Ph 406: Elementary Particle Physics
Problem Set 9

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1. **Double-Beta Decay**

   a. In 1935, Goeppert-Mayer made a calculation[1] of the process $A \to A'ee\overline{\nu}\overline{\nu}$, so-called **2-neutrino double-beta decay**, which was first definitively observed in 1987.[2] Make a quick order-of-magnitude **estimate** (not a Fermi-theory calculation) of the lifetime of a nucleus against double-beta decay, starting from the fact that the lifetime of the neutron is 880 s. *What is the characteristic time scale/frequency for the collision of two quarks within a nucleon of radius ≈ 1 fermi? What fraction of these collisions result in a beta-decay of a neutron?*

   b. An important comment by Majorana[3] was that neutrinos might not be "Dirac" particles as considered in the Notes (Lectures 6, 7, 19, *etc.*),[4] but rather what are called the (Dirac) neutrino and antineutrino are combined in a single particle, **the** neutrino, which then is its own antiparticle (and which implies that lepton number is not conserved).

   It was realized by Furry[5] that, if neutrinos behave according to Majorana's view, there could exist the phenomenon of **neutrinoless double-beta decay**, $A \to A'ee$. Further, if the nuclear matrix elements were similar for 2-neutrino double-beta decay and neutrinoless double-beta decay, then the rate for the latter would be much larger (and the lifetime much shorter), because the 3-body phase space (Lecture 11 of the Notes) for the final state of the latter reaction is much larger than the 5-body phase space for the former.

[1] M. Goeppert-Mayer, *Double Beta-Disintegration*, Phys. Rev. **48**, 512 (1935),
http://kirkmcd.princeton.edu/examples/EP/goeppert-mayer_pr_48_512_35.pdf.

[2] S.R. Elliott *et al.*, *Direct Evidence for Two-Neutrino Double-Beta Decay in* ^{82}*Se*, Phys. Rev. Lett. **59**, 2020 (1987), http://kirkmcd.princeton.edu/examples/EP/elliott_prl_59_2020_87.pdf.

[3] E. Majorana, *Teoria simmetrica dell'elettrone e del positrone*, Nuovo Cim. **14**, 171 (1937),
http://kirkmcd.princeton.edu/examples/EP/majorana_nc_14_171_37.pdf.
http://kirkmcd.princeton.edu/examples/EP/majorana_nc_14_171_37_english.pdf.
F. Wilczek, *Majorana Returns*, Nature Phys. **5**, 614 (2009),
http://kirkmcd.princeton.edu/examples/neutrinos/wilczek_np_5_614_09.pdf.

[4] Massless Dirac fermions are sometimes called Weyl fermions, and the theory of massless Dirac neutrino is often called the "two component" theory, following H. Weyl, *Elektron und Gravitation. I*, Z. Phys. **56**, 330 (1929), http://kirkmcd.princeton.edu/examples/EP/weyl_zp_56_330_29.pdf.
Gravitation and the Electron, Proc. Nat. Acad. Sci. **15**, 323 (1929),
http://kirkmcd.princeton.edu/examples/GR/weyl_pnas_15_323_29.pdf.

[5] W.H. Furry, *On Transition Probabilities in Double Beta-Disintegration*, Phys. Rev. **56**, 1184 (1939),
http://kirkmcd.princeton.edu/examples/EP/furry_pr_56_1184_39.pdf.

To date, no neutrinoless double-beta decay has been observed, with limits on the lifetime ($> 10^{25}$ yr) being longer than that for observed 2-neutrino decays.[6]

Draw a Feynman diagram for neutrinoless double-beta decay, as due to the decay of two d-quarks. Include arrows on the spin-1/2 particle lines indicating the nominal helicities expected for the V-A model of the weak interaction (with $1-\gamma_5$ coupling).[7] Then, recalling the spin-1/2 propagator (p. 28, Lecture 3 of the Notes), the matrix element includes a factor (between the final-state electron spinors),

$$(1-\gamma_5)\frac{q^\mu\gamma_\mu - m_\nu}{q^2 - m_\nu^2}(1-\gamma_5). \tag{1}$$

Show that this reduces to

$$\frac{-2m_\nu}{q^2 - m_\nu^2}(1-\gamma_5) \approx \frac{-2m_\nu}{q^2}(1-\gamma_5). \tag{2}$$

Thus, the matrix element is proportional to the mass m_ν of the neutrino, and the decay rate is proportional to the square of the mass, so the rate vanishes if the neutrino were massless.[8]

In neutrinoless double-beta decay the four momentum q^μ of the virtual neutrino is small, but we expect that $q^2 \gg m_\nu^2$ while also $q^2 \ll m_e^2$. This suggests that the rate for neutrinoless double-beta decay is suppressed relative to that for 2-neutrino double-beta decay by a factor $(m_\nu/E)^2$ where E is very roughly of order 1 keV. This suppression may be sufficient to overcompensate for the larger phase space for neutrinoless double-beta decay, and make its rate smaller than that for 2-neutrino double-beta decay.

In searches for neutrinoless double-beta decay (negative to date), a factor of two improvement in the limit on the neutrino mass requires a factor of four improvement in the limit on the lifetime, *etc.*

While Majorana neutrinos could lead to neutrinoless double-beta decay, this need not be the only explanation for the nonconservation of lepton number in the decay. If neutrinoless double-beta decay were detected, additional effort would still be required to identify its cause.

The virtual neutrino in neutrinoless double-beta decay can be a mixture of the three mass eigenstates ν_i. This implies that the m_ν written above is actually $\sum_i U_{ei}^2 m_i$, where $U_{\alpha i}$ is the CKMNS mixing matrix for neutrinos (p. 356, Lecture 19 of the Notes). Hence, it may be that neutrinoless double-beta decay is further suppressed by the numerical values of the neutrino-mixing matrix.

[6]See, for example, R. Henning, *Current status of neutrinoless double-beta decay searches*, Rep. Phys. **1**, 29 (2016), http://kirkmcd.princeton.edu/examples/neutrinos/henning_rp_1_29_16.pdf.

[7]R.P. Feynman and M. Gell-Mann, *Theory of the Fermi Interaction*, Phys. Rev. **109**, 193 (1958), http://kirkmcd.princeton.edu/examples/EP/feynman_pr_109_193_58.pdf.

[8]This result was perhaps first obtained, by a different argument, on p. 314 of K.M. Case, *Reformulation of the Majorana Theory of the Neutrino*, Phys. Rev. **107**, 307 (1957), some 9 months before the emergence of the V-A theory, http://kirkmcd.princeton.edu/examples/neutrinos/case_pr_107_307_57.pdf.

2. **Helicity of Neutrinos.** (M. Goldhaber *et al.*, Phys. Rev. **109**, 1015 (1958), http://kirkmcd.princeton.edu/examples/EP/goldhaber_pr_109_1015_58.pdf; see also p. 1423 ff in *Alpha-, Beta-, and Gamma-Ray Spectroscopy*, ed. by K. Siegbahn.)

   Suppose a spin-0 nucleus A decays via electron capture, $A + e^- \to B^\star + \nu_e$, to a spin-1 excited state $B^\star$ of nucleus B along with emission of a neutrino (Gamow-Teller transition). Then, suppose the spin-1 state $B^\star$ decays via emission of an electric-dipole photon to the spin-0 ground state of B. We wish to infer the helicity of the neutrino by observation of the angular distribution of the photon. To do this, the direction of the neutrino in the laboratory must be singled out by some feature of the apparatus. This is possible by an ingenious argument of Goldhaber *et al.*

   Take the z axis along the direction of the neutrino in the decay $A \to B^\star$, such that the neutrino has $S_z^{(\nu)} = \pm\frac{1}{2}$ (with $S_z^{(\nu)} = -\frac{1}{2}$ in the $V - A$ theory of the weak interaction, which is to be confirmed). The nucleus $B^\star$ could have $S_z^{(B^\star)} = -1,\ 0$ or 1, leading to J_z for the final state of $-\frac{3}{2},\ -\frac{1}{2},\ \frac{1}{2}$ or $\frac{3}{2}$. (Recall that $L_z = 0$ for any two-body state where the z axis is along the direction of momentum in the rest frame.) But, the initial state has only spin $\frac{1}{2}$ from the electron, assuming electron capture from an S-wave orbital (K capture), so if $S_z^{(\nu)} = -\frac{1}{2}$ only $S_z^{(B^\star)} = 0$ or 1 are possible for nucleus $B^\star$, while if $S_z^{(\nu)} = \frac{1}{2}$ only $S_z^{(B^\star)} = -1$ or 0 are possible

   Let θ' be the angle of emission of the photon with respect to the $-z$ axis in the rest frame of state $B^\star$. (In the lab frame, $B^\star$ moves along the $-z$ axis.) Use the spin-1 rotation matrix, http://pdg.lbl.gov/2015/reviews/rpp2015-rev-clebsch-gordan-coefs.pdf, to show that the angular distribution of the $E1$ photons is $\sin^2\theta'$ if $S_z^{(B^\star)} = 0$, and $(1+\cos^2\theta')/2$ when $S_z^{(B^\star)} = 1$. Remember that the photon can only have $S_{z'}^{(\gamma)} = \pm 1$ along the z' axis, which is along the photon's direction in the $B^\star$ rest frame.

   In particular, show that photons emitted at $\theta' = 0$ or 180° can only have $S_z^{(\gamma)} = +1$ if $S_z^{(\nu)} = -\frac{1}{2}$, and only $S_z^{(\gamma)} = -1$ if $S_z^{(\nu)} = \frac{1}{2}$.

   Let E_K be the energy of the neutrino emitted in the (K-capture) decay of A, and E_0 be the excitation energy of $B^\star$ with respect to ground state B ($M_{B^\star} = M_B + E_0 \gg E_0,\ E_K$). Deduce the energy E'_γ of the photon in the rest frame of the $B^\star$, and transform this to the lab frame, expressing E_γ as a function of $\theta',\ E_K$ and E_0. What are the minimum and maximum values for E_γ? You should find that the highest photon energy occurs for $\theta' = 0$, for which these photons have $S_z = 1$ (and negative helicity as these photons are moving along the $-z$ axis) if the neutrino has negative helicity; and that if the neutrino had positive helicity, the highest-energy photons would have positive helicity also.

   So if we can measure the helicity of the highest-energy photons, we determine the helicity of the neutrino.

   The helicity of photons can be determined by passing them through a filter consisting of magnetized iron, which attenuates photons of +1 and −1 helicity by different amounts. The reaction here is just Compton scattering of polarized electrons and photons. Because the electron has spin $\frac{1}{2}$, an electron can only absorb a photon whose spin is opposite, which flips the spin of the intermediate electron prior to the radiation (scattering) of the final photon.

We want to determine the helicity of only the highest-energy photons, so a final trick is needed. Suppose the photons from the $B^\star$ decay impinge upon other ground-state B nuclei that are at rest in the lab. Calculate the energy of the photons such that a nucleus B at rest can be excited to the level $B^\star$. The latter states decay back to the ground state B by photon emission, in effect scattering only a certain subset of the photons from the first $B^\star$ decay into the detector.

For the historical experiment, $A = \mathrm{Eu}^{152}$, and $B = \mathrm{Sm}^{152}$, for which $E_K = 840$ keV, while $E_0 = 961$ keV. Due to recoil effects, you should have found that even the highest-energy photons from the first $B^\star$ decay have insufficient energy to re-excite B nuclei at rest. However, the lifetime of the spin-1 Sm^{152} excited state was measured to be 7×10^{-14} s. Convert this to a width in eV. What fraction of the Breit-Wigner mass/energy distribution (recall p. 11, Lecture 1 of the Notes) of the short-lived $B^\star$ in the reaction $\gamma + B \to B^\star$ lies with the energy distribution of the photons from the first $B^\star \to B + \gamma$ decay (assuming that distribution to be flat between its min and max energies)?

If the lifetime of the $B^\star$ level had been too long, the overlap would be too small for the experiment to work. Also, if the lifetime were long, the $B^\star$ atom might have collided with another atom and changed its momentum prior to the photon decay. Then, the correlation between the decay-photon helicity and the neutrino helicity would have been lost.

So it's a small miracle that any system exists in nature which permits this measurement!

Helicity of Neutrinos*

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(Received December 11, 1957)

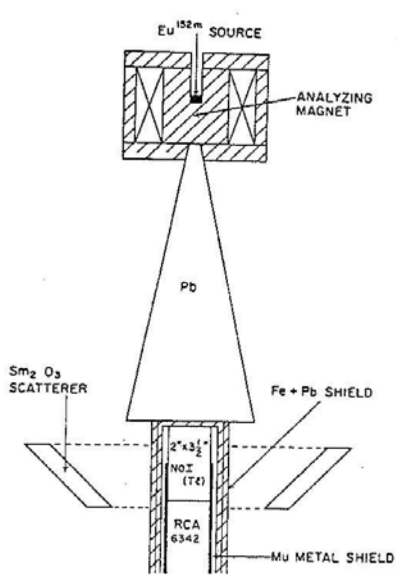


Fig. 6. Schematical arrangement of neutrino helicity experiment. (From Goldhaber *et al.*[1])

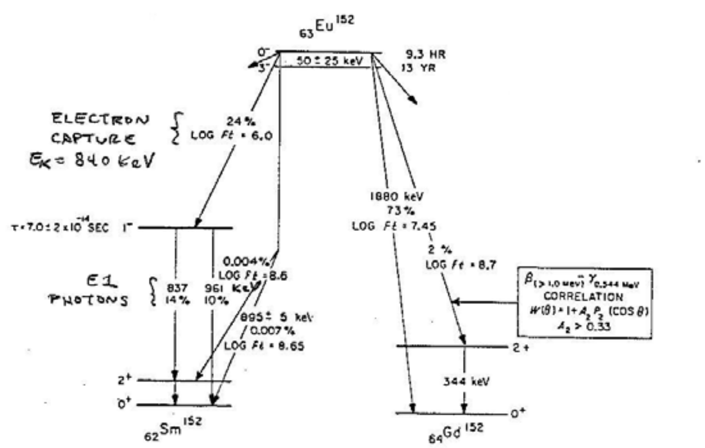


Fig. 4. Partial decay scheme of Eu^{152m}

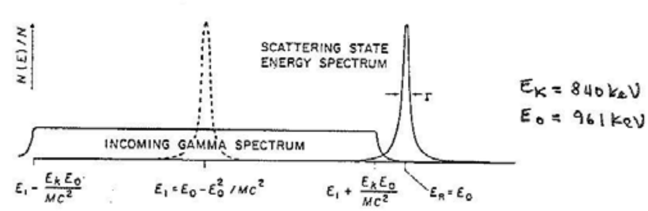


Fig. 5. Schematic diagram of incoming photon spectrum and resonance level width. (Not to scale)

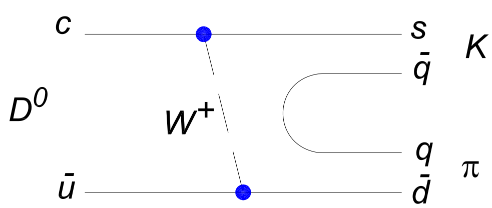
3. **Spurions**

   The spurion concept[9] is that in weak decays of an s quark the hadronic isospin character of the final state can be predicted by supposing the s quark absorbs a fictitious particle, the spurion, that is an isospin state $|I = \frac{1}{2}, I_3 = -\frac{1}{2}\rangle$ (and that an $\overline{s}$ quark absorbs an antispurion that is an isospin state $|I = \frac{1}{2}, I_3 = \frac{1}{2}\rangle$). Use this model to predict

$$\frac{\Gamma_{\Xi^0 \to \Lambda\pi^0}}{\Gamma_{\Xi^- \to \Lambda\pi^-}}, \qquad \text{and} \qquad \frac{\Gamma_{\Omega^- \to \Xi^0\pi^-}}{\Gamma_{\Omega^- \to \Xi^-\pi^0}}. \tag{3}$$

   Compare with data reported at http://pdg.lbl.gov/2013/tables/contents_tables_baryons.html.

4. The charmed meson D^0 can decay to $K\pi$ via the Cabibbo-favored W-exchange diagram (with gluons not shown),



   If this were the only possible diagram, predict the ratio of branching ratios:

$$\frac{\Gamma(D^0 \to K^-\pi^+)}{\Gamma(D^0 \to \overline{K}^0\pi^0)}. \tag{4}$$

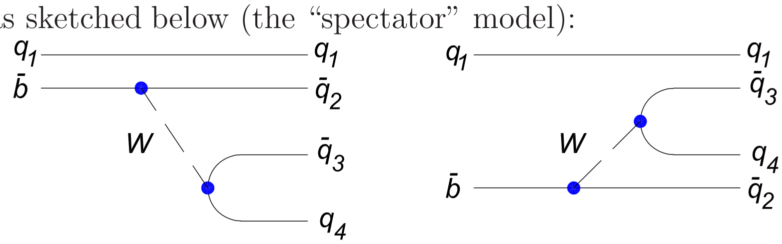
   Draw any other Cabibbo-favored diagrams for these decays.

   Assuming that the decay $D^0 \to K^+\pi^-$ proceeds via a diagram similar to the above, predict the ratio of branching ratios:

$$\frac{\Gamma(D^0 \to K^+\pi^-)}{\Gamma(D^0 \to K^-\pi^-)}. \tag{5}$$

   Compare with data reported at http://pdg.lbl.gov/2013/tables/rpp2013-tab-mesons-charm.pdf.

5. There are four spin-0 mesons that contain one bottom quark: $B_u^+ = u\bar{b}$, $B_d^0 = d\bar{b}$, $B_s^0 = s\bar{b}$, and $B_c^+ = c\bar{b}$. These decay via the weak interaction by two graphs with roughly equal strength, as sketched below (the "spectator" model):



   Here we consider only nonleptonic final states. Suppose the four final-state quarks form exactly two mesons (as happens a few percent of the time). List the two dominant two-body decays for each of the four bottom mesons.

[9] Attributed to G. Wentzel, *Heavy-Meson Decays and the Selection Rule $\Delta I \le 1/2$*, Phys. Rev. **101**, 1215 (1956), http://kirkmcd.princeton.edu/examples/EP/wentzel_pr_101_1215_56.pdf.

A complication arises for the B_c meson. The charm quark has a slightly shorter lifetime than the bottom quark. Hence, there are two more prominent two-body decays of the B_c involving $c \to Wq$ rather than $b \to Wq$ transitions. List these.

According to the measured values of the C-K-M matrix elements

$$\frac{V_{ub}}{V_{cb}} \approx \frac{V_{us}}{V_{ud}} \approx \frac{V_{cd}}{V_{cs}} \approx \lambda = \text{Cabibbo angle}.$$

List (or indicate on diagrams) the prominent two-body nonleptonic decays of the four bottom mesons that are suppressed by one power of λ in the matrix element (and hence by $\lambda^2 \approx 1/25$ in rate).

Note that $D^+ = c\bar{d}$, $D^0 = c\bar{u}$, and $D_s^0 = c\bar{s}$. If a meson is produced from, say, a $d\bar{d}$ state it could be a π^0, η, ρ^0, or ω^0. Here it is sufficient to list only the ρ^0 as empirically ρ produced seems favored...

Compare with data reported at `http://pdg.lbl.gov/2013/tables/rpp2013-tab-mesons-bottom.pdf`.

6. Both the $B_d^0 = d\bar{b}$ and $\bar{B}_d^0 = \bar{d}b$ can decay to common final states, such as $J/\psi K_S^0$ as you found in Prob. 5. Hence, there are transitions between B^0 and $\bar{B}^0$ and so these are not the states of definite mass and lifetime, which latter states can be written as

$$B_\pm^0 = pB^0 \pm q\overline{B}^0, \qquad \text{where} \qquad |p|^2 + |q|^2 = 1. \tag{6}$$

Taking into account the weak interactions, one writes the 2×2 Hamiltonian (in the $|B^0\rangle$-$|\overline{B}^0\rangle$ basis) as

$$\mathsf{H} = \mathsf{M} - \frac{i}{2}\mathsf{\Gamma}, \tag{7}$$

where the mass matrix M and the decay matrix $\mathsf{\Gamma}$ are Hermitian. (Since neutral B's decay, H itself is not Hermitian.) CPT invariance implies that the diagonal components of H are equal, and if CP is conserved M and $\mathsf{\Gamma}$ are real. Allowing for the possibility of CP violation, the Hamiltonian can be written as

$$\mathsf{H} = \begin{bmatrix} m & M_{12} \\ M_{12}^* & m \end{bmatrix} - \frac{i}{2}\begin{bmatrix} \gamma & \Gamma_{12} \\ \Gamma_{12}^* & \gamma \end{bmatrix}. \tag{8}$$

Unlike the eigenstates K_L^0 and K_S^0 of the neutral K system, the eigenstates $B_\pm^0$ of the neutral B system have roughly equal lifetimes, such that we can approximate matrix element Γ_{12} as zero.

Show that this implies that q/p is a pure phase (with magnitude 1), so that

$$B_\pm^0 \approx e^{i\phi_\pm}\frac{B^0 \pm \overline{B}^0}{\sqrt{2}}. \tag{9}$$

The time evolution of the eigenstates can then be written as $|B_\pm\rangle \to e^{-\Gamma t/2}e^{im_\pm t}|B_\pm\rangle$. Show that the time evolution of initially pure B^0 and $\bar{B}^0$ states can be written as

$$|B^0\rangle \to F_c(t)|B^0\rangle + i\frac{q}{p}F_s(t)|\overline{B}^0\rangle, \qquad |\overline{B}^0\rangle \to i\frac{p}{q}F_s(t)|B^0\rangle + F_c(t)|\overline{B}^0\rangle, \tag{10}$$

where F_c and F_s are proportional to the cosine and sine of $\Delta m\, t/2$, respectively, with $\Delta m \equiv m_+ - m_-$.

If a B^0-$\overline{B}^0$ pair is produced at an electron-positron collider via the reaction $e^+e^- \to B^0\overline{B}^0$, it is produced in the entangled[10] state $(|B_1^0\rangle|\overline{B}_2^0\rangle - |\overline{B}_1^0\rangle|B_2^0\rangle)/\sqrt{2}$ with negative charge conjugation. Later, B_1 decays at time t_1 and B_2 decays at time t_2 (where t_1 can be larger than t_2 if we suppose that B_1 travels to the "left" and B_2 travels to the "right"). Consider the time evolution of the entangled state from (0,0) to (t_1, t_2) to show if the two $B's$ happened to decay at the same time $t_1 = t_2$ then one must be a $\overline{B}^0$ and the other a B^0 (although if $t_1 \neq t_2$ they have nonzero amplitudes to be both B^0's or both $\overline{B}^0$'s).[11]

Such persistence of the initial quantum correlation for spacelike-separated events puzzled Einstein, Podolsky and Rosen,[12] and remains ever impressive.[13]

A subtle consequence of this "EPR" quantum correlation is that if the lab frame of the e^+e^- is the same as the center-of-mass frame, as typically the case, CP violation could not be detected in certain decay modes. To make such decay modes useful for the study of CP violation, so-called asymmetric B-factories were built in which the center of mass is in motion in the lab frame.

[10] The term entangled was introduced in a quantum context by E. Schrödinger (1935) in sec. 14 of his famous "cat" paper, http://kirkmcd.princeton.edu/examples/QM/schroedinger_cat.pdf. Schrödinger lived with two women, one of whom was his legal wife. The concept of quantum-mechanical entanglement first appeared in J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, 1932), *Mathematical Foundations of Quantum Mechanics*, (Princeton U. Press, 1955). For commentary by the author, see Prob. 5 of http://kirkmcd.princeton.edu/examples/ph410problems.pdf.

[11] This effect also occurs for K^0-$\overline{K}^0$ produced in the reaction $e^+e^- \to \phi \to K^0\overline{K}^0$, as discussed by H.J. Lipkin, *CP Violation and Coherent Decays of Kaon Pairs*, Phys. Rev. **176**, 1715 (1968), http://kirkmcd.princeton.edu/examples/EP/lipkin_pr_176_1715_68.pdf.

[12] A. Einstein, B. Podolsky and N. Rosen, *Can Quantum Mechanical Description of Physical Reality Be Considered Complete?* Phys. Rev. **47**, 777 (1935), http://kirkmcd.princeton.edu/examples/QM/einstein_pr_47_777_35.pdf.

[13] Some people say things like the decay of one B instantaneously forces the "other" B to be its antiparticle, and the "other" B then subsequently evolves according to the single-B prescription (10). This is not a good way to think about the nature of quantum correlations in entangled states.

Solutions

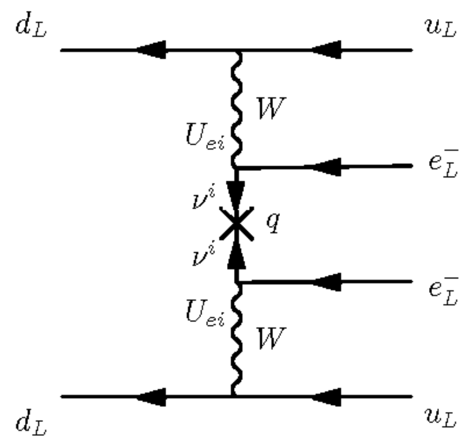
1. a. **2-Neutrino Double-Beta Decay**

A neutron has radius $r \approx 1$ fermi, so the time scale for collisions of quarks inside it is $r/c \approx 3 \times 10^{-23}$ s. The neutron lifetime is about 10^3 s, so the probability of a weak interaction occurring during each collision is about 3×10^{-26}.

The probability of two such weak interactions occuring in a nucleus within a single collision time is the square of this, and hence the lifetime for double-beta decay is about 3×10^{25} times the neutron lifetime, *i.e.*, $\approx 3 \times 10^{28}$ s. Recalling that a year contains about $\pi \times 10^7$ s, we estimate that the lifetime against double-beta decay is 10^{21} years (which agrees with experiment to within an order of magnitude).

b. **Neutrinoless Double-Beta Decay**

A Feynman diagram for neutrinoless double-beta decay, $A \to A'ee$ from two d quarks of neutrons in the nucleus A, is, in the V-A theory of the weak interaction,



As such, the matrix element includes a factor, related to the exchanged virtual neutrino ν_i, of

$$\sum_i U_{ei}^2 (1 - \gamma_5) \frac{q^\mu \gamma_\mu - m_i}{q^2 - m_i^2} (1 - \gamma_5), \tag{11}$$

recalling the form of the spin-1/2 propagator mentioned on p. 28, Lecture 3 of the Notes. Here, we consider 3 neutrino mass eigenstates ν_i, which are related to the neutrino flavor states ν_α via the 3-neutrino CKMNS matrix $U_{\alpha i}$. The factors $1-\gamma_5$ indicate that the virtual neutrino couples to lefthanded (chirality) electrons, e_L^-.

In eq. (11), the term in the numerator involving the neutrino mass m_i is,

$$(1 - \gamma_5) m_i (1 - \gamma_5) = m_i (1 - 2\gamma_5 + \gamma_5^2) = 2 m_i (1 - \gamma_5), \tag{12}$$

recalling from p. 116, Lecture 7 of the Notes that $\gamma_5^2 = 1$.

The other term in the numerator of eq. (11) includes the product

$$(1 - \gamma_5) \gamma_\mu (1 - \gamma_5) = \gamma_\mu - \gamma_5 \gamma_\mu - \gamma_\mu \gamma_5 + \gamma_5 \gamma_\mu \gamma_5 = \gamma_\mu - \gamma_5^2 \gamma_\mu = 0, \tag{13}$$

recalling from p. 116, Lecture 7 of the Notes that $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$. Hence, the factor (11) reduces to

$$-2\sum_i \frac{U_{ei}^2 m_i}{q^2 - m_i^2}(1-\gamma_5) \approx -2\sum_i \frac{U_{ei}^2 m_i}{q^2}(1-\gamma_5)\,, \tag{14}$$

with the implication that the rate for neutrinoless double-beta decay varies as the square of the effective neutrino mass $\sum_i U_{ei}^2 m_i$.

If we interpret the lower beta-decay as involving emission of the virtual neutrino, then in the V-A theory this is a righthanded (chirality) Dirac antineutrino. On the other hand, in the upper beta-decay, the virtual neutrino is absorbed, and must be a lefthanded Dirac neutrino. Since a Dirac antineutrino is distinct from a Dirac neutrino, the neutrinoless double-beta decay is not possible for Dirac neutrinos. But it is possible for Majorana neutrinos, for which a lefthanded "neutrino" is a 50-50 combination of a lefthanded Dirac neutrino and a righthanded Dirac antineutrino (as discussed in the Digression below).

It remains that the arrowheads on the two ends of the virtual-neutrino line point in opposite directions in the diagram. If we interpret the arrowheads as indicating the needed helicity, then the diagram indicates that the antineutrino is emitted as chirality state v_R in the lower beta-decay with nominally positive helicity, but must be absorbed in the upper beta-decay (as a neutrino in chirality state u_L) with nominally negative helicity.

An argument is sometimes made[14] that a righthanded chirality antiparticle state, v_R, has a small relative amplitude, $\approx m/2E$, to have negative helicity, rather than the nominal positive helicity (recall eq. (70), Prob. 2, Set 4). This suggests that there is an amplitude $\approx m_i/2q_0$ that the arrowhead for the lower neutrino in the diagram actually points downward, which implies that the amplitude for neutrinoless double-beta decay varies as neutrino mass, so that the rate varies as the square of the neutrino mass.

However, as discussed in the Digressions below, a Majorana neutrino from a beta-decay is a 50-50 mixture of a lefthanded Dirac neutrino and a righthanded Dirac antineutrino, so there is no suppression of the type speculated above. The author's view is that it's better to stick with to the discussion based on the form of the spin-1/2 propagator.

Digression: Majorana 4-Spinors.[15]

We saw in the digression on p. 5, Prob. 2, Set 4 that the charge-conjugation transformation $\tilde\psi = i\gamma_2\psi^*$ of a 4-spinor state ψ leads to its antiparticle state $\tilde\psi$.

[14]See, for example, p. 17 of B. Kayser, *Neutrino Mass, Mixing and Flavor Change* (Feb. 2002), https://arxiv.org/abs/hep-ph/0211134.
This argument may have been inspired by Case, footnote 8.

[15]This digression follows Prob. 7.51 of Griffiths' text:
http://kirkmcd.princeton.edu/examples/EP/griffiths_particles_08.pdf
http://kirkmcd.princeton.edu/examples/EP/griffiths_particles_08_solutions.pdf.

Majorana spinors (when constructed from spinors that obey the Dirac equation) are their own antiparticles, which suggests that they are combinations of Dirac particles u and antiparticles v. So, recalling eqs. (17)-(18) of Set 4, we consider a general Majorana spinor of the form,

$$\frac{\psi}{\sqrt{E_+}} = au + bv = a\, e^{-ipx} \begin{pmatrix} \chi \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\chi \end{pmatrix} + b\, e^{ipx} \begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\tilde{\chi} \\ \tilde{\chi} \end{pmatrix}, \tag{15}$$

where χ and $\tilde{\chi} = -i\sigma_2\chi^*$ ($\chi = i\sigma_2\tilde{\chi}^*$, from eq. (28), Prob. 2, Set 4) are 2-spinors with unit normalization, $E_+ = E + m$, and $|a|^2 + |b|^2 = 1$, so that $\bar{\psi}\psi = 2m$. Then, the requirement that this state be its own antiparticle, $\psi = \tilde{\psi} = i\gamma_2\psi^*$, implies

$$\begin{aligned}
\frac{\psi}{\sqrt{E_+}} &= \frac{\tilde{\psi}}{\sqrt{E_+}} = \tilde{au} + \tilde{bv} = a^*\tilde{u} + b^*\tilde{v} \\
&= \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \left[a^*\, e^{ipx} \begin{pmatrix} \chi^* \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}^*}{E+m}\chi^* \end{pmatrix} + b^*\, e^{-ipx} \begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}^*}{E+m}\tilde{\chi}^* \\ \tilde{\chi}^* \end{pmatrix} \right] \\
&= a^*\, e^{ipx} \begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}(-i\sigma_2\chi^*) \\ -i\sigma_2\chi^* \end{pmatrix} + b^*\, e^{-ipx} \begin{pmatrix} i\sigma_2\tilde{\chi}^* \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}(i\sigma_2\tilde{\chi}^*) \end{pmatrix}
\end{aligned} \tag{16}$$

recalling that $\sigma_2\boldsymbol{\sigma}^* = -\boldsymbol{\sigma}\sigma_2$. Hence, $b = a^*$ $(= 1/\sqrt{2})$, and $v = \tilde{u}$ $(u = \tilde{v})$, such that Majorana 4-spinors have only 2 independent components (and always contain both a Dirac-particle and -antiparticle spinor).[16] A consequence is that the top and bottom 2-spinors, χ_t and χ_b, of a Majorana 4-spinor state ψ are related by

$$\frac{\psi}{\sqrt{E_+}} = \begin{pmatrix} \chi_t \\ \chi_b \end{pmatrix}, \qquad \chi_b = -i\sigma_2\chi_t^*, \qquad \chi_t = i\sigma_2\chi_b^*. \tag{17}$$

First, we consider the spin-up/down helicity 2-spinors $\chi_\pm$ (eq. (46) of Set 4),

$$\chi_+ = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}, \qquad \chi_- = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\phi/2} \\ \cos\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}, \tag{18}$$

for which $\tilde{\chi}_\pm = -i\sigma_2\chi_\pm^* = \pm\chi_\mp$ (as found in eqs. (29)-(30) of Set 4). Taking $a = b = 1/\sqrt{2}$, the **Majorana helicity 4-spinors** $\psi_\pm$ are their own antiparticles, $\psi_\pm = \tilde{\psi}_\pm$, with

$$\frac{\psi_\pm}{\sqrt{E_+/2}} = e^{-ipx} \begin{pmatrix} \chi_\pm \\ \pm\frac{p}{E+m}\chi_\pm \end{pmatrix} + e^{ipx} \begin{pmatrix} -\frac{p}{E+m}\chi_\mp \\ \pm\chi_\mp \end{pmatrix}. \tag{19}$$

[16]The Majorana two-component theory applies to particle with mass (but without electric charge), unlike the Weyl two-component theory of massless fermions (pp. 14-15 of Set 4), and footnote 4 above.

Majorana chirality 4-spinors.

Following Case, footnote 8, we define these states, for a Majorana state $\psi = \tilde{\psi}$, as

$$\psi_{R,L} = \frac{1 \pm \gamma_5}{2}\psi. \tag{20}$$

However, these are not quite their own antiparticles. Rather (recalling that $\gamma_\mu\gamma_5 = -\gamma_5\gamma_\mu$),

$$\tilde{\psi}_{R,L} = i\gamma_2\psi^*_{R,L} = i\gamma_2\frac{1 \pm \gamma_5}{2}\psi^* = \frac{1 \mp \gamma_5}{2}(i\gamma_2\psi^*) = \frac{1 \mp \gamma_5}{2}\psi = \psi_{L,R}. \tag{21}$$

Applying the chirality projection operators $(1 \pm \gamma_5)/2$ to the Dirac equation for a Majorana state ψ yields the coupled equations,

$$\frac{1 \pm \gamma_5}{2}i\partial^\mu\gamma_\mu\psi = i\partial^\mu\gamma_\mu\frac{1 \mp \gamma_5}{2}\psi = \frac{1 \pm \gamma_5}{2}m\,\psi, \qquad i\partial^\mu\gamma_\mu\psi_{L,R} = m\,\psi_{R,L}, \tag{22}$$

while their sum $\psi = \psi_R + \psi_L$ does satisfy the Dirac equation $i\partial^\mu\gamma_\mu\psi = m\,\psi$. It remains that Majorana's requirement is that states which obey the Dirac equation are their own antiparticles.

Also (recalling eq. (15)), the righthanded Majorana spinor ψ_R consists of a righthanded Dirac-particle spinor u_R together with a lefthanded Dirac-antiparticle spinor v_L. That is, if we write a Majorana spinor (15) as $\psi \propto u + v$, where the Dirac spinors u and v are related by $v = \tilde{u}$ as seen above (and so have equal amplitudes), then the corresponding Majorana chirality spinors would be

$$\psi_{R,L} = \frac{1 \pm \gamma_5}{2}\psi = \frac{u_{R,L} + v_{L,R}}{\sqrt{2}} = \frac{\sqrt{E_+}}{2\sqrt{2}}\begin{pmatrix} \left(1 \pm \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\right)(e^{-ipx}\,\chi \pm e^{ipx}\,\tilde{\chi}) \\ \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \pm 1\right)(e^{-ipx}\,\chi \pm e^{ipx}\,\tilde{\chi}) \end{pmatrix}, \tag{23}$$

recalling the states $u_{R,L}$ and $v_{R,L}$ displayed in eq. (77) of Set 4. Then, u_R and v_L have equal amplitudes in ψ_R, and likewise for u_L and v_R in ψ_L.[17] According to eq. (86) of Set 4, $\tilde{u}_{R,L} = v_{R,L}$ (and $\tilde{v}_{L,R} = u_{L,R}$), such that indeed

$$\tilde{\psi}_{R,L} \propto \tilde{u}_{R,L} + \tilde{v}_{L,R} = v_{R,L} + u_{L,R} = \psi_{L,R}, \tag{25}$$

as in eq. (21).

In a beta-decay $n \to p\,e^-\,\nu$ that produces a Majorana neutrino, the V-A interaction, with its $1-\gamma_5$ coupling, produces a lefthanded Majorana neutrino ψ_L that has equal amplitudes to be a lefthanded Dirac-particle spinor u_L and a righthanded Dirac-antiparticle spinor v_R as in eq. (23).[18] Similarly, in the reaction $\nu\,n \to p\,e^-$,

[17] Recalling eqs. (82) and (85) of Set 4, the Majorana chirality spinors obey

$$\bar{\psi}_R\psi_R = \bar{\psi}_L\psi_L = 0, \qquad \bar{\psi}_R\psi_L = \bar{\psi}_L\psi_R = m. \tag{24}$$

[18] Since a righthanded Majorana neutrino, ψ_R, has no weak, electromagnetic or strong coupling, it is a "sterile" neutrino, as is also the case for a righthanded Dirac neutrino.

the Majorana neutrino needs to be lefthanded. Hence, if the virtual neutrino in a "neutrinoless" double-beta decay is a Majorana state, the wavefunction of the neutrino created in the "first" beta-decay "matches" the wavefunction of the neutrino absorbed in the second beta-decay, and there is no suppression by a factor m/E as sometimes argued (p. 9, above).

In an analysis via 4-spinors, the factor of m in the matrix element for neutrinoless double-beta decay comes from the spin-1/2 propagator of the virtual neutrino, and not from a supposed mismatch between the wavefunctions of the virtual neutrino as created and absorbed.

Digression. Another convention for Majorana chirality 4-spinors.

In a different convention, Majorana chirality states are their own antiparticles. These could be obtained from eq. (20) by symmetrizing with respect to their antiparticles states,

$$\psi_{R,L}^{(2)} = \frac{\psi_{R,L} + \tilde{\psi}_{R,L}}{\sqrt{2}}. \tag{26}$$

Such states will contain all four of 4-spinors u_R, u_L, v_R and v_L, and so are not well aligned with the $V - A$ theory of the weak interaction, in which only spinors u_L and v_R participate.

For completeness, we return to consideration of a Dirac particle 4-spinor u and its antiparticle $v = \tilde{u}$, for which the Dirac chirality 4-spinors are,

$$u_{R,L} = \frac{1 \pm \gamma_5}{2} u, \qquad v_{L,R} = \frac{1 \pm \gamma_5}{2} v = \tilde{u}_{L,R}. \tag{27}$$

Then for the general 4-spinor $\phi = au + bv$, where $|a|^2 + |b|^2 = 1$, we have

$$\phi_{R,L} = au_{R,L} + bv_{L,R}, \qquad \tilde{\phi}_{R,L} = a^* v_{R,L} + b^* u_{L,R}. \tag{28}$$

We now define the self-conjugate states[19]

$$\psi_{R,L}^{(2)} = \frac{\phi_{R,L} + \tilde{\phi}_{R,L}}{\sqrt{2}} = \frac{au_{R,L} + bv_{L,R} + a^* v_{R,L} + b^* u_{L,R}}{\sqrt{2}} \qquad \text{(Convention 2).} \tag{29}$$

The Dirac-type equations of motion for the $\psi_{R,L}^{(2)}$ are

$$\begin{aligned} i\partial^\mu \gamma_\mu \psi_{R,L}^{(2)} &= i\partial^\mu \gamma_\mu \frac{au_{R,L} + bv_{L,R} + a^* v_{R,L} + b^* u_{L,R}}{\sqrt{2}} \\ &= m \frac{au_{L,R} + bv_{R,L} + a^* v_{L,R} + b^* u_{R,L}}{\sqrt{2}} = m\psi_{L,R}^{(2)}, \end{aligned} \tag{30}$$

[19]Convention 2 follows, for example, T.P. Cheng and L.-F. Li, *Neutrino masses, mixings, and oscillations in SU(2) × U(1) models of electroweak interactions*, Phys. Rev. D **22**, 2860 (1980), http://kirkmcd.princeton.edu/examples/neutrinos/cheng_prd_22_2860_80.pdf, and L.F. Li and F. Wilczek, *Physical processes involving Majorana neutrinos*, Phys. Rev. D **25**, 143 (1982), http://kirkmcd.princeton.edu/examples/neutrinos/li_prd_25_143_82.pdf.

recalling eq. (75) of Set 4. That is, both the chirality states (20) and (29) obey the same coupled Dirac-type equations, (22) and (30). Since these equations can be deduced from the Lagrangian for noninteracting Majorana particles, that Lagrangian alone does not distinguish between the two conventions.

Note that if $a = b = 1/\sqrt{2}$, then $\psi_R^{(2)} = \psi_L^{(2)} = (u_R + u_L + v_R + v_L)/2$.

In the $V - A$ theory of the weak interaction, only 4-spinors u_L and v_R participate, so it seems that the Majorana states $\psi_{R,L}^{(2)}$ are not well "matched" to this theory, and we prefer to use the convention (20) for Majorana chirality states.

Digression: Majorana 2-Spinors.

As found in eq. (16), there are only two independent Majorana states, which implies that they can be represented by 2-spinors rather than 4-spinors.[20,21] For example, we could represent Majorana states by the 2-spinor $\phi = \sqrt{E}\,\chi_t$, using the top 2-spinor χ_t of a Majorana 4-spinor, as in eq. (17) and assuming that $\chi_t^\dagger \chi_t = 1$.

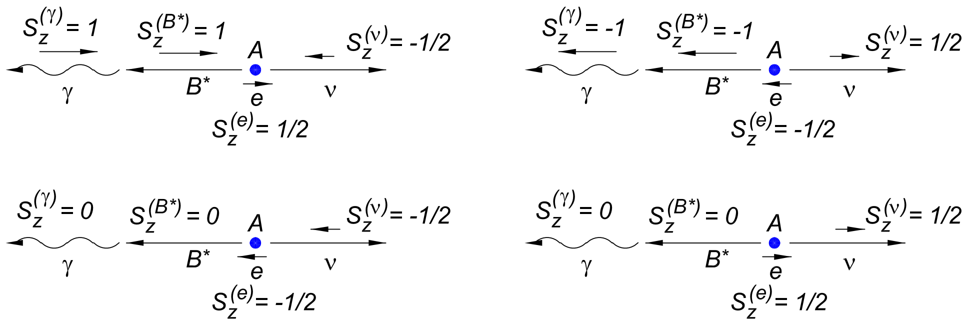
For what it's worth, the Majorana chirality 2-spinors $\phi_{R,L}$, obtained from the top 2-spinors of eq. (23), obey coupled Dirac-type equations, $i\partial^\mu \sigma_\mu \phi_{R,L} = m\phi_{L,R}$, where $\sigma_\mu = (I, \boldsymbol{\sigma})$. Note that the lefthanded Majorana 2-spinor ϕ_L is, to a first approximation, an equal mixture of a negative-helicity particle and a positive-helicity antiparticle.

2. **Helicity of Neutrinos**

In the E1 decay $B^\star \to B\gamma$ of a spin-1 excited state $B^\star$ of a spin-0 nucleus B, the final state can only have $S_{z'} = \pm 1$ along the z'-axis of the photon's momentum. The distribution of angles θ' of the photon with respect to some z-axis in the rest frame of the spin-1 $B^\star$ nucleus can be determined from the relevant components of the spin-1 rotation matrix. In particular, if the B* has $S_z = 0$, the amplitude to decay to a photon with $S_{z'} = \pm 1$ is $d^1_{\pm 1,0} = \mp \sin\theta'/\sqrt{2}$, and the angular distribution is $(d^1_{1,0})^2 + (d^1_{-1,0})^2 = \sin^2\theta'$. Similarly, if the B* has $S_z = 1$, the amplitude to decay to a photon with $S_{z'} = \pm 1$ is $d^1_{\pm 1,1} = (1 \pm \cos\theta')/2$, and the angular distribution is $(d^1_{1,1})^2 + (d^1_{-1,1})^2 = (1 + \cos^2\theta')/2$. Of course, if the B* has $S_z = -1$, the amplitude to decay to a photon with $S_{z'} = \pm 1$ is $d^1_{\pm 1,-1} = (1 \pm \cos\theta')/2$, and the angular distribution is $(d^1_{1,-1})^2 + (d^1_{-1,-1})^2 = (1 + \cos^2\theta')/2$ also.

[20] To discuss Majorana states in, say, beta-decay such as $n \to pe^-\overline{\nu}_e$, where the n, p and e^- are described by Dirac 4-spinors, it seems best to describe the antineutrino $\overline{\nu}_e$, possibly a Majorana state, as a 4-spinor as well.

[21] Massless neutrinos can be described as two-component Weyl fermions, which also can be represented by 2-spinors (although this is not very useful in calculations where the other spinors are 4-spinors).



The spin-1 nucleus $B^\star$ comes from electron capture, $A + e^- \to B^* + \nu_e$, of a spin-0 nucleus A, so S_z of the initial state is that of the electron. If the photon is emitted by the $B^\star$ opposite to the direction of the ν_e (or in the same direction as the ν_e), and the spins of the e and ν_e were aligned, then $S_z^{(B^\star)} = 0 = S_z^{(\gamma)}$, as shown in the figure below, which is impossible for a real photon. Rather, the direction of the photon and the neutrino can be parallel only if the spins of the e and ν_e are antiparallel, in which case the spins of the photon and neutrino are antiparallel.

Turning to the kinematics of the reactions, we start in the rest frame of the $B^\star$ nucleus, which decays to $B + \gamma$. In this frame the photon has energy $E'_\gamma = P'_\gamma = P'_B$, and the total energy of the B nucleus is

$$E'_B = \sqrt{M_B^2 + P_B^2} \approx M_B \frac{P_B^2}{2M_B} = M_B + \frac{E'^2_\gamma}{2M_B}\,, \tag{31}$$

and the photon energy E'_γ is related to the excitation energy $E_0 = M_{B^\star} - M_B$ by

$$E_0 = E'_\gamma + E'_B - M_B \approx E'_\gamma + \frac{E_0^2}{2M_B} \qquad E'_\gamma \approx E_0 - \frac{E_0^2}{2M_B}\,. \tag{32}$$

The energy-momentum 4-vector of the photon in the rest frame of the $B^\star$ is

$$E'_\gamma(1, \sin\theta', 0, -\cos\theta') \approx E_0(1, \sin\theta', 0, -\cos\theta') \tag{33}$$

in the convention that angle θ' is measured with respect to the $-z$ axis (and that the $B^\star$ moves along the $-z$ axis). To transform this energy to the lab frame, we note that $P_{B^\star} = P_\nu = E_\nu = E_K$, and the energy of the $B^\star$ is

$$E_{B^\star} = \sqrt{M_{B^\star}^2 + P_{B^\star}^2} \approx M_{B^\star}\left(1 + \frac{P_{B^\star}^2}{2M_{B^\star}^2}\right) = M_{B^\star}\left(1 + \frac{E_K^2}{2M_{B^\star}^2}\right) \approx M_{B^\star}, \tag{34}$$

so the velocity of the nucleus $B^\star$ in the lab frame (where it moves in the $-z$ direction) is related by

$$\beta_{B^\star} = \frac{-P_{B^\star}}{E_{B^\star}} = -\frac{P_\nu}{E_{B^\star}} = -\frac{E_\nu}{E_{B^\star}} = -\frac{E_K}{E_{B^\star}} \approx -\frac{E_K}{M_{B^\star}}, \qquad |\beta_{B^\star}| \ll 1. \tag{35}$$

Hence, the Lorentz transformation from the lab frame to the rest frame of the $B^\star$ has "boost" $\gamma = 1/\sqrt{1 - \beta_{B^\star}^2} \approx 1$, and the energy of the photon in the lab frame is

$$E_\gamma \approx E'_\gamma(1 - \beta_{B^\star}\cos\theta') \approx E_0 - \frac{E_0^2}{2M_{B^\star}} + \frac{E_K E_0}{M_{B^\star}}\cos\theta'. \tag{36}$$

The range of possible photon energies in the lab frame is therefore

$$E_0 - \frac{E_0^2}{2M_B} - \frac{E_K E_0}{M_{B^\star}} < E_\gamma < E_0 - \frac{E_0^2}{2M_B} + \frac{E_K E_0}{M_{B^\star}}\,, \tag{37}$$

as indicated in Fig. 5, p. 3 (where $E_1 = E'_\gamma$ and both M_B and $M_{B^\star}$ are written as M).

For the reaction $\gamma + B \to B^\star$ to proceed in the lab frame, energy-momentum conservation implies that

$$\begin{aligned} p_\gamma + p_B &= p_{B^\star}, \qquad p_\gamma^2 + 2p_\gamma \cdot p_B + p_B^2 = p_{B^\star}^2, \\ 2E_\gamma M_B + M_B^2 &= M_{B^\star}^2 = (M_B + E_0)^2, \qquad E_\gamma = E_0 + \frac{E_0^2}{2M_B}\,. \end{aligned} \tag{38}$$

To have $E_\gamma < E_{\gamma,\rm max}$ from eq. (37), we need

$$E_0 - \frac{E_0^2}{2M_B} + \frac{E_K E_0}{M_{B^\star}} > E_0 + \frac{E_0^2}{2M_B}\,, \qquad E_K > E_0 \frac{M_{B^\star}}{M_B} \approx E_0. \tag{39}$$

However, for $A = \mathrm{Eu}^{152}$ and $B = \mathrm{Sm}^{152}$ we have $E_K = 840$ keV, while $E_0 = 961$ keV, so the reaction could not proceed if the excited state $B^\star$ of Sm^{152} were long lived, and had a well-defined mass/energy.

But, since the lifetime of the spin-1 Sm^{152} excited state is $\tau = 7 \times 10^{-14}$ s, its decay width is

$$\Gamma = \frac{\hbar c}{c\tau} = \frac{200\ \text{MeV-fermi}}{3 \times 10^{23}\ \text{fermi/s} \cdot 7 \times 10^{-14}\ \text{s}} = 10^{-2}\ \text{eV}, \tag{40}$$

Hence, there is some probability that the mass energy of the short-lived excited nucleus is small enough that the reaction can proceed.

From p. 11 of the Notes,

$$P(E) \propto \frac{1}{(E - E_R)^2 + \Gamma^2/4}\,, \tag{41}$$

is the (Breit-Wigner) probability that the short-lived state with nominal (resonant) mass/energy E_R appears to have mass/energy E. In the present example, the central/resonant mass/energy E_R of the $B^\star$ in the reaction $\gamma + B \to B^\star$ is, from eq. (38),

$$E_R = E_0 + \frac{E_0^2}{2M_B} \approx E_0 + \frac{(961\ \text{keV})^2}{2 \cdot 152 \mathrm{M_p}} = E_0 + 3.2\ \text{eV}, \tag{42}$$

while the highest-energy photon from the decay in flight $B^\star \to B + \gamma$ has, from eq. (37),

$$\begin{aligned} E_{\gamma,\rm max} &= E_0 - \frac{E_0^2}{2M_B} + \frac{E_K E_0}{M_{B^\star}} \approx E_0 - 3.2\ \text{eV} + \frac{(840\ \text{kev})(961\ \text{keV})}{152 \cdot 940\ \text{MeV}} \\ &= E_0 - 3.2\ \text{eV} + 5.6\ \text{eV} = \mathrm{E_0} + 2.4\ \text{eV} = \mathrm{E_R} - 0.8\ \text{eV}. \end{aligned} \tag{43}$$

That is, the maximum photon energy $E_{\gamma,\rm max}$ available from the decay in flight, $B^\star \to B + \gamma$, is 0.8 eV below the central energy E_R for the reaction $B^\star \to B + \gamma$. We note that the minimum photon energy $E_{\gamma,\min}$ available from the decay in flight, $B^\star \to B+\gamma$, is

$$\begin{aligned} E_{\gamma,\min} &= E_0 - \frac{E_0^2}{2M_B} - \frac{E_K E_0}{M_{B^\star}} \approx E_0 - 3.2 \text{ eV} - 5.6 \text{ eV} = \text{E}_0 - 8.8 \text{ eV} \\ &= E_R - 12.0 \text{ eV}. \end{aligned} \tag{44}$$

Under the (good) assumption that the distribution of photons from the decay in flight, $B^\star \to B+\gamma$, is uniform in energy, the fraction f of the Breit-Wigner distribution (41) of the energies of the $B^\star$ in the subsequent reaction $\gamma + B \to B^\star$ that is overlapped by the decay photons is

$$\begin{aligned} f &= \int_{E_{\gamma,\min}}^{E_{\gamma,\max}} \frac{dE}{(E-E_R)^2+\Gamma^2/4} \Big/ \int_{-\infty}^{\infty} \frac{dE}{(E-E_R)^2+\Gamma^2/4} \\ &= \int_{-12 \text{ eV}}^{-0.8 \text{ eV}} \frac{dE}{E^2+\Gamma^2/4} \Big/ \int_{-\infty}^{\infty} \frac{dE}{E^2+\Gamma^2/4} \\ &= \frac{2}{\Gamma}\left(\tan^{-1}\frac{2\cdot -0.8 \text{ eV}}{\Gamma} - \tan^{-1}\frac{2\cdot -12.0 \text{ eV}}{\Gamma}\right) \Big/ \frac{2\pi}{\Gamma} \\ &= \frac{1}{\pi}\left(\tan^{-1}\frac{\Gamma}{1.6 \text{ eV}} - \tan^{-1}\frac{\Gamma}{24 \text{ eV}}\right) \approx \frac{\Gamma}{1.6\pi \text{ eV}} \approx \frac{10^{-2}}{1.6\pi} \approx 0.002. \end{aligned} \tag{45}$$

3. The Ξ^0 is an isospin state $|I = \frac{1}{2}, I_3 = \frac{1}{2}\rangle$ that contains an s quark, so in the spurion model it combines with the spurion state $|I = \frac{1}{2}, I_3 = -\frac{1}{2}\rangle$, leading to $(|1,0\rangle + |0,0\rangle)/\sqrt{2}$, while the Ξ^- is an isospin state $|\frac{1}{2}, -\frac{1}{2}\rangle$, so it combines with the spurion leading to $|1,-1\rangle$. The final state $\Lambda\pi^0$ is the isospin state $|1,0\rangle$, while $\Lambda\pi^-$ is the isospin state $|1,-1\rangle$. Hence, the spurion model predicts

$$\frac{\Gamma_{\Xi^0\to\Lambda\pi^0}}{\Gamma_{\Xi^-\to\Lambda\pi^-}} = \frac{1}{2}\,. \tag{46}$$

The Ξ baryons decay almost exclusively to $\Lambda\pi$, so

$$\frac{\Gamma_{\Xi^0\to\Lambda\pi^0}}{\Gamma_{\Xi^-\to\Lambda\pi^-}} \approx \frac{\tau_\Xi^-}{\tau_\Xi^0} = \frac{1.64\times 10^{-10}}{2.90\times 10^{-10}} = 0.57. \tag{47}$$

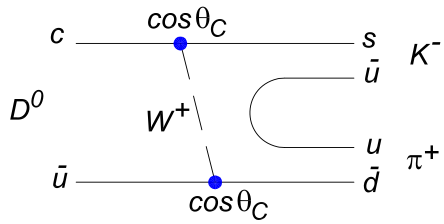
The Ω^- is an isospin state $|0,0\rangle$ that contains two s quarks, so in the spurion model it combines with the spurion state $|\frac{1}{2}, -\frac{1}{2}\rangle$, leading to $|\frac{1}{2}, -\frac{1}{2}\rangle$. The final state $\Xi^0\pi^-$ is the isospin combination $|\frac{1}{2}, \frac{1}{2}\rangle|1,-1\rangle = (|\frac{3}{2}, -\frac{1}{2}\rangle - \sqrt{2}|\frac{1}{2}, -\frac{1}{2}\rangle)/\sqrt{3}$, while the state $\Xi^-\pi^0$ is the isospin combination $|\frac{1}{2}, -\frac{1}{2}\rangle|1,0\rangle = (\sqrt{2}|\frac{3}{2}, -\frac{1}{2}\rangle + |\frac{1}{2}, -\frac{1}{2}\rangle)/\sqrt{3}$. Hence, the spurion model predicts

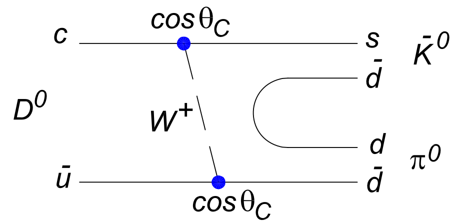
$$\frac{\Gamma_{\Omega^-\to\Xi^0\pi^-}}{\Gamma_{\Omega^-\to\Xi^-\pi^0}} = 2\,. \tag{48}$$

The data are that

$$\frac{\Gamma_{\Omega^-\to\Xi^0\pi^-}}{\Gamma_{\Omega^-\to\Xi^-\pi^0}} = \frac{23.6}{8.6} = 2.74. \tag{49}$$

4. The Cabibbo-favored W-exchange diagrams for the decays $D^0 \to K^-\pi^+$ and $D^0 \to \overline{K}^0\pi^0$ are





The decay amplitudes are identical except that the π^0 is not $u\overline{u}$ but $(u\overline{u} - d\overline{d})/\sqrt{2}$, so $A(D^0 \to \overline{K}^0\pi^0) = A(D^0 \to K^-\pi^+)/\sqrt{2}$, and if only these diagrams are relevant,

$$\frac{\Gamma(D^0 \to K^-\pi^+)}{\Gamma(D^0 \to \overline{K}^0\pi^0)} = 2. \tag{50}$$

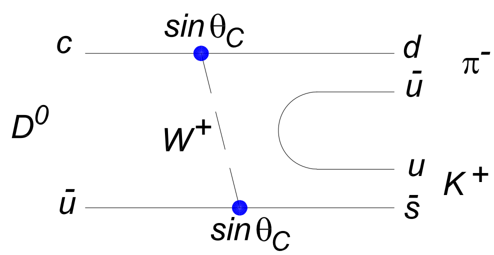
The present data for D^0 branching fractions are

$$\begin{aligned} B(D^0 \to K^-\pi^+) &= 3.88 \pm 0.05\%, &(51)\\ B(D^0 \to K^0_S\pi^0) &= 1.19 \pm 0.04\%, &(52)\\ B(D^0 \to K^0_L\pi^0) &= 1.0 \pm 0.07\%, &(53) \end{aligned}$$

Taking $\overline{K}^0 \approx (K^0_S - K^0_L)/\sqrt{2}$, we add the two branching fractions to $K\pi^0$, and have that

$$\frac{\Gamma(D^0 \to K^-\pi^+)}{\Gamma(D^0 \to \overline{K}^0\pi^0)} = 1.77 \pm 0.13. \tag{54}$$

The decay $D^0 \to K^+\pi^-$ is doubly Cabibbo suppressed, with graph:



Hence, we predict

$$\frac{\Gamma(D^0 \to K^+\pi^-)}{\Gamma(D^0 \to K^-\pi^+)} = \frac{\sin^4\theta_C}{\cos^4\theta_C} = \tan^4 13.02^\circ = 0.0029. \tag{55}$$

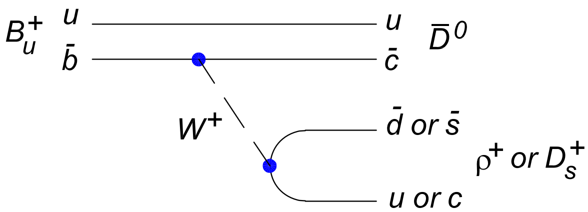
The branching fraction $B(D^0 \to K^+\pi^-)$ is $0.0137 \pm 0.0006\%$, so the data are that

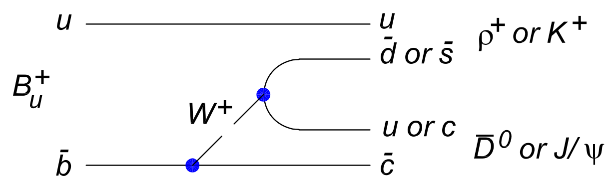
$$\frac{\Gamma(D^0 \to K^+\pi^-)}{\Gamma(D^0 \to K^-\pi^+)} = 0.0035 \pm 0.00015. \tag{56}$$

5. The Cabibbo-favored weak decay of the $\overline{b}$ quark is to $\overline{c}$, so $\overline{q}_2 = \overline{c}$.

   The mass of the $\overline{b}$ quark is high enough that the W^+ emitted in a weak decay can materialize as either $u\overline{d}$ or $c\overline{s}$ with Cabibbo-favored coupling $\cos\theta_C$. That is, either $\overline{q}_3 = \overline{d}$ and $q_4 = u$ or $\overline{q}_3 = \overline{s}$ and $q_4 = c$.

   For the B_u^+ with $q_1 = u$, the left diagram on p. 4 leads to $q_1\overline{q}_2 = u\overline{c} = \overline{D}^0$ and $\overline{q}_3 q_4 = u\overline{d} = \rho^+$ (π^+) or $\overline{q}_3 q_4 = c\overline{s} = D^+$. The right diagram leads to either $q_1\overline{q}_3 = u\overline{d} = \rho^+$ (π^+) and $q_4\overline{q}_2 = u\overline{c} = \overline{D}^0$ or $q_1\overline{q}_3 = u\overline{s} = K^+$ $(K^{\star +})$ and $q_4\overline{q}_2 = c\overline{c} = J/\psi$ (η_c).





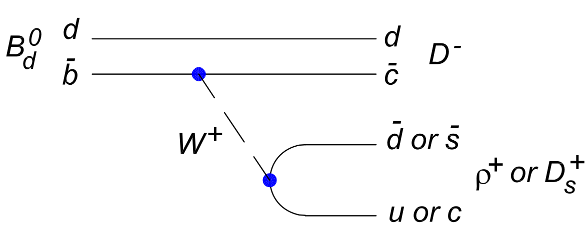
   The spectator model predicts that the leading 2-body hadronic decay modes of the B_u^+ are $\overline{D}^0\rho^+(\pi^+)$, $\overline{D}^0 D_s^+$ and $K^+(K^{\star +})J/\psi(\eta_c)$. We anticipate that the decays to two Ds and to K J/ψ are suppressed somewhat due to reduced final-state phase space.

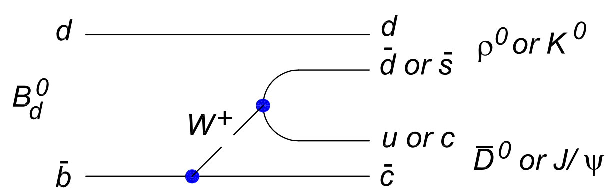
   At http://pdg.lbl.gov/2013/tables/rpp2013-tab-mesons-charm.pdf we learn

$$
\begin{aligned}
B(B_u^+ \to \overline{D}^0\rho^+) &= 1.34 \pm 0.18 \times 10^{-2}, \\
B(B_u^+ \to \overline{D}^0\pi^+) &= 4.83 \pm 0.15 \times 10^{-3}, \\
B(B_u^+ \to \overline{D}^0 D_s^+) &= 1.00 \pm 0.17 \times 10^{-2}, \\
B(B_u^+ \to K^{\star +}J/\psi) &= 1.44 \pm 0.08 \times 10^{-3}, \\
B(B_u^+ \to K^+J/\psi) &= 1.03 \pm 0.03 \times 10^{-3}, \\
B(B_u^+ \to K^{\star +}\eta_c) &= 1.0 \pm 0.5 \times 10^{-3}, \\
B(B_u^+ \to K^+\eta_c) &= 0.96 \pm 0.11 \times 10^{-3}.
\end{aligned} \tag{57}
$$

   Thus, it seems slightly favored to produce vector mesons over scalar mesons.

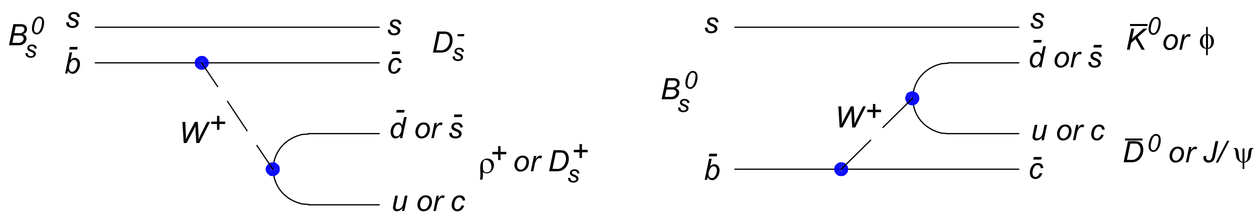
   The leading spectator diagrams for hadronic 2-body decays of the $B_d^0 = d\overline{b}$ meson are





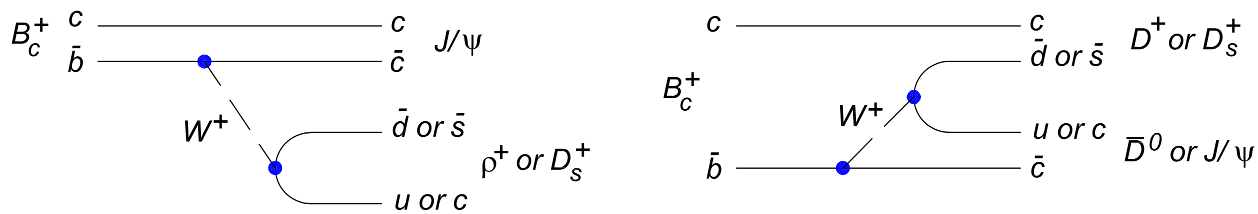
   We infer that the most prominent decays are to $D^-\rho^+$, $\overline{D}^0\rho^0$, $D^-D_s^+$ and $K^{\star 0}J/\psi$. Of these, the first and third have very similar branches, while $\overline{D}^0\rho^0$ is somewhat suppressed, being roughly equal to $K^{\star 0}J/\psi$ which latter has reduced phase space.

   The leading spectator diagrams for hadronic 2-body decays of the $B_s^0 = s\overline{b}$ meson are



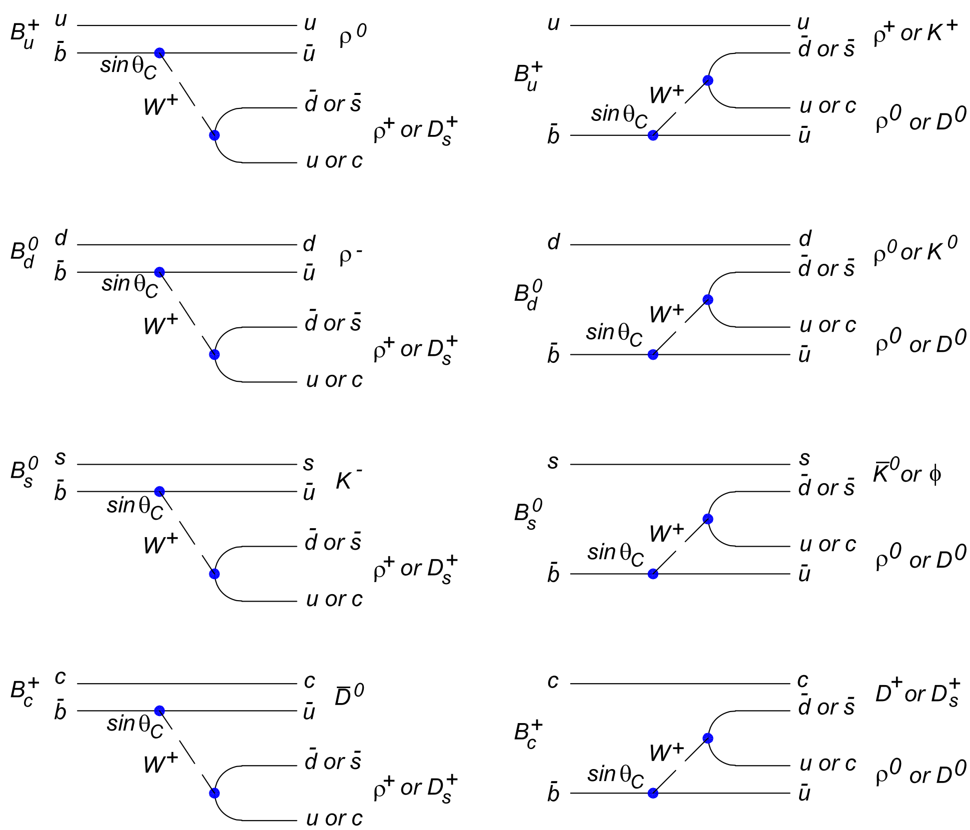
We infer that the most prominent decays are to $D_s^-\rho^+$, $D_s^-D_s^+$, $\phi J/\psi$ and $\overline{K}^{\star 0}\overline{D}^0$. These branches are all similar, and decreasing in the order listed.

The leading spectator diagrams for hadronic 2-body decays of the $B_c^0 = c\overline{b}$ meson are

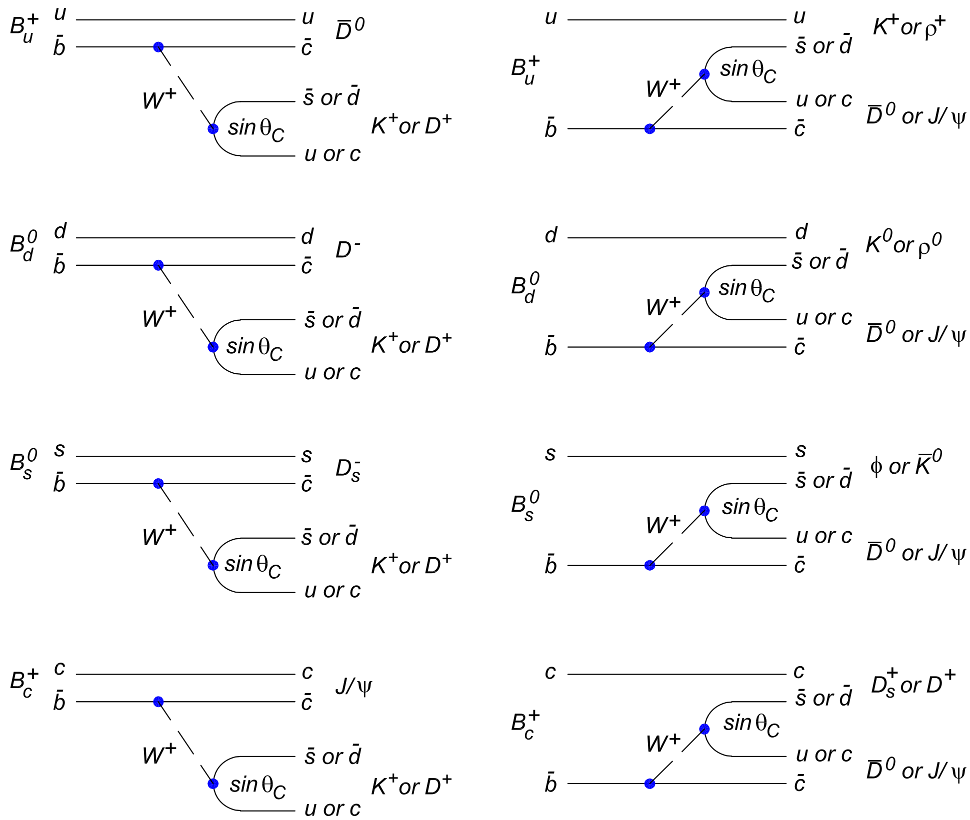


We infer that the most prominent decays are to $D^+\overline{D}^0$, $\rho^+ J\psi$ and D_s^+J/ψ. Of these, only the first is reasonably large.

Cabibbo-suppressed decays have diagrams in which $\overline{b} \to \overline{u}$ or the $W+ \to u\overline{s}$ or $c\overline{d}$. The leading spectator decays for these are



and



6. The 2×2 Hamiltonian (in the $|B^0\rangle$-$|\overline{B}^0\rangle$ basis) is

$$\mathsf{H} = \mathsf{M} - \frac{i}{2}\Gamma = \begin{bmatrix} m & M_{12} \\ M_{12}^* & m \end{bmatrix} - \frac{i}{2}\begin{bmatrix} \gamma & \Gamma_{12} \\ \Gamma_{12}^* & \gamma \end{bmatrix}. \tag{58}$$

Solving $|\mathsf{H} - \lambda \mathsf{I}| = \lambda^2 - 2H_{11}\lambda + H_{11}^2 - H_{12}H_{21} = 0$ for the eigenvalues λ, we find

$$\lambda = H_{11} \pm \sqrt{H_{12}H_{21}} = m \pm \mathrm{Re}\sqrt{H_{12}H_{21}} - \frac{i}{2}\left(\gamma \mp 2\,\mathrm{Im}\sqrt{H_{12}H_{21}}\right). \tag{59}$$

The weak eigenstates of the B^0-$\overline{B}^0$ system have essentially identical lifetimes, which implies that $\Gamma_{12} = \Gamma_{21} = 0$, and hence $H_{12}H_{21} = M_{12}M_{12}^\star = |M_{12}|^2$ is real. CP is violated if $H_{12} = M_{12}$ has an imaginary part.

The masses of the weak eigenstates are

$$m_\pm = m \pm \frac{\Delta m}{2} = m \pm |M_{12}|^{1/2}, \tag{60}$$

and their common decay rate is just $\Gamma = \gamma$.

We define the eigenstates via the complex coefficients p and q, where $|p|^2 + |q|^2 = 1$, according to

$$|B^0_\pm\rangle = p|B^0\rangle \pm q|\overline{B}^0\rangle, \qquad |B^0\rangle = \frac{|B^0_+\rangle + |B^0_-\rangle}{2p}, \qquad |\overline{B}^0\rangle = \frac{|B^0_+\rangle - |B^0_-\rangle}{2q}. \tag{61}$$

The eigenstate $|B^0_+\rangle$ obeys $\mathsf{H}|B^0_+\rangle = (m + |M_{12}|^{1/2})|B^0_+\rangle$, and, say, the $|B^0\rangle$ component of this relation implies that

$$m\,p + M_{12}\,q = \left(m + |M_{12}|^{1/2}\right)p, \qquad \frac{q}{p} = \frac{|M_{12}|^{1/2}}{M_{12}} = \sqrt{\frac{M_{12}^\star}{M_{12}}} = e^{-i\phi_{12}}\,, \tag{62}$$

writing $M_{12} = |M_{12}|\; e^{i\phi_{12}}$. We can satisfy the condition $|p|^2 + |q|^2 = 1$ by taking

$$p = \frac{e^{i\phi_{12}/2}}{\sqrt{2}}\,, \qquad q = \frac{e^{-i\phi_{12}/2}}{\sqrt{2}}\,, \tag{63}$$

so the weak eigenstates are

$$|B^0_\pm\rangle = e^{\pm i\phi_{12}/2}\frac{|B^0\rangle \pm |\overline{B}^0\rangle}{\sqrt{2}}\,. \tag{64}$$

That is, the phases $\phi_\pm$ in eq. (9) are $\pm\phi_{12}$.
It is noteworthy that p and q differ only by a phase factor.

The time dependences of the eigenstates can then be written as

$$|B_\pm(t)\rangle = e^{-\Gamma t/2}e^{im_\pm t}|B_\pm(0)\rangle, \qquad \text{or as} \qquad |B_\pm\rangle \to e^{-\Gamma t/2}e^{im_\pm t}|B_\pm\rangle. \tag{65}$$

Then according to eq. (61), the time evolution of an initially pure B^0 is

$$\begin{aligned} |B^0\rangle &= \frac{|B^0_+\rangle + |B^0_-\rangle}{2p} \to \frac{e^{-\Gamma t/2}}{2p}[e^{im_+ t}|B_+\rangle + e^{im_- t}|B_-\rangle] \\ &= \frac{e^{-\Gamma t/2}}{2p}[e^{i(m+\Delta m/2)t}(p|B^0\rangle + q|\overline{B}^0\rangle) + e^{i(m-\Delta m/2)t}(p|B^0\rangle - q|\overline{B}^0\rangle] \\ &= e^{-imt}e^{-\Gamma t/2}\left[\cos\frac{\Delta m t}{2}|B^0\rangle + i\frac{q}{p}\sin\frac{\Delta m t}{2}|\overline{B}^0\rangle\right] = F_c(t)|B^0\rangle + i\frac{q}{p}F_s(t)|\overline{B}^0\rangle, \end{aligned} \tag{66}$$

and similarly the time evolution of an initially pure $\overline{B}^0$ is

$$\begin{aligned} |\overline{B}^0\rangle &= \frac{|B^0_+\rangle - |B^0_-\rangle}{2q} \to \frac{e^{-\Gamma t/2}}{2q}[e^{im_+ t}|B_+(0)\rangle - e^{im_- t}|B_-(0)\rangle] \\ &= \frac{e^{-\Gamma t/2}}{2q}[e^{i(m+\Delta m/2)t}(p|B^0(0)\rangle + q|\overline{B}^0(0)\rangle) - e^{i(m-\Delta m/2)t}(p|B^0(0)\rangle - q|\overline{B}^0(0)\rangle] \\ &= e^{-imt}e^{-\Gamma t/2}\left[i\frac{q}{p}\sin\frac{\Delta m t}{2}|B^0\rangle + \cos\frac{\Delta m t}{2}|\overline{B}^0\rangle\right] = i\frac{p}{q}F_s(t)|B^0\rangle + F_c(t)|\overline{B}^0\rangle, \end{aligned}$$

with

$$F_c(t) = e^{-imt}e^{-\Gamma t/2}\cos\frac{\Delta m\,t}{2}\,, \qquad F_s(t) = e^{-imt}e^{-\Gamma t/2}\sin\frac{\Delta m\,t}{2}\,. \tag{67}$$

The time evolution of the entangled initial B^0-$\overline{B}^0$ pair produced at an e^+e^- collider has negative charge conjugation, and its time evolution is given by

$$|B_1^0\rangle|\overline{B}_2^0\rangle \rightarrow \left(F_c(t_1)|B_1^0\rangle + i\frac{q}{p}F_s(t_1)|\overline{B}_1^0\rangle\right)\left(i\frac{p}{q}F_s(t_2)|B_2^0\rangle + F_c(t_2)|\overline{B}_2^0\rangle\right), \quad (68)$$

$$|\overline{B}_1^0\rangle|B_2^0\rangle \rightarrow \left(i\frac{p}{q}F_s(t_1)|B_1^0\rangle + F_c(t_1)|\overline{B}_1^0\rangle\right)\left(F_c(t_2)|B_2^0\rangle + i\frac{q}{p}F_s(t_2)|\overline{B}_2^0\rangle\right), \quad (69)$$

$$\begin{aligned}
\frac{|B_1^0\rangle|\overline{B}_2^0\rangle - |\overline{B}_1^0\rangle|B_2^0\rangle}{\sqrt{2}} \rightarrow\ & i\frac{p}{q}[F_c(t_1)F_s(t_2) - F_s(t_1)F_c(t_2)]\frac{|B_1^0\rangle|B_2^0\rangle}{\sqrt{2}} \\
& +[F_c(t_1)F_c(t_2) + F_s(t_1)F_s(t_2)]\frac{|B_1^0\rangle|\overline{B}_2^0\rangle}{\sqrt{2}} \\
& -[F_s(t_1)F_s(t_2) + F_c(t_1)F_c(t_2)]\frac{|\overline{B}_1^0\rangle|B_2^0\rangle}{\sqrt{2}} \\
& +i\frac{q}{p}[F_s(t_1)F_c(t_2) - F_c(t_1)F_s(t_2)]\frac{|\overline{B}_1^0\rangle|\overline{B}_2^0\rangle}{\sqrt{2}} \\
& = e^{im(t_1+t_2)}\, e^{-(t_1+t_2)/2}\Big\{-\frac{ip}{q}\sin\frac{\Delta m(t_1-t_2)}{2}\frac{|B_1^0\rangle|B_2^0\rangle}{\sqrt{2}} \\
& +\cos\frac{\Delta m(t_1-t_2)}{2}\frac{|B_1^0\rangle|\overline{B}_2^0\rangle - |\overline{B}_1^0\rangle|B_2^0\rangle}{\sqrt{2}} \\
& +\frac{iq}{p}\sin\frac{\Delta m(t_1-t_2)}{2}\frac{|\overline{B}_1^0\rangle|\overline{B}_2^0\rangle}{\sqrt{2}}\Big\}. \qquad (70)
\end{aligned}$$

Note that within eq. (70) the amplitudes vanish for the combinations of two like particles, $|B_1^0\rangle|B_2^0\rangle$ and $|\overline{B}_1^0\rangle|\overline{B}_2^0\rangle$, at times $t_1 = t_2$ if the B^0-$\overline{B}^0$ is produced in a CP-odd state. That is, if both B's decay at the same time, one must be a B^0 and the other a $\overline{B}^0$.