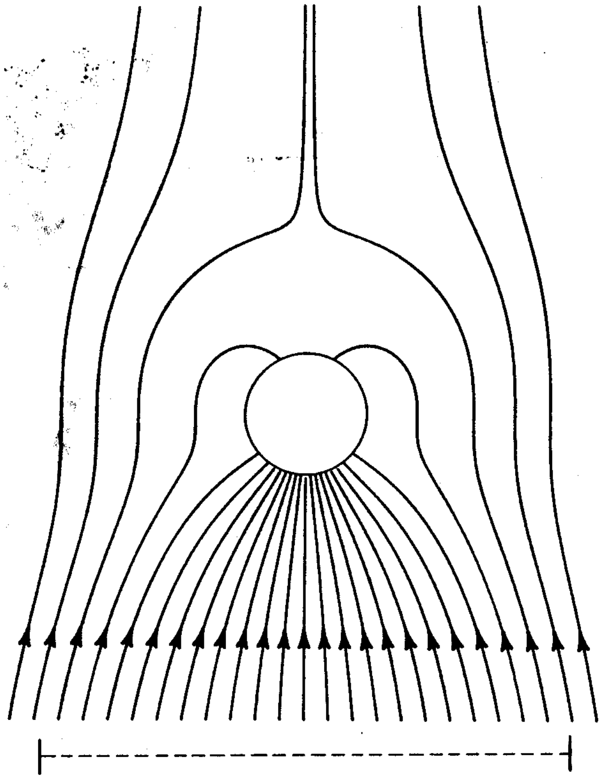


Fig. 1. Field lines of the total Poynting vector (excluding that scattered) around a small aluminum sphere illuminated by light of energy 8.8 eV. The dashed vertical line indicates the effective size of the sphere for absorption of incident light.

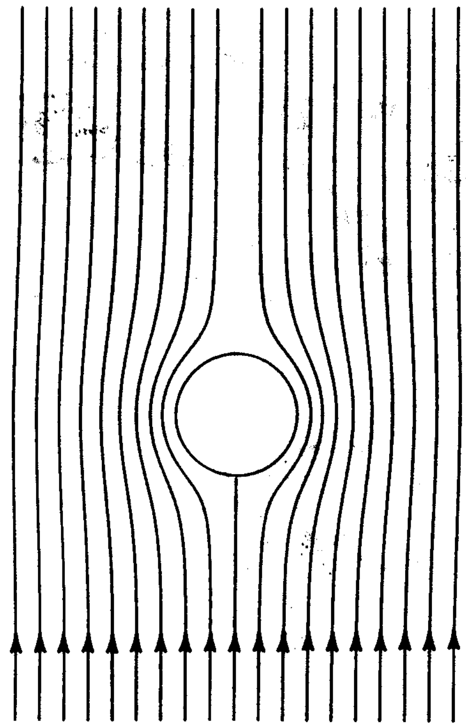


Fig. 2. Field lines of the total Poynting vector (excluding that scattered) around a small aluminum sphere illuminated by light of energy 5 eV.

APPENDIX: PROOF OF THE OPTICAL THEOREM

FOR COMPLETENESS SAKE WE SKETCH A DERIVATION.

A PLANE WAVE IS INCIDENT ON AN OBJECT WHICH 'SCATTERS' THE WAVE, AS WELL AS POSSIBLY 'ABSORBING' SOME OF THE WAVE.

BY 'SCATTERS' WE MEAN THAT CHARGES IN THE OBJECT OSCILLATE DUE TO THE INCIDENT ELECTROMAGNETIC FIELD, AND THEREFORE EMIT RADIATION.

BY 'ABSORBS' WE MEAN THAT A DAMPING MECHANISM IN THE OBJECT CONVERTS SOME OF THE INCIDENT WAVE ENERGY INTO HEAT.

WE EXAMINE THE SCATTERED WAVE FAR ENOUGH AWAY FROM THE OBJECT THAT WE MAY REGARD THIS WAVE AS A SPHERICAL WAVE.

THEN $\bar{E} = \bar{E}_i + \bar{E}_{SCAT} = E_{IN}\left[\hat{x}\, e^{i(kz-\omega t)} + \bar{f}(\theta,\varphi)\,\frac{e^{i(kr-\omega t)}}{r}\right]$

AND $\bar{B} = \bar{B}_i + \bar{B}_{SCAT} = E_{IN}\left[\hat{y}\, e^{i(kz-\omega t)} + \hat{n}\times\bar{f}(\theta,\varphi)\,\frac{e^{i(kr-\omega t)}}{r}\right]$

WE HAVE INTRODUCED $\bar{f}(\theta,\varphi) \equiv$ RELATIVE SCATTERING AMPLITUDE.

$\hat{n}\cdot\bar{f} = 0$, WHERE $\hat{n}$ POINTS ALONG THE NORMAL TO THE SCATTERED SPHERICAL WAVE.

NOTE THAT THE SCATTERING CROSS SECTION IS

$$\frac{d\sigma_{SCAT}}{d\Omega} = \frac{\langle r^2 \bar{S}_{SCAT}\cdot\hat{n}\rangle}{|\bar{S}_{IN}|} = |\bar{f}(\theta,\varphi)|^2 = |f(\theta,\varphi)|^2 .$$

NEXT WE MUST RECALL OUR ORIGINAL DERIVATION OF THE POYNTING VECTOR IN LECTURE 10, pp 116-117.

WE FOUND THAT THE POWER WHICH MUST BE ADDED TO A VOLUME TO PRODUCE CHANGES IN THE FIELDS IS

$$-\int \bar{J}\cdot\bar{E}\, dvol = \frac{d}{dt}\int \frac{\bar{E}\cdot\bar{D} + \bar{B}\cdot\bar{H}}{8\pi}\, dvol + \frac{c}{4\pi}\int_{SURFACE} \bar{E}\times\bar{H}\cdot d\overline{Area} .$$

IN THE PRESENT CASE THERE IS NO BATTERY TO ADD POWER. RATHER, THE OBJECT ABSORBS POWER FROM THE FIELDS AT RATE

$$P_{ABSORBED} = \left\langle \int \bar{J}\cdot E\, dvol \right\rangle ,$$

SO THAT $\sigma_{ABS} = \frac{1}{|\bar{S}_{IN}|}\left\langle \int \bar{J}\cdot\bar{E}\, dvol\right\rangle$, WITH $\langle \bar{S}_{IN}\rangle = \frac{c}{8\pi}E_{IN}^2$.

FOR A STEADY-STATE SCATTERING SITUATION, THE FIELD ENERGY INSIDE A SURFACE SURROUNDING THE OBJECT WILL BE CONSTANT IN TIME.

THUS, $-\sigma_{ABS} = \frac{1}{E_{IN}^2}\int_{SURFACE} Re(E\times H^*)\cdot d\overline{Area}$, TAKING THE TIME AVERAGE.

WE CHOOSE AS THE SURFACE A SPHERE OF RADIUS LARGE ENOUGH THAT THE EXPRESSIONS FOR $\bar{E}$ AND $\bar{B}$ HOLD AS ABOVE.

THEN $\bar{H} = \bar{B}$, ASSUMING VACUUM OUTSIDE THE SCATTERER.

$$\int = \mathrm{Re}\int d\overline{\text{AREA}}\cdot\left[\hat{z} + \frac{\bar{f}\times(\hat{n}\times\bar{f}^*)}{r^2} + \frac{\hat{x}\times(\hat{n}\times\bar{f}^*)}{r}e^{ik(z-r)} + \frac{\bar{f}\times\hat{y}}{r}e^{-ik(z-r)}\right].$$

THE FIRST TERM, $\int d\overline{\text{AREA}}\cdot\hat{z} = 0$.

IN THE 2ND TERM, WE CAN EXPAND $\bar{f}\times(\hat{n}\times\bar{f}^*) = |f|^2\hat{n} - \underbrace{(\hat{n}\cdot\bar{f})}_{0}\bar{f}^*$,

SO WE HAVE $\int d\overline{\text{AREA}}\cdot\hat{n}\,\frac{|f|^2}{r^2} = \int d\Omega|f|^2 = \int d\Omega\,\frac{d\sigma_{SCAT}}{d\Omega} = \sigma_{SCAT}$.

IN THE 3RD AND 4RT TERMS WE SET $z = r\cos\theta$ SO THAT $e^{ik(z-r)} = e^{-ikr(1-\cos\theta)}$.

THIS FACTOR OSCILLATES WILDLY UNLESS $\theta\sim 0$. THUS THE ONLY CONTRIBUTION TO THE INTEGRAL IS IN THE VERY FORWARD DIRECTION. WE CAN REGARD THE REST OF THE INTEGRAND AS ESSENTIALLY CONSTANT, AND PUT $kr(1-\cos\theta)\sim kr\,\theta^2/2$.

THUS, FOR EXAMPLE $\int d\overline{\text{AREA}}\cdot\bar{F}(\theta,\varphi)\,e^{ikr(1-\cos\theta)}$

$$\approx 2\pi r^2\,\hat{z}\cdot\bar{F}(0)\int_0^\pi \underbrace{\sin\theta\,d\theta}_{\sim d\theta^2/2}\,e^{ikr\theta^2/2} \approx \frac{2\pi i r}{k}\,\hat{z}\cdot\bar{F}(0).$$

SO THE INTEGRAL OF THE 3RD AND 4RTH TERMS BECOMES

$$\frac{2\pi}{k}\,\mathrm{Re}\,i\,\hat{z}\cdot\left(-\hat{y}\times\bar{f}^*(0) - \bar{f}(0)\times\hat{y}\right) = \frac{2\pi}{k}\,\mathrm{Re}\,i\,\hat{x}\cdot(\bar{f}(0)-\bar{f}^*(0))$$

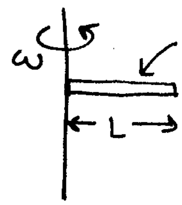
$$= -\frac{4\pi}{k}\,\mathrm{Im}(\hat{x}\cdot\bar{f}(0)).$$

ALTOGETHER, $-\sigma_{ABS} = \sigma_{SCAT} - \frac{4\pi}{k}\,\mathrm{Im}(\hat{x}\cdot\bar{f}(0))$,

OR $\frac{4\pi}{k}\,\mathrm{Im}(\hat{x}\cdot\bar{f}(0)) = \sigma_{SCAT} + \sigma_{ABS} = \sigma_{TOTAL}$.

THE OPTICAL THEOREM.

PRELIMS PROBLEM



MASS M, CHARGE Q

WHAT IS RATE OF ELECTRIC & MAGNETIC DIPOLE RADIATION?

IN GENERAL $P = \frac{dU}{dt} = \frac{2}{3}\frac{\ddot{d}^2 + \ddot{m}^2}{c^3}$ $\quad d$ = ELECTRIC DIPOLE MOMENT, m = MAGNETIC " "

ELECTRIC DIPOLE RADIATION IS BIGGER: SO DO FIRST.

$$d = \int_0^L \rho\, x\, dx = \frac{Q}{L}\int_0^L x\, dx = \frac{QL}{2} \qquad \ddot{d} = \omega^2 d = \frac{\omega^2 QL}{2} \quad \text{FOR CIRCULAR MOTION}$$

$$\boxed{P_{E1} = \frac{\omega^4 Q^2 L^2}{6c^3}}$$

$$m_{LOOP} = \frac{IA}{c} \qquad m = \frac{1}{c}\int_0^L \underbrace{\frac{Q\,dx}{L}}_{dQ}\ \underbrace{\frac{\omega}{2\pi}}_{1/dT} \cdot \underbrace{\pi x^2}_{A} = \frac{\omega Q L^2}{6c}, \qquad \ddot{m} = \frac{\ddot{\omega}\, Q L^2}{6c}$$

ω = CONST $\Rightarrow P_{M1} = 0$.

BUT, ENERGY OF ROTATION IS DECREASING DUE TO E1 RADIATION:

$$U_{ROT} = \frac{1}{2} I \omega^2 = \frac{1}{6} m L^2 \omega^2 \qquad \left[I = \int_0^L \rho_m x^2 dx = \frac{mL^3}{3} \right]$$

$$\frac{dU}{dt} = \frac{mL^2\omega\dot{\omega}}{3} = -P_{E1} = -\frac{\omega^4 Q^2 L^2}{6c^3} \Rightarrow \dot{\omega} = -\frac{\omega^3 Q^2}{2mc^3}$$

$$\ddot{\omega} = -\frac{3\omega^2 \dot{\omega} Q^2}{2mc^3} = \frac{3}{4}\frac{\omega^5 Q^4}{m^2 c^6}$$

$$\ddot{m} = \frac{1}{8}\frac{\omega^5 Q^5 L^2}{m^2 c^7}$$

$$\boxed{P_{M1} = \frac{1}{96}\frac{\omega^{10} Q^{10} L^4}{m^4 c^{14}}} \approx P_{E1} \cdot \frac{\omega^6 Q^8 L^2}{M^4 c^{14}} = P_{E1}\frac{Q^8}{L^4 M^4 c^8}\frac{\omega^6 L^6}{c^6}$$

COMPARE TO P_{E1}:

$$= P_{E1}\left[\frac{Q^2/L}{Mc^2}\right]^4 \left(\frac{v}{c}\right)^6$$

M1 RADIATION TYPICALLY SUPPRESSED BY SOME POWERS OF $\frac{v}{c} = \frac{\omega L}{c}$

$U_{EM} \sim Q^2/L$

$U_{MECH} = U_{REST} = Mc^2$

$$P_{M1} \approx P_{E1}\left[\frac{U_{EM}}{U_{MECH}}\right]^4 \left(\frac{v}{c}\right)^6$$