PH [redacted] – E[redacted]RICITY AND MAGNETISM –

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TEXT: ELECTROMAGNETIC FIELDS AND INTERACTIONS BY R. BECKER (DOVER)

LECTURE TOPICS:

1. INTRODUCTION; ELECTROSTATIC FIELD AND POTENTIAL
2. CONDUCTORS, DIELECTRICS
3. ENERGY IN ELECTROSTATICS; MAXWELL STRESSES
4. POTENTIAL PROBLEMS: IMAGES; SEPARATION OF VARIABLES
5. POTENTIAL PROBLEMS IN 2 DIMENSIONS; IN SPHERICAL COORDS
6. POTENTIAL PROBLEMS IN CYLINDRICAL COORDS; OTHER METHODS
7. STEADY CURRENTS; MAGNETOSTATICS
8. MAGNETIC MATERIALS, ETC.
9. FARADAY'S LAW
10. ENERGY CONSIDERATIONS
11. INTRODUCTION TO ELECTROMAGNETIC WAVES
12. WAVES IN DIELECTRICS
13. WAVES IN CONDUCTORS; TRANSMISSION LINES
14. WAVES IN BOXES AND PIPES

PH 501 ↓

15. RETARDED POTENTIALS; RADIATION
16. MULTIPOLE RADIATION; ANTENNAS
17. OPTICS AND DIFFRACTION
18. SPECIAL RELATIVITY
19. A DIGRESSION ON THE WAVE EQUATION; RADIATION OF A MOVING CHARGE
20. RELATIVISTIC RADIATION: BREMSSTRAHLUNG; SYNCHROTRON RADIATION
21. CERENKOV RADIATION; TRANSITION RADIATION
22. ELECTROMAGNETIC MASS; RADIATION REACTION
23. MORE ABOUT CLASSICAL ELECTRON THEORY
24. ELECTROMAGNETISM, MECHANICS, GRAVITY

THE ORGANIZATION OF THE COURSE IS BASED ON THE BOOK BY PANOFSKY & PHILLIPS. TYPICAL UNDERGRADUATE E & M TEXTS INCLUDE:

CORSON & LORRAIN	
MARION	RECENT TEXTS: GRIFFITHS
SCHWARTZ	HEALD & MARION
SOMMERFELD	LOW

SPECIAL NOTICE SHOULD BE MADE OF THE FEYNMAN LECTURES VOL II (& SOME OF I). THE TYPICAL GRADUATE TEXT IS JACKSON. LANDAU & LIFSHITZ ARE OF COURSE SUPERIOR: SEE VOL 2 AND TO SOME EXTENT VOL 8 FOR PH 206.

IN LECTURE 18 WE SUMMARIZE MANY RESULTS OF SPECIAL RELATIVITY, BUT ASSUME YOU ARE FAMILIAR WITH THE BASICS: RELATIVITY OF SIMULTANEITY; LORENTZ TRANSFORMATION; TIME DILATION; LORENTZ CONTRACTION; ADDITION OF VELOCITIES, ETC. THIS IS COVERED IN SECTIONS 73-77 OF BECKER – OR CONSULT YOUR FAVORITE SOURCE. WE WILL NOT USE IMAGINARY TIME ($x_4 = ict$).

MY TEACHING SITE: http://puhep1.princeton.edu/~mcdonald/examples/

Introduction to Electricity and Magnetism

In Ph 205 we considered some of the properties of matter and motion within the framework of "classical mechanics". Among the various forces causing motion we paid particular attention to the spring force, and that of gravity. The spring force typically appeared in considerations of extended objects, and often was only an approximation to a more complex behavior. The force of gravity appears to be a more fundamental concept: it could be applied consistently to "elementary" point particles as well as large objects. However, these forces do not seem to be the driving mechanism of most phenomena within our everyday experience.

We now take up the study of electricity and magnetism, which in the physicist's view are the fundamental forces governing most facets of our experience. This is a rather grand claim, and can hardly be substantiated in a one semester course. Indeed, pursuit of understanding of electricity and magnetism eventually led people beyond 'classical' mechanics and into 'quantum' mechanics. Our present considerations will be limited to the context of classical mechanics (plus relativity – a 'classical' subject in this sense). As such, our understanding of many of the critical applications of the concepts of electricity and magnetism to macroscopic objects will not be completely satisfactory. To put on a reasonably good show, we will tend to restrict our examples of electromagnetic phenomena to cases of small numbers of free point charges, or idealized macroscopic collections of charges in the form of 'dielectrics' and 'conductors'.

Our purpose is to present a fairly complete picture of the successes of classical electricity and magnetism prior to the study of quantum mechanics in Ph 305-6 etc. The restriction to idealized situations, and the heavy use of mathematical formalism to compress argumentation, will no doubt give the impression that this is what physics is 'really' about. But to the contrary, the true task of our endeavors in physics is the detailed explanation of the phenomena of our experience.

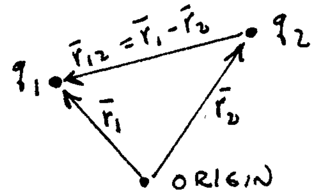
Our idealized discussion of electricity and magnetism will depart quite a bit from the concerns of the physicists of the 19th century who developed this theory. They were primarily concerned with the properties of matter in bulk, and considered such concepts as an 'electron' or 'atom' as theoretical tools without direct verifiable relevance. With the benefit of deeper investigation into the microworld performed in the 20th century, we will base our considerations on such concepts.

Ph 206 Lecture 1

The Field Concept ; Units

The researches of Coulomb and others (Priestley, Franklin....) led to the statement of the electrostatic force law for the case of two point charges at rest

$$\bar{F}_{on 1} = K \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12}$$



q = charge

K = a constant

This is very similar in form to the law of gravitation, although both attractive and repulsive electrical forces occur, leading us to suppose both positive and negative charges exist.

Concerning the measurement of the quantities appearing in this law, there is an ambiguity. 'Charge' is a new concept whose units have not been established. The size and dimensions of the constant K depend on the choice of units!

Two choices remain common today, leading to possible confusion! In the so-called 'practical' or MKSA units charge is considered a 4th independent quantity, measured in coulombs. All charges observed so far turn out to be integral multiples of the charge of the electron

$$q_e \sim 1.6 \times 10^{-19} \text{ coulombs}$$

In the MKSA units, K is called $\frac{1}{4\pi\epsilon_0} = 8.89 \times 10^{9} \frac{N m^2}{C^2}$ as measured by Cavendish.

In the other common system, called CGS or GAUSSIAN units

K is taken to be 1 and dimensionless, so that charge is not an independent unit. $[q] = [m]^{1/2} [l]^{3/2} [t]$

The CGS unit of charge is often called the ESU or STATCOULOMB

Then 1 Coulomb of charge $= 3 \times 10^{9}$ ESU

so $q_{electron} \sim 4.8 \times 10^{-10}$ ESU.

We will mainly use the GAUSSIAN system of units, in which the synthesis of electricity and magnestism into electromagnetism is more symmetrical, and in which the deep relation of the theory to the speed of light is prominently displayed.

THE FORM OF THE ELECTROSTATIC FORCE LAW, AND THE INTUITION OF SUCH WORKERS AS FARADAY AND MAXWELL LED TO THE FIELD CONCEPT. THEY WROTE

$$\bar{F}_{on\,1} = q_1 \bar{E}$$

WHERE $\bar{E}$ IS THE <u>ELECTRIC</u> <u>FIELD</u> DUE TO CHARGE 2.

THIS SEPARATION OF THE FORCE LAW INTO FACTORS TURNS OUT TO BE FRUITFUL BECAUSE ELECTRICITY PROVED TO OBEY A <u>PRINCIPLE OF SUPERPOSITION</u>. IF MANY CHARGES ARE PRESENT

$$\bar{F}_{on\,1} = q_1 \sum_{j=2}^{N} \frac{k q_j}{r_{1j}^2} \hat{r}_{1j} \qquad \text{SO} \qquad \bar{E} = \sum_{j=2}^{N} \frac{k q_j}{r_{1j}} \hat{r}_{1j}$$

IS A CONSISTENT FORMULATION.

A GREAT ABSTRACTION IS TO SUPPOSE THAT $\bar{E}$ IS DEFINED AT EACH POINT IN SPACE, WHETHER OR NOT A CHARGE IS LOCATED THERE.

SINCE $\bar{E} = \dfrac{\bar{F}}{q} \rightarrow \dfrac{\text{FORCE}}{\text{CHARGE}}$, WE NOTE $[E] = \left[\dfrac{m\,l}{t^2 q}\right]$ DIMENSIONALLY.

OUR PRESENT CONCEPTION OF MAGNETIC FORCES DERIVES ALSO FROM COULOMB (FOLLOWING MICHELL). HE NOTED THAT POSITIVE AND NEGATIVE MAGNETISM ALWAYS OCCURED TOGETHER IN A BODY, UNLIKE ELECTRICITY. HOWEVER, HE FOUND THAT BY USING LONG MAGNETISED NEEDLES, ONE COULD IMAGINE THAT ALL THE MAGNETIC EFFECT IS CONCENTRATED IN THE TWO TIPS, ONE A POSITIVE 'POLE', THE OTHER NEGATIVE. EXPERIMENTATION SHOWED THAT THE POLES OBEYED AN INVERSE SQUARE FORCE LAW

$$\bar{F}_{on\,1} = \frac{k' P_1 P_2}{r_{12}^2} \hat{r}_{12}$$

SINCE ISOLATED POLES (MONOPOLES) DO NOT SEEM TO EXIST (ALTHOUGH PEOPLE ARE MADLY SEARCHING FOR THEM), THERE IS NO COMMON NAME FOR THE UNIT OF POLE STRENGTH. IN MKSA UNITS,

$$k' = \frac{\mu_0}{4\pi} = 10^{-7} \text{(MKSA)} \qquad \text{WHILE IN CGS UNITS}, \ k' = 1.$$

AGAIN THE FIELD CONCEPT WAS INTRODUCED, SO THAT

$$\bar{F}_1 = P_1 \bar{B} \qquad \bar{B} \equiv \text{MAGNETIC FIELD}$$

$$[B] = \left[\frac{F}{P}\right] = \left[\frac{m\,l}{t^2 P}\right]$$

The Utility of the Field Concept

In electrostatics the field $\bar{E} = \bar{F}/q$ can be regarded as a convenient computational device, not necessarily having 'physical reality'.

The field concept plays a more crucial role in electrodynamics, situations that change in time. This arises due to the result of relativity that no physical effect can propagate faster than the speed of light.

The question of action at a distance becomes very acute: changing the arrangement of charges 'here' results in a change of the forces on other charges 'there', but only at a later time.

How can this be unless there is 'something' between 'here' and 'there' that participates in these forces?

The electromagnetic field is a candidate for the 'something'.

Charges here act on the nearby field, which in turn acts on the field farther away, and the speed of propagation of this action is limited to the speed of light. Eventually the distant field acts on the distant charges.

Throughout all this, Newton's 3rd law is observed, and the field communicates reaction forces back onto the charges 'here'.

In the above view, the field plays much the role of the famous 'aether' of the 19th century. Maxwell felt that the field was a kind of strain on another medium – which medium has never been detected.

In the quantum view, the field can be thought of as consisting of particles – photons – that carry localised energy and momentum. (Static electric fields don't carry momentum, and can be said to consist of 'virtual' photons).

The quantum view can provide a picturesque explanation for the 'strain' of Maxwell: the field acts on the quantum-mechanical sea of 'virtual' electron-positron pairs.

This view has a realizable consequence: If the field adds energy mc^2 to the virtual pair when the + and – charges separate by their quantum wavelength (Compton wavelength = $\hbar/mc$) then the 'virtual' pair becomes real: the vacuum sparks!

The condition: $e \cdot E \cdot \frac{\hbar}{mc} = mc^2$ or $E = \frac{m^2c^3}{e\hbar} = 1.6 \times 10^{16}$ V/cm = QED critical field

WE MAY NOTE A DIMENSIONAL CONNECTION BETWEEN $\bar{E}$ AND $\bar{B}$ BY REMARKING ON A KEY INSIGHT OF AMPERE BETWEEN ELECTRICITY AND MAGNETISM, NAMELY THAT A LOOP OF CURRENT IS EQUIVALENT TO A MAGNETIC DIPOLE! THIS WAS THE FIRST STEP IN THE SYNTHESIS OF ELECTRICITY WITH MAGNETISM, WHICH WAS COMPLETED BY MAXWELL.

IN MKSA UNITS, THE ARGUMENT OF AMPERE IS THAT FOR THE CURRENT LOOP

$$I \cdot \text{AREA} \sim \text{MAGNETIC DIPOLE STRENGTH} \sim P \cdot L$$

DIMENSIONALLY $\left[\frac{q\,\ell^2}{t}\right] = [p\,\ell]$ OR $[p] = \left[\frac{q\ell}{t}\right]$

OR $[p] = [q] \cdot \text{VELOCITY}$

HENCE $[B] = \left[\frac{M\ell}{t^2 p}\right] = \left[\frac{m}{t\,q}\right]$

WHILE $[E] = \left[\frac{M\ell}{t^2 q}\right]$

SO $[E] = [B] \cdot \left[\frac{\ell}{t}\right] = [B] \cdot \text{VELOCITY}$ MKSA UNITS

THIS DIMENSIONAL ARGUMENT ANTICIPATES THE INSIGHTS OF MAXWELL AND EINSTEIN THAT A SPECIAL VELOCITY - THE VELOCITY OF LIGHT - RELATES $\bar{E}$ AND $\bar{B}$. THIS RELATION IS CONTAINED IN THE OBSERVATION THAT

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \text{SPEED OF LIGHT}$$

WHERE ϵ_0 AND μ_0 APPEAR IN THE MKSA FORM OF THE ELECTRIC AND MAGNETIC FORCE LAWS.

ANOTHER REFLECTION OF THE DIMENSIONAL RELATION BETWEEN $\bar{E}$ AND $\bar{B}$ IS CONTAINED IN THE LORENTZ FORCE LAW FOR MOVING CHARGES

$$\bar{F} = q(\bar{E} + \bar{v} \times \bar{B}) \qquad \text{(MKSA)}$$

AND $\bar{F} = p\left(\bar{B} - \frac{\bar{v}}{c^2} \times \bar{E}\right)$ FOR MONOPOLES.

NOW THAT THE SPECIAL ROLE OF C IS BETTER APPRECIATED, WE FIND THAT THIS IS MORE NATURALLY INCORPORATED IN THE CGS SYSTEM OF UNITS THAN THE MKSA.

IN CGS UNITS $[q] = [p]$ SO $[E] = [B]$ CGS

THE LORENTZ FORCE LAWS ARE

$$\bar{F} = q\left(\bar{E} + \frac{\bar{v}}{c} \times \bar{B}\right) \quad \text{AND} \quad \bar{F} = p\left(\bar{B} - \frac{\bar{v}}{c} \times \bar{E}\right) \qquad \text{CGS}$$

WE WILL USE THE CGS SYSTEM IN THIS COURSE.

WE NOTE THAT AN EVEN MORE 'RELATIVISTIC' SYSTEM OF UNITS COULD BE USED. WE COULD REDEFINE OUR UNIT OF LENGTH, OR OF TIME, SUCH THAT $c = 1$. THIS WOULD RESULT IN THE ULTIMATE SIMPLIFICATION OF THE FORM OF ELECTROMAGNETIC EQUATIONS, AND IS MUCH BELOVED BY THEORETICIANS. (THO SOME PEOPLE ALSO LIKE TO GET RID OF THE 4π'S IN MAXWELL'S EQUATIONS...)

THE MAXWELLIAN SYNTHESIS

MAXWELL COMBINED THE FIELD CONCEPT SO FRUITFUL TO FARADAY WITH THE INSIGHT OF AMPERE THAT ALL MAGNETIC EFFECTS ARE DUE TO CHARGES IN MOTION, AND PRODUCED HIS FIELD EQUATIONS. WE INTRODUCE THEM FROM A MICROSCOPIC VIEWPOINT, SUPPOSING THAT ALL CHARGES ARE KNOWN, AND THAT NO MAGNETIC MONOPOLES EXIST. IN MODERN NOTATION, AND CGS UNITS, MAXWELL SAID

$$\bar{\nabla} \cdot \bar{E} = 4\pi \rho \qquad \bar{\nabla} \cdot \bar{B} = 0$$

$$\bar{\nabla} \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t} \qquad \bar{\nabla} \times \bar{B} = \frac{1}{c}\frac{\partial \bar{E}}{\partial t} + \frac{4\pi}{c}\bar{j}$$

WHERE ρ = CHARGE DENSITY = $\frac{\text{CHARGE}}{\text{VOLUME}}$, $\bar{j}$ = CURRENT DENSITY = $\frac{\text{CURRENT}}{\text{AREA}}$

IF THESE EQUATIONS ARE SOLVED FOR $\bar{E}$ AND $\bar{B}$ IN TERMS OF THE CHARGES ρ AND CURRENTS $\bar{J}$, THEN WE MAY USE THE LORENTZ FORCE LAW TO CALCULATE MOTIONS, ETC.

WE RECALL BRIEFLY SOME FEATURES OF VECTOR CALCULUS IMPLIED IN THE NOTATION $\bar{\nabla} \cdot \bar{E}$, $\bar{\nabla} \times \bar{E}$, ETC.

FOR ANY VECTOR FIELD $\bar{A}$,

$$\int_{\text{CLOSED SURFACE}} \bar{A} \cdot d\bar{s} = \int_{\text{VOLUME}} \bar{\nabla} \cdot \bar{A} \, d\text{vol} \qquad \text{(GAUSS)}$$

$$\oint_{\text{LOOP}} \bar{A} \cdot d\bar{l} = \int_{\text{SURFACE OF LOOP}} \bar{\nabla} \times \bar{A} \cdot d\bar{s} \qquad \text{(STOKES)}$$

THUS $\int \bar{\nabla}\cdot\bar{E}\, d\text{vol} = \int 4\pi\rho\, d\text{vol} = 4\pi\, Q_{\text{INSIDE}}$

$$\| $$

$$\int \bar{E}\cdot d\bar{S} = 4\pi\, Q_{\text{INSIDE}} \qquad (\underline{\text{GAUSS'}}\ \underline{\text{LAW}})$$

THIS HAS A GEOMETRIC INTERPRETATION THAT 'LINES OF FLUX' OF THE VECTOR FIELD $\bar{E}$ CAN ONLY BEGIN AND END ON CHARGES.

THEN $\bar{\nabla}\cdot\bar{B} = 0 \implies \int \bar{B}\cdot d\bar{S} = 0$

LINES OF FLUX OF MAGNETIC FIELD $\bar{B}$ CANNOT TERMINATE ANYWHERE — BUT ALWAYS FORM CLOSED LOOPS. (NO MONOPOLES)

ALSO $\int \bar{\nabla}\times\bar{E}\cdot d\bar{S} = -\frac{1}{c}\frac{\partial}{\partial t}\int \bar{B}\cdot d\bar{S}$

$$\oint \bar{E}\cdot d\bar{l} = -\frac{1}{c}\frac{\partial \Phi}{\partial t} \qquad \Phi = \int_{\text{SURFACE}} \bar{B}\cdot d\bar{S}$$

OR $\quad$ E.M.F. $= -\frac{1}{c}\frac{\partial}{\partial t}$ (MAGNETIC FLUX) $\quad$ ($\underline{\text{FARADAY'S}}$ $\underline{\text{LAW}}$)

LIKEWISE $\int \bar{\nabla}\times\bar{B}\cdot d\bar{S} = \frac{4\pi}{c}\int \bar{j}\cdot d\bar{S} + \frac{1}{c}\frac{\partial}{\partial t}\int \bar{E}\cdot d\bar{S}$

$$\underbrace{\oint_{\text{LOOP}} \bar{B}\cdot d\bar{l} = \frac{4\pi}{c} I_{\text{THRU LOOP}}}_{\text{AMPERE'S LAW}} + \underbrace{\frac{1}{c}\frac{\partial}{\partial t}(\text{ELECTRIC FLUX THRU LOOP})}_{\text{MAXWELL'S 'DISPLACEMENT CURRENT'}}$$

ANOTHER RESULT CONTAINED IN MAXWELL'S EQUATIONS IS <u>CHARGE CONSERVATION</u>.

FOR ANY VECTOR FIELD $\bar{A}$, $\bar{\nabla}\cdot(\bar{\nabla}\times\bar{A}) = 0$

SO $\nabla\cdot(\bar{\nabla}\times\bar{B}) = 0 = \frac{4\pi}{c}\bar{\nabla}\cdot\bar{j} + \frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot E = \frac{4\pi}{c}\left(\bar{\nabla}\cdot\bar{j} + \frac{\partial\rho}{\partial t}\right)$

SO WE MUST HAVE $\bar{\nabla}\cdot\bar{j} = -\frac{\partial\rho}{\partial t}$

BUT $\int \bar{\nabla}\cdot\bar{j}\, d\text{vol} = \int_{\text{SURFACE}} \bar{j}\cdot d\bar{S} = -\frac{\partial}{\partial t}\int \rho\, d\text{vol} = -\frac{d}{dt} Q_{\text{INSIDE}}$

SO FLOW OF CHARGE <u>OUTWARDS</u> ACROSS THE SURFACE = LOSS OF CHARGE INSIDE

$\implies$ CHARGE IS CONSERVED.

AS OUR FINAL INTRODUCTORY COMMENT ON MAXWELL'S EQUATIONS, WE NOTE THAT THEY CONTAIN WAVE PHENOMENA.

FOR ANY VECTOR FIELD $\bar{A}$, $\bar{\nabla}\times(\bar{\nabla}\times\bar{A}) = \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \nabla^2\bar{A}$

THUS $$\bar{\nabla}\times(\bar{\nabla}\times\bar{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\bar{\nabla}\times\bar{B}) = -\frac{4\pi}{c^2}\frac{\partial\bar{J}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

$$\parallel$$

$$\bar{\nabla}(\underbrace{\bar{\nabla}\cdot\bar{E}}_{4\pi\rho}) - \nabla^2\bar{E}$$

SO $$\nabla^2\bar{E} - \frac{1}{c^2}\frac{\partial^2\bar{E}}{\partial t^2} = 4\pi\bar{\nabla}\rho + \frac{4\pi}{c^2}\frac{\partial\bar{J}}{\partial t}$$

SIMILARLY $$\nabla\times(\bar{\nabla}\times\bar{B}) = \frac{4\pi}{c}\bar{\nabla}\times\bar{J} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\bar{E})$$

$$\parallel$$

$$\bar{\nabla}(\underbrace{\bar{\nabla}\cdot\bar{B}}_{0}) - \nabla^2\bar{B} = \frac{4\pi}{c}\bar{\nabla}\times\bar{J} - \frac{1}{c^2}\frac{\partial^2\bar{B}}{\partial t^2}$$

SO $$\nabla^2\bar{B} - \frac{1}{c^2}\frac{\partial^2\bar{B}}{\partial t^2} = -\frac{4\pi}{c}\bar{\nabla}\times\bar{J}$$

IN REGIONS WHERE $\rho = 0 = \bar{J}$, BOTH FIELDS $\bar{E}$ AND $\bar{B}$ OBEY THE WAVE EQUATION $$\nabla^2\begin{Bmatrix}\bar{E}\\ \bar{B}\end{Bmatrix} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\begin{Bmatrix}\bar{E}\\ \bar{B}\end{Bmatrix} = 0$$

THE VELOCITY OF THE WAVE SOLUTIONS IS, OF COURSE, c, THE SPEED OF LIGHT.

IN THE FIRST PART OF THE COURSE WE WILL REVIEW SOME OF THE CONSIDERATIONS WHICH LED TO MAXWELL'S SYNTHESIS (10 LECTURES). THEN WE WILL EXPLORE SOME OF THE RICHES CONTAIN IN ELECTROMAGNETIC WAVE PHENOMENA (14 LECTURES).

HELMHOLTZ' THEOREM - A MATHEMATICAL DIGRESSION (BECKER SEC. 12)

AS A WARM-UP EXERCISE IN VECTOR CALCULUS WE CONSIDER AN AMUSING QUESTION: MAXWELL GIVES US A PRESCRIPTION FOR $\bar{\nabla}\cdot\bar{E}$ AND $\bar{\nabla}\times\bar{E}$, NOT FOR $\bar{E}$ DIRECTLY. CAN $\bar{E}$ IN FACT BE [UNIQUELY] CONSTRUCTED FROM THIS INFORMATION?

YES, SAYS HELMHOLTZ! $\bar{E} = -\bar{\nabla}\phi + \bar{\nabla}\times\bar{A}$

WHERE $\phi = \frac{1}{4\pi}\int \frac{\bar{\nabla}\cdot\bar{E}}{r}\, dvol$ AND $\bar{A} = \frac{1}{4\pi}\int\frac{\bar{\nabla}\times\bar{E}}{r}\, dvol$

SKETCH OF A PROOF: NOTE THAT $\frac{\bar{\nabla}\cdot\bar{E}}{r} = \bar{\nabla}\cdot\left(\frac{\bar{E}}{r}\right) - \bar{E}\cdot\bar{\nabla}\frac{1}{r}$

SO $\phi = -\frac{1}{4\pi}\int \bar{E}\cdot\bar{\nabla}\left(\frac{1}{r}\right) dvol + \frac{1}{4\pi}\int\bar{\nabla}\cdot\left(\frac{\bar{E}}{r}\right) dvol$

GAUSS' THEOREM SAYS $\int_{VOL}\bar{\nabla}\cdot\left(\frac{\bar{E}}{r}\right) dvol = \int_{SURFACE}\frac{\bar{E}}{r}\cdot d\bar{S} \rightarrow 0$ AS $S\rightarrow\infty$ (IF $\bar{E}$ WELL BEHAVED)

SIMILARLY $\frac{\bar{\nabla}\times\bar{E}}{r} = \bar{\nabla}\times\left(\frac{\bar{E}}{r}\right) + \bar{E}\times\bar{\nabla}\left(\frac{1}{r}\right)$

SO $\bar{A} = \frac{1}{4\pi}\int\bar{E}\times\bar{\nabla}\left(\frac{1}{r}\right) dvol + \frac{1}{4\pi}\int\bar{\nabla}\times\left(\frac{\bar{E}}{r}\right) dvol$

BY A VARIANT OF STOKES' THEOREM

$= \int_{SURFACE}\left(\frac{\bar{E}}{r}\right)\times d\bar{S} \rightarrow 0$ AS $S\rightarrow\infty$

NOW CONSIDER $\bar{W}(1) \equiv \frac{1}{4\pi}\int\left(\frac{\bar{E}(2)}{r_{12}}\right) dvol_2$, SO THAT $\phi = \bar{\nabla}\cdot\bar{W}$ AND $\bar{A} = \bar{\nabla}\times\bar{W}$

(NOTING THAT $\bar{\nabla}(1)\frac{1}{r_{12}} = -\nabla(2)\frac{1}{r_{12}}$).

THEN $\nabla^2\bar{W}(1) = \frac{1}{4\pi}\int\nabla^2_{(1)}\left(\frac{\bar{E}(2)}{r_{12}}\right) dvol_2 = \frac{1}{4\pi}\int\bar{E}(2)\,\nabla^2_{(1)}\frac{1}{r_{12}}\, dvol_2$

$$= -\int\bar{E}(2)\,\delta^3(\bar{r}_1-\bar{r}_2)\, dvol_2 = -\bar{E}(1)$$

(CAN YOU SHOW THAT $\nabla^2\frac{1}{r} = -4\pi\,\delta^3(r)$ VIA GAUSS' THEOREM .... ?)

FINALLY, $-\nabla\phi + \bar{\nabla}\times\bar{A} = -\bar{\nabla}(\bar{\nabla}\cdot\bar{W}) + \bar{\nabla}\times(\bar{\nabla}\times\bar{W})$

$$= -\bar{\nabla}(\bar{\nabla}\cdot\bar{W}) + \bar{\nabla}(\bar{\nabla}\cdot\bar{W}) - \nabla^2\bar{W} = -\nabla^2\bar{W} = \bar{E}$$

AS DESIRED. [UNIQUENESS? $\bar{W}$ IS NOT UNIQUE, BEING AN INTEGRAL; BUT WE USE ONLY $\nabla^2\bar{W}$, WHICH IS UNIQUE.]

ELECTRO STATICS

WE NOW START OVER AND SKETCH HOW MAXWELL'S LAWS CAN BE DERIVED FROM THE PROPERTIES OF ELECTRICITY AND MAGNETISM AS THEY WERE INITIALLY PERCEIVED.

WE BEGIN WITH A SITUATION IN WHICH ALL CHARGES ARE AT REST — ELECTROSTATICS

THE KEY FACT FROM EXPERIMENT IS COULOMB'S LAW

$$\bar{F} = q_1 q_2 \frac{\hat{r}_{12}}{r_{12}^2} = q_1 q_2 \frac{\bar{r}_{12}}{r_{12}^3}$$

WE NOTE THAT $\bar{\nabla}_1\left(\frac{1}{r_{12}}\right) = -\frac{1}{r^2}\bar{\nabla} r = -\frac{1}{r^2}\frac{\bar{r}}{r}$

SO COULOMB'S LAW CAN BE WRITTEN $\bar{F} = -q_1 q_2 \bar{\nabla}\left(\frac{1}{r}\right)$

INTRODUCING THE ELECTROMAGNETIC FIELD $\bar{E}(1) = \frac{\bar{F}}{q_1} = -q_2 \bar{\nabla}_1\left(\frac{1}{r_{12}}\right)$

THEN AT ONCE $\underline{\bar{\nabla}\times\bar{E} = 0}$ BY A VECTOR IDENTITY.

AND $\bar{\nabla}\cdot\bar{E} = -q_2 \nabla^2\left(\frac{1}{r_{12}}\right) = 4\pi q_2 \delta^3(\bar{r}_{12})$

TO SEE THAT $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\bar{r})$, NOTE THAT FOR $r > 0$,

$$\nabla^2\left(\frac{1}{r}\right) = \bar{\nabla}\cdot\bar{\nabla}\left(\frac{1}{r}\right) = -\bar{\nabla}\cdot\left(\frac{\bar{r}}{r^3}\right) = -\frac{\bar{\nabla}\cdot\bar{r}}{r^3} + \frac{3}{r^4}\bar{r}\cdot\nabla\bar{r} = -\frac{3}{r^3} + \frac{3}{r^4}\bar{r}\cdot\frac{\bar{r}}{r^2} = 0$$

AT $r=0$ $\frac{1}{r}$ BLOWS UP, SO $\nabla^2\left(\frac{1}{r}\right)$ DOES ALSO. BUT CONSIDER $\int\nabla^2\left(\frac{1}{r}\right)d\mathrm{vol}$ OVER A SMALL SPHERE ABOUT THE ORIGIN.

$$\int \bar{\nabla}\cdot\bar{\nabla}\left(\frac{1}{r}\right)d\mathrm{vol} = \int\bar{\nabla}\left(\frac{1}{r}\right)\cdot d\bar{S} = -\int\frac{\hat{r}}{r^2}\cdot d\bar{S} = -\int\frac{\hat{r}\cdot\hat{r}\,r^2 d\Omega}{r^2} = -4\pi$$

THIS IS EXACTLY WHAT IS MEANT BY $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\bar{r})$

IF MANY CHARGES ARE PRESENT IN A SMALL VOLUME, THEN

$$\int\bar{\nabla}\cdot\bar{E}\,d\mathrm{vol} = 4\pi\int\sum_i q_i\,\delta^3(\bar{r}_{1i})\,d\mathrm{vol} = 4\pi\, q_{\text{INSIDE}}$$ [GAUSS' LAW]

IN THE LIMIT OF SMALL VOLUMES. $\underline{\bar{\nabla}\cdot\bar{E}} = 4\pi\frac{q_{\text{INSIDE}}}{d\mathrm{vol}} \equiv \underline{4\pi\rho}$

GREEN'S DERIVATION THAT $\nabla^2\phi = -4\pi\rho$ (1828)

$$\phi = \int \frac{\rho}{r}\, d\mathrm{vol}$$

$$\nabla^2\phi = \int \rho\, \nabla^2\left(\frac{1}{r}\right) d\mathrm{vol}$$

AWAY FROM $r=0$, $\nabla^2 \frac{1}{r} = 0$. SO THE MATHEMATICAL DIFFICULTY COMES ONLY WHEN EVALUATING $\nabla^2\phi$ INSIDE A CHARGE DISTRIBUTION ρ.

GREEN SUGGESTS DIVIDING THE VOLUME INTO A SMALL SPHERE CONTAINING THE OBSERVER + REST OF VOL.

$$\nabla^2\phi = \int_{\substack{\text{SMALL}\\\text{SPHERE}}} \rho\, \nabla^2\frac{1}{r}\, d\mathrm{vol} + \underbrace{\int_{\text{REST}} \rho\, \nabla^2\frac{1}{r}\, d\mathrm{vol}}_{=0 \text{ SINCE } r \text{ IS NEVER } 0 \text{ IN THIS VOLUME}}$$

DO <u>NOT</u> PUT THE OBSERVER ~~...~~ AT THE ORIGIN; RATHER AT RADIUS r

a = RADIUS OF SPHERE.

THEN FOR SMALL ENOUGH a, ρ IS ESSENTIALLY CONSTANT

TO CALCULATE THE POTENTIAL INSIDE THE SPHERE, NOTE THAT

$$E_r = \frac{4}{3}\pi r\rho \quad (r<a) \qquad \text{AND} \qquad E_r = \frac{4}{3}\pi \frac{a^3\rho}{r^2} \quad r>a$$

$$\phi = -\int_\infty^r E_r\,dr = \int_r^\infty E_r\,dr = \int_r^a E_r + \int_a^\infty E_r$$

$$= \frac{2}{3}\pi(a^2-r^2)\rho + \frac{4}{3}\pi a^2\rho = 2\pi a^2\rho - \frac{2}{3}\pi r^2\rho$$

FINALLY, $\nabla^2\phi = -4\pi\rho$ FOLLOWS WITHOUT ANY CONTROVERSY ABOUT $\frac{1}{r}$ AS $r \to 0$.

Thus we see how Coulomb's law for a point charge, plus superposition leads to the usual Maxwell equation for a continuous charge distribution.

The Electric Potential

By Stokes theorem, $\oint_{\text{LOOP}} \bar{E}\cdot d\bar{l} = \int_{\text{SURFACE}} \bar{\nabla}\times\bar{E}\cdot d\bar{S} = 0$,

so we may say that the electric field is 'conservative'.

Then we expect that $\bar{E}$ can be derived from a potential.

i.e., $\bar{E} = -\bar{\nabla}\phi \iff \phi(\bar{r}) = \phi_0 - \underbrace{\int_0^{\bar{r}} \bar{E}\cdot d\bar{l}}_{\text{INDEPENDENT OF PATH SINCE } \bar{\nabla}\times\bar{E}=0.}$

In fact, from p. 9, $\bar{E} = -\bar{\nabla}\left(\frac{q}{r}\right)$,

so the potential of a point charge is just

$$\phi = \frac{q}{r}.$$

The limit of a continuous charge distribution is clearly

$$\phi = \int \frac{\rho\, d\text{vol}}{r},$$

which always solves for ϕ in principle.

In practice, it is often useful to note a differential equation for the potential:

$$\bar{\nabla}\cdot\bar{E} = 4\pi\rho = \bar{\nabla}\cdot(-\bar{\nabla}\phi) \Rightarrow \underline{\nabla^2\phi = -4\pi\rho} \quad \underline{\text{POISSON'S EQ.}}$$

In charge-free regions this reduces to

$$\underline{\nabla^2\phi = 0} \quad \underline{\text{LAPLACE'S EQUATION}}.$$

In lecture's 4-6 we will discuss various tricks for solving for the potential.

For now, we mention only one method, especially useful if the charge distribution is localized.

MULTIPOLE EXPANSION OF THE POTENTIAL (BECKER SEC 24)

A COMMON ELECTROSTATICS PROBLEM IS THE CALCULATION OF THE ELECTRIC FIELD DUE TO A LOCALIZED CHARGE DISTRIBUTION, AS SEEN BY AN OBSERVER FAR FROM THE CHARGES.

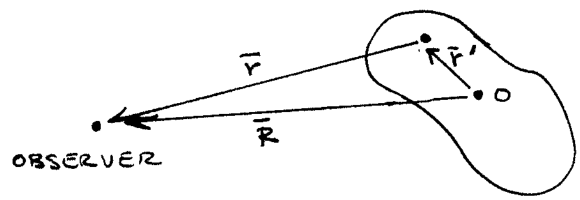
RATHER THAN DO THE 3 INTEGRALS IMPLIED IN THE EXPRESSION

$$\bar{E} = \int \frac{\rho \hat{r}}{r^2}\, d\text{vol}$$

WE CONSIDER THE SCALAR POTENTIAL

$$\phi = \int \frac{\rho}{r}\, d\text{vol} \qquad \text{AND THEN CALCULATE} \qquad \bar{E} = -\bar{\nabla}\phi .$$

WE CHOOSE THE ORIGIN TO BE AT A FIXED POINT INSIDE THE CHARGE DISTRIBUTION. IT REMAINS TO BE SEEN IF THERE IS A NATURAL CHOICE FOR THE ORIGIN, SUCH AS A "CENTER OF CHARGE."



THE DISTANCE $\bar{r}$ FROM AN ARBITRARY ELEMENT OF CHARGE TO THE OBSERVER CAN THEN BE WRITTEN

$$\bar{r} = \bar{R} - \bar{r}' \quad \text{AND SO} \quad \frac{1}{r} = \frac{1}{|\bar{R}-\bar{r}'|} = \left(R^2 - 2\bar{R}\cdot\bar{r}' + r'^2\right)^{-\frac{1}{2}} = \frac{1}{R}\left(1 - 2\frac{\hat{R}\cdot\bar{r}'}{R} + \left(\frac{r'}{R}\right)^2\right)^{-\frac{1}{2}}$$

WHERE $R = |\bar{R}|$ ETC. FOR $r' \ll R$ A TAYLOR SERIES EXPANSION OF $1/r$ CONVERGES RAPIDLY:

$$\frac{1}{r} = \frac{1}{R}\left(1 - \frac{1}{2}\left[-2\frac{\hat{R}\cdot\bar{r}'}{R} + \left(\frac{r'}{R}\right)^2\right] + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left[-2\frac{\hat{R}\cdot\bar{r}'}{R} + \left(\frac{r'}{R}\right)^2\right]^2 + \cdots\right)$$

$$= \frac{1}{R} + \frac{\hat{R}\cdot\bar{r}'}{R^2} + \frac{3(\hat{R}\cdot\bar{r}')^2 - r'^2}{2R^3} + \frac{5(\hat{R}\cdot\bar{r}')^3 - 3(\hat{R}\cdot\bar{r}')r'^2}{2R^4} + \cdots$$

THE POTENTIAL IS THEN

$$\phi(\bar{R}) = \frac{1}{R}\int \rho\, d\text{vol} + \frac{\hat{R}}{R^2}\cdot\int \bar{r}'\rho\, d\text{vol} + \frac{\sum_{ij}\hat{R}_i\hat{R}_j\int(3r_i'r_j' - \delta_{ij}r'^2)\rho\, d\text{vol}}{2R^3} + \cdots$$

$$\equiv \frac{Q}{R} + \frac{\bar{P}\cdot\hat{R}}{R^2} + \frac{\hat{R}\cdot\bar{\bar{Q}}\cdot\hat{R}}{2R^3} + \cdots$$

$Q = \int \rho\, d\text{vol}$ = TOTAL CHARGE

$\bar{P} = \int \bar{r}'\rho\, d\text{vol}$ = DIPOLE MOMENT

$\bar{\bar{Q}} = Q_{ij} = \int(3r_i'r_j' - \delta_{ij}r'^2)\rho\, d\text{vol}$ = QUADRUPOLE MOMENT (TENSOR)

THIS IS THE MULTIPOLE EXPANSION.

Since charges can have both signs, we often have interesting cases with $Q_{TOTAL} = 0$ (atoms, people, etc).

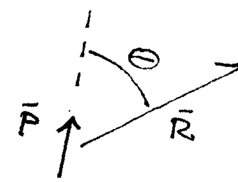
Can we set the dipole term, $\bar{P}$, to zero by a suitable choice of origin – as we did when defining the center of mass?

By analogy we would write $Q\bar{R}' = \int \rho \bar{r}' \, dvol$

to define the center of charge $\bar{R}'$. Then the dipole moment calculated about a shifted origin at $\bar{R}'$ will vanish. But note that if $Q = 0$, $\bar{R}'$ cannot be defined! Thus the dipole concept, which played no important role in systems of masses, represents the first non-trivial departure of a charge distribution from that of a point charge.

We may picture a purely dipole charge distribution as equivalent to a pair of charges $\pm q$ separated by a distance $\bar{d}$, such that $\bar{P} = q\bar{d}$

The dipole potential is



$$\phi_{DIPOLE} = \frac{\bar{P} \cdot \hat{R}}{R^2} = \frac{\bar{P} \cdot \bar{R}}{R^3} = \frac{P\cos\theta}{R^2} = \frac{P \, P_1(\cos\theta)}{R^2}$$

where $P_1(z) = z$ is the 1st Legendre polynomial.

Note that $\phi_{DIPOLE} \sim \dfrac{1}{R^2}$.

The electric field is then $\bar{E} = -\bar{\nabla}\phi = -\dfrac{(\bar{P} - 3\hat{R}(\bar{P}\cdot\hat{R}))}{R^3} \sim \dfrac{1}{R^3}$ (R>0

On problem ④ of the homework set you will consider the limit $R = 0$. Then to get consistent behavior of $\int \bar{E} \, dvol$ you should find an additional piece $-\frac{4\pi \bar{P}}{3}\delta(\bar{R})$ which may be thought of as the very strong field between the + and – charges of the dipole.

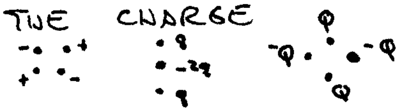
The next order of complexity of a charge distribution is described by its quadrupole moment:

$$Q_{ij} = \int (3 r_i' r_j' - \delta_{ij} r'^2) \rho \, dvol$$

$$\phi_{QUADRUPOLE} \sim \frac{1}{R^3} \Rightarrow \bar{E}_{QUADRUPOLE} \sim \frac{1}{R^4}$$

EXAMPLES OF SIMPLE DISTRIBUTIONS WITH NON-VANISHING QUADRUPOLE MOMENTS ARE A RING OF CHARGE AND A LINE OF UNIFORM CHARGE. NOTE THAT A SPHERE OF CHARGE HAS NO QUADRUPOLE MOMENT, HOWEVER. A SINGLE CHARGE HAS NON ZERO MULTIPOLE MOMENTS OF ALL ORDER IF REFERENCE VECTOR $\bar{R}$ DOES NOT BEGIN ON THAT CHARGE!

THE QUADRUPOLE MOMENT TENSOR IS ROUGHLY THE CHARGE EQUIVALENT OF THE INERTIA TENSOR. PURE QUADRUPOLES:



THE QUADRUPOLE TENSOR IS <u>SYMMETRIC</u>: $Q_{ij} = Q_{ji}$.

BUT WE KNOW THAT ANY SYMMETRIC MATRIX CAN BE DIAGONALISED BY A SUITABLE CHOICE OF ORIENTATION OF THE COORDINATE AXES (i.e., USE THE PRINCIPAL AXES). SO Q_{ij} HAS ONLY 3 INDEPENDENT COMPONENTS.

IN ADDITION, Q_{ij} IS <u>TRACELESS</u>: $Q_{ii} \equiv \sum_i Q_{ii} = \int (3r_i' r_i' - \delta_{ii} r'^2) = 0$

HENCE ONLY 2 COMPONENTS OF Q_{ij} ARE TRULY INDEPENDENT IN GENERAL.

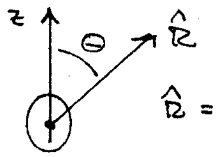
MOST SIMPLE EXAMPLES OF QUADRUPOLE DISTRIBUTIONS WILL ALSO BE <u>ROTATIONALLY</u> <u>SYMMETRIC</u> — SAY ABOUT THE z AXIS

THEN $Q_{xx} = Q_{yy} = -\frac{1}{2} Q_{zz}$ SINCE $Q_{xx} + Q_{yy} + Q_{zz} = 0$ ALWAYS.

IN THIS CASE THERE IS ONLY ONE INDEPENDENT COMPONENT $Q \equiv Q_{zz}$

(WE TRY NOT TO CONFUSE Q WITH THE TOTAL CHARGE!)

$$Q_{ij} = \begin{pmatrix} -Q/2 & & 0 \\ & -Q/2 & \\ 0 & & Q \end{pmatrix}$$



$\hat{R} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\text{THEN } \phi_{\text{QUADRUPOLE}} = \frac{\hat{R}\cdot\bar{\bar{Q}}\cdot\hat{R}}{2R^3} = \frac{Q}{2R^3}\left(\cos^2\theta - \frac{1}{2}\sin^2\theta\right) = \frac{Q}{2R^3}\left(\frac{3\cos^2\theta - 1}{2}\right)$$

$$= \frac{Q}{2R^3} P_2(\cos\theta)$$

↖ THE 2ND ORDER LEGENDRE POLYNOMIAL.

$$\bar{E}_{\text{QUAD}} = -\bar{\nabla}\phi_{\text{QUAD}} \Rightarrow E_i = -\frac{\partial}{\partial x_i} \frac{x_j Q_{jk} x_k}{2R^5}$$

$$= -\frac{\delta_{ij} Q_{jk} x_k}{2R^5} - \frac{x_j Q_{jk} \delta_{ik}}{2R^5} + \frac{5 x_i (x_j Q_{jk} x_k)}{2R^7}$$

$$\text{SO } \bar{E}_{\text{QUAD}} = \frac{5\hat{R}(\hat{R}\cdot\bar{\bar{Q}}\cdot\hat{R}) - 2\bar{\bar{Q}}\cdot\hat{R}}{2R^4}$$

FOR A SYMMETRIC QUADRUPOLE: $\bar{E}_{\text{QUAD}} = \frac{Q}{2R^4}\left(5\hat{R}P_2(\cos\theta) - 3\hat{z}P_1(\cos\theta) + \hat{R}\right)$

CHAPTER G II

Electrodynamics

The Gaussian CGS system and the Giorgi MKSA system ar employed. If a given formula is different in these two systems, then in the following, both formulae will be given side by side.

1. The field equations and the constitutive equations

	CGS System	MKSA System
Maxwell equations:	$\mathrm{curl}\,\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{g}$	$\mathrm{curl}\,\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{g}$
	$\mathrm{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$	$\mathrm{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
	$\mathrm{div}\,\mathbf{D} = 4\pi\rho$	$\mathrm{div}\,\mathbf{D} = \rho$
	$\mathrm{div}\,\mathbf{B} = 0$	$\mathrm{div}\,\mathbf{B} = 0$
Constitutive equations:	$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$	$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$
	$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$	$\mathbf{B} = \mu_0\mathbf{H} + \mathbf{M}$
Equation of continuity:	$\mathrm{div}\,\mathbf{g} + \frac{\partial \rho}{\partial t} = 0$	$\mathrm{div}\,\mathbf{g} + \frac{\partial \rho}{\partial t} = 0$

If the quantities **D** and **H**, considered in electron theory to be secondary quantities, are eliminated from the Maxwell equations, we obtain the new equations:

CGS	MKSA
$\mathrm{curl}\,\mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}(\mathbf{g} + \mathbf{g}_P + \mathbf{g}_M)$	$\frac{1}{\mu_0}\mathrm{curl}\,\mathbf{B} = \varepsilon_0\frac{\partial \mathbf{E}}{\partial t} + (\mathbf{g} + \mathbf{g}_P + \mathbf{g}_M)$
$\mathrm{div}\,\mathbf{E} = 4\pi(\rho + \rho_P)$	$\varepsilon_0\,\mathrm{div}\,\mathbf{E} = \rho + \rho_P$

The meanings here are:

$\rho_P = -\mathrm{div}\,\mathbf{P}$, the density of polarization charge

$\mathbf{g}_P = \frac{\partial \mathbf{P}}{\partial t}$, the density of polarization current

CGS	MKSA
$\mathbf{g}_M = c\,\mathrm{curl}\,\mathbf{M}$	$\mathbf{g}_M = \frac{1}{\mu_0}\mathrm{curl}\,\mathbf{M}$, the density of magnetization current

2. The material constants

Normal isotropic substances (i.e. non-ferroelectric non-ferromagnetic substances) are, in their relations to static fields, characterized by special material constants:

	CGS	MKSA
Electrical conductivity:	$\mathbf{g} = \sigma(\mathbf{E} + \mathbf{E}^{(e)})$	$\mathbf{g} = \sigma(\mathbf{E} + \mathbf{E}^{(e)})$
Permittivity (dielectric constant)	$\mathbf{D} = \varepsilon\mathbf{E}$	$\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E}$
or electric susceptibility:	$\mathbf{P} = \chi\mathbf{E}$	$\mathbf{P} = \chi\varepsilon_0\mathbf{E}$
	$\varepsilon = 1 + 4\pi\chi$	$\varepsilon = 1 + \chi$
Permeability	$\mathbf{B} = \mu\mathbf{H}$	$\mathbf{B} = \mu\mu_0\mathbf{H}$
or magnetic susceptibility:	$\mathbf{M} = \kappa\mathbf{H}$	$\mathbf{M} = \kappa\mu_0\mathbf{H}$
	$\mu = 1 + 4\pi\kappa$	$\mu = 1 + \kappa$

In fields changing with time, σ, ε, and μ are in general frequency-dependent; the foregoing relationships then only hold between the Fourier components of the field strengths.

3. Energy and force expressions

Energy theorem: $\frac{\partial u}{\partial t} + \mathrm{div}\,\mathbf{S} = -\mathbf{g}.\mathbf{E} = -\frac{\mathbf{g}^2}{\sigma} + \mathbf{g}.\mathbf{E}^{(e)}$

	CGS	MKSA
Energy density of the field:	$du = \frac{1}{4\pi}(\mathbf{E}.d\mathbf{D} + \mathbf{H}.d\mathbf{B})$	$du = \mathbf{E}.d\mathbf{D} + \mathbf{H}.d\mathbf{B}$
specially, for normal substances:	$u = \frac{1}{8\pi}(\mathbf{E}.\mathbf{D} + \mathbf{H}.\mathbf{B})$	$u = \frac{1}{2}(\mathbf{E}.\mathbf{D} + \mathbf{H}.\mathbf{B})$
Poynting vector:	$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}$	$\mathbf{S} = \mathbf{E} \times \mathbf{H}$
Force on moving charge:	$\mathbf{F} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$	$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
Force density in matter with $\varepsilon = \mu = 1$:	$\mathbf{f} = \rho\mathbf{E} + \frac{\mathbf{g}}{c} \times \mathbf{B}$	$\mathbf{f} = \rho\mathbf{E} + \mathbf{g} \times \mathbf{B}$

Force density in matter with $\mu = 1$, $\sigma = 0$ (conductivity):

$$\mathbf{f} = \rho\mathbf{E} - \frac{1}{8\pi}\mathbf{E}^2\,\mathrm{grad}\,\varepsilon + \frac{1}{8\pi}\mathrm{grad}\left(\mathbf{E}^2\frac{d\varepsilon}{d\sigma}\sigma\right)$$

(The σ in this formula is the matter density.)

The force density can always be represented as the divergence of the Maxwell stress tensor.

4. Wave propagation

For normal homogeneous uncharged media (ε, μ, σ constant in space, $\mathbf{E}^{(e)}=0$, $\rho=0$) every field component F satisfies the wave equation

$$\nabla^2 F = \frac{\varepsilon\mu}{c^2}\frac{\partial^2 F}{\partial t^2} + \frac{4\pi\sigma\mu}{c^2}\frac{\partial F}{\partial t} \qquad \nabla^2 F = \mu\mu_0\left(\varepsilon\varepsilon_0\frac{\partial^2 F}{\partial t^2} + \sigma\frac{\partial F}{\partial t}\right)$$

For material constants that are frequency-dependent this equation has meaning only for the individual temporal Fourier components of F.

From this, the wave velocity *in vacuo*	$c = 1/\sqrt{(\varepsilon_0\mu_0)}$
Wave velocity in insulators	$c/n = c/\sqrt{(\varepsilon\mu)}$
Index of refraction	$n = \sqrt{(\varepsilon\mu)}$

The calculation of fields *in vacuo* is facilitated by going over to the potentials A and ϕ by means of

$$\left\{\begin{aligned} \mathbf{B} &= \operatorname{curl}\mathbf{A} \\ \mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \operatorname{grad}\phi \end{aligned}\right. \qquad \begin{aligned} \mathbf{B} &= \operatorname{curl}\mathbf{A} \\ \mathbf{E} &= -\frac{\partial\mathbf{A}}{\partial t} - \operatorname{grad}\phi \end{aligned}$$

Potential equations:

$$\left\{\begin{aligned} \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\frac{4\pi\mathbf{g}}{c} \\ \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} &= -4\pi\rho \end{aligned}\right. \qquad \begin{aligned} \nabla^2\mathbf{A} - \varepsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\mu_0\mathbf{g} \\ \nabla^2\phi - \varepsilon_0\mu_0\frac{\partial^2\phi}{\partial t^2} &= -\frac{\rho}{\varepsilon_0} \end{aligned}$$

Lorentz convention

$$\operatorname{div}\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0 \qquad \operatorname{div}\mathbf{A} + \varepsilon_0\mu_0\frac{\partial\phi}{\partial t} = 0$$

General integral of the ϕ-equation in the CGS system:

$$\phi(\mathbf{r}, t) = \int\frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)dv'}{|\mathbf{r}-\mathbf{r}'|}$$

5. Electrotechnical concepts

Capacitance:	$C = Q/V$	$C = Q/V$
Parallel-plate capacitor (plate separation d, surface S):	$\dfrac{S\varepsilon}{4\pi d}$	$\dfrac{S\varepsilon\varepsilon_0}{d}$
Cylindrical capacitor (radii $r_1 < r_2$; length l):	$\dfrac{l\varepsilon}{2\ln(r_2/r_1)}$	$\dfrac{2\pi l\varepsilon\varepsilon_0}{\ln(r_2/r_1)}$
Capacitance of a sphere (radius r):	$r\varepsilon$	$4\pi r\varepsilon\varepsilon_0$
Ellipsoid of rotation (axes $a > b$):	$\dfrac{\sqrt{(a^2-b^2)}\varepsilon}{\ln[a+\sqrt{(a^2-b^2)}]-\ln b}$	$\dfrac{4\pi\sqrt{(a^2-b^2)}\varepsilon\varepsilon_0}{\ln[a+\sqrt{(a^2-b^2)}]-\ln b}$
Inductance:	$L = \dfrac{\mu}{c^2}\oint\oint\dfrac{d\mathbf{r}_1 . d\mathbf{r}_2}{r_{12}}$	$L = \dfrac{\mu\mu_0}{4\pi}\oint\oint\dfrac{d\mathbf{r}_1 . d\mathbf{r}_2}{r_{12}}$
Self-inductance of a coil (length l, cross-section a, n turns):	$\dfrac{4\pi n^2\mu a}{l}$	$\dfrac{n^2\mu\mu_0 a}{l}$

Resistance: $R = V/I$

Field energy of a capacitor:

$$U_{\text{el}} = \tfrac{1}{2}QV = \tfrac{1}{2}CV^2$$

Field energy of a current system:

$$U_{\text{mag}} = \tfrac{1}{2}\sum\sum L_{jk}I_jI_k$$

Oscillatory circuit with capacitance, self-inductance, and resistance:

Oscillation equation: $L\dfrac{d^2I}{dt^2} + R\dfrac{dI}{dt} + \dfrac{I}{C} = 0$

Eigen frequency (resonance frequency) of the undamped circuit:

$$\omega_0 = 1/\sqrt{(LC)}$$

Logarithmic decrement: $D = \pi R\sqrt{(C/L)}$

6. Conversion table: MKSA units to Gaussian

In this table, as in certain places in the text, the symbols for quantities in the MKSA system have been temporarily written, for distinction, with a small star. If the symbols in the two systems represent different physical quantities, then in the comparison of units we have an arrow instead of an equal sign. In this case the unit quantity in the MKSA system is not equal to the similarly named quantity in the Gaussian system given on the right, but corresponds to it only in the sense of the relationship given in the second column of the table.

The numerical factor 3 in the table should, more accurately, be replaced by 2·99790, the number associated with the velocity of light.

Quantity of electricity	$Q^* = Q$	1 Coulomb (C)	$= 3\times10^9$	CGS units
Electric current	$I^* = I$	1 Ampere (A)	$= 3\times10^9$	CGS units
Electric potential	$V^* = V$	1 Volt (V)	$= \frac{1}{300}$	CGS units
Electric field	$E^* = E$	1 V/m	$= \frac{1}{30{,}000}$	CGS units
Electric displacement	$D^* = \frac{1}{4\pi}D$	1 As/m^2	$\rightarrow 4\pi\times3\times10^5$	CGS units
Dielectric polarization	$P^* = P$	1 As/m^2	$= 3\times10^5$	CGS units
Magnetic field	$H^* = \frac{c}{4\pi}H$	1 A/m	$\rightarrow 4\pi\times10^{-3}$	Oersted (Oe)
Magnetic induction	$B^* = \frac{1}{c}B$	1 Vs/m^2	$\rightarrow 1\times10^4$	Gauss (G)
Magnetization	$M^* = \frac{4\pi}{c}M$	1 Vs/m^2	$\rightarrow \frac{1}{4\pi}\times10^4$	Gauss (G)
Magnetic flux	$\Phi^* = \frac{1}{c}\Phi$	1 Weber (Wb)	$\rightarrow 1\times10^8$	Maxwell (Mx)
Power	$I^*V^* = IV$	1 Watt (W)	$= 1\times10^7$	erg/s
Capacitance	$C^* = C$	1 Farad (F)	$= 9\times10^{11}$	cm
Inductance	$L^* = L$	1 Henry (H)	$= \frac{1}{9}\times10^{-11}$	s^2/cm
Resistance	$R^* = R$	1 Ohm (Ω)	$= \frac{1}{9}\times10^{-11}$	s/cm
Conductivity	$\sigma^* = \sigma$	1 Siemens (S)	$= 9\times10^9$	1/s

Permittivity of a vacuum:

$$\varepsilon_0^* = \varepsilon_0 = 8{\cdot}86\times10^{-12}\ \mathrm{As/Vm} = \frac{1}{4\pi\times9\times10^9}\ \mathrm{As/Vm} = \frac{1}{4\pi}$$

Permeability of a vacuum:

$$\mu_0^* = \mu_0 = 4\pi\times10^{-7}\ \mathrm{Vs/Am} = \frac{4\pi}{c^2} = \frac{4\pi}{9}\times10^{-20}\ \mathrm{s^2/cm^2}$$