# Ph 406 Problem Set 4
Due March 5, 1992

1. Meson Theory of 'Hyperdeuterons.'

Estimate the relative binding energies of the 64 possible pairs of baryons in the basic octet: n, p, Λ, Σ^-, Σ^0, Σ^+, Ξ^-, Ξ^0.

For this use the 'one-pion-exchange' model of nuclear forces – a fancy name for Yukawa's idea, taking into account the existence of three charge states of the pion: π^-, π^0, π^+. This model uses isospin analysis to codify the 'charge independence' of the nuclear force.

For example, the force between a pp (or nn) pair is due to π^0 exchange,
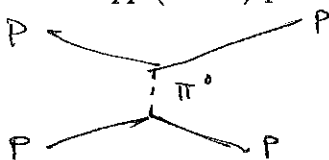
while the force between a np pair is due both to π^0 and π^+ (or π^- depending on the time ordering):
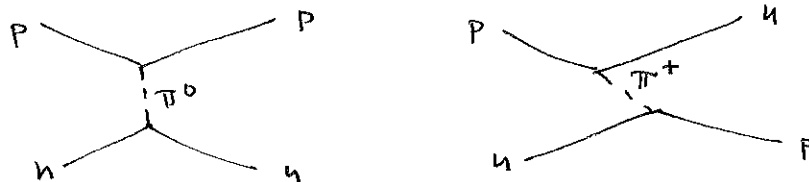
But the force should be the same.

We want to describe the coupling strength with the aid of isospin. Charge independence means that the matrix element should depend on total isospin rather isospin components. We write the (vector) isospin operator on one particle as $\vec{\tau}$. In our two-body problem we seek scalar operators composed of $\vec{\tau}_1$ and $\vec{\tau}_2$, where 1 and 2 label the first and second particles, such that the operator actually only depends on the magnitude of the total isospin $\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2$. Perhaps you can convince yourself that the only distinct possibilities are the operators 1 and $\vec{\tau}_1 \cdot \vec{\tau}_2$.

Next you need to convince yourself that one-pion exhange is described by $\vec{\tau}_1 \cdot \vec{\tau}_2$. Certainly a transition like $p \to n\pi^+$ cannot be described by the unit operator. A useful argument involves the so-called raising and lowering operators:

$$\tau_+ \equiv \tau_x + i\tau_y \qquad \tau_- \equiv \tau_x - i\tau_y,$$

where x, y, and z refer to directions in isospin space rather than ordinary space. You may know that these operators obey

$$\tau_+|t, t_z\rangle = \sqrt{(t + t_z + 1)(t - t_z)}|t, t_z + 1\rangle,$$

$$\tau_-|t, t_z\rangle = \sqrt{(t + t_z)(t - t_z + 1)}|t, t_z - 1\rangle,$$

Thus $\tau_+|n\rangle = |p\rangle$, and $\tau_-|p\rangle = |n\rangle$. Of course, $\tau_z|p\rangle = \frac{1}{2}|p\rangle$ and $\tau_z|n\rangle = -\frac{1}{2}|n\rangle$. (That is, the τ operators can be represented by Pauli spin matrices times 1/2.)

Verify that

$$\vec{\tau}_1 \cdot \vec{\tau}_2 = \frac{1}{2}(\tau_{+,1}\tau_{-,2} + \tau_{-,1}\tau_{+,2}) + \tau_{z,1}\tau_{z,2}.$$

We can now use $\vec{\tau}_1 \cdot \vec{\tau}_2$ to evaluate the strength of each of the diagrams above. The pp diagram gets a factor $\frac{1}{2}$ at each vertex for an overall strength of $\frac{1}{4}$ (times g^2 which we ignore when comparing strengths). For the nn diagram we get $(-\frac{1}{2})(-\frac{1}{2}) = \frac{1}{4}$ also. Now for the np case we get $(-\frac{1}{2})(\frac{1}{2}) = -\frac{1}{4}$ for the diagram with π^0 exchange, and $(\frac{1}{2})(1)(1) = \frac{1}{2}$ for the diagram with π^+ exchange. But these two diagrams interfere, so we add the amplitudes, and the result is again $\frac{1}{4}$. Charge independence!

We have glossed over an important point in the above. There are really two kinds of np states: $\sqrt{\frac{1}{2}}(|np\rangle + |pn\rangle)$ and $\sqrt{\frac{1}{2}}(|np\rangle - |pn\rangle)$. The charge-independent result above is for the first state, which we recognize as the partner of $|pp\rangle$ and $|nn\rangle$ in the symmetric isospin-1 triplet. The second np state is the antisymmetric isospin-0 singlet. Show that this state is an eigenstate of $\vec{\tau}_1 \cdot \vec{\tau}_2$ with eigenvalue $-\frac{3}{4}$.

As for electricity, we infer that a negative matrix element $\langle NN|\vec{\tau}_1 \cdot \vec{\tau}_2|NN\rangle$ implies an attractive force, while a positive matrix element implies repulsion. The matrix element (of the interaction Hamiltonian) is proportional to the strength of the binding (or lack thereof). Hence we infer that only the np isosinglet is bound, and with a strength (binding energy) of $-\frac{3}{4}$ units, while the NN isotriplet states are unbound, but might exist as continuum states of energy $\frac{1}{4}$ units.

Experimentally, the binding energy of the deuteron is 2.25 MeV, and indeed there are no bound pp or nn states. By our model we expect an excited state of the deuteron (with pp and nn partners) with positive energy (2.25 MeV)/3 = 750 keV. Experimentally such states exist, but with positive energy of only 75 keV.

Verify that a two-nucleon (NN) isotriplet state with angular momentum $L = 0$ must be a spin singlet to satisfy Fermi statistics. Since the nuclear force is known to depend on ordinary spin, this may account for the discrepancy between our model and the data. Still, the one-pion-exchange model has got the basic features right.

Having examined 4 of the 64 dibaryon states you can now turn to the other 60, again ignoring the dependence of the nuclear force on ordinary spin. But you may want to note a few time-saving tricks. Consider the eigenvalues of the operator $\vec{\tau}^2 = (\tau_1 + \tau_2)^2$. Also, charge independence means you don't have to look at each of the 64 pairs separately, but you can more simply consider pairs of isospin multiplets, each of which leads to one or more multiplets of total isospin exactly as for combinations of ordinary spin. For this note that the nucleons, N, and the cascade particles, Ξ, each form an isodoublet, the Λ is an isosinglet, and the Σ's form an isotriplet.

I found that 11 of the 64 pairs should have bounds states, and that none of these would be more weakly bound than the deuteron. Some of the proposed bound states have an interesting feature in their isospin wave function...

No dibaryon bound state other than the deuteron has ever been observed, but I'm not sure anyone has ever looked for the other nine predicted states....

In the model above we have ignored the ordinary-spin dependence of the nuclear force. When considering pairs of baryons from different isospin mutiplets, the restrictions on the ordinary-spin wave function due to Fermi statistics no longer apply. It is possible that for total ordinary spins other than those tacitly assumed above one-pion exchange leads to additional bound states..... Perhaps we should also consider two-pion exchange....

2. πp Scattering at the Δ Resonance.

Another example of the use of isospin is a comparison of the cross sections

$$\sigma(\pi^+ p \to \pi^+ p), \qquad \sigma(\pi^- p \to \pi^- p), \qquad \text{and} \qquad \sigma(\pi^- p \to \pi^0 n),$$

at a center-of-mass energy 1236 MeV, where the scattering is dominated by the s-channel diagram:
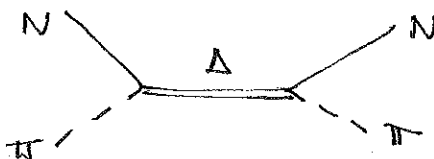
Here the Δ is the isospin $\frac{3}{2}$ 'resonance' of a pion and nucleon.

By charge indepence of the nuclear force, all vertex factors in the relevant diagrams are the same. Use the Clebsch-Gordan decomposition of isospin states to predict the relatives sizes of the three cross sections.

Recall how $|1,1\rangle|\frac{1}{2},\frac{1}{2}\rangle = |\frac{3}{2},\frac{3}{2}\rangle$, but $|1,0\rangle|\frac{1}{2},\frac{1}{2}\rangle = C_1|\frac{3}{2},\frac{1}{2}\rangle + C_2|\frac{1}{2},\frac{1}{2}\rangle$, where C_1 and C_2 are the Clebsch-Gordan coefficients....

3.

   (a) Show that the magnetic interaction energy between two magnetic dipoles $\mu\sigma_1$ and $\mu\sigma_2$ is of the form $V_T(r)\Omega_T$ with $V_T(r) = -(\mu_0/4\pi)\mu^2/r^3$ (μ_0 is the permeability of the vacuum.)

   (b) Verify that equation (3.8) includes terms in the nucleon–nucleon potential of tensor form.

4.

Using the semi-empirical mass formula show that the energy S_n required to separate a neutron from the nucleus (A, Z) is given approximately by:

$$S_n \approx a_v - \tfrac{2}{3}a_s A^{-1/3} - a_a[1 - 4Z^2/A(A-1)]$$

Estimate the mass number of the Na nucleus which is just stable against neutron emission.

To use the notation of Cottingham and Greenwood, $a_v = a = 15.8$ MeV, $a_s = b = 18.3$ MeV, and $a_a = s = 23.2$ MeV.

$5.$ (a) Show that in the model of §5.2 the total kinetic energy of a nucleus containing N neutrons and Z protons is

$$[\tfrac{3}{5}NE_n^F + \tfrac{3}{5}Z(E_p^F - \bar{U})].$$

(b) For $(N-Z) \ll A$, this expression may be Taylor expanded about $N_0 = A/2$, $Z_0 = A/2$, $E_0^F \approx 38$ MeV. Show that

$$E_n^F \approx E_0^F \left[1 + \frac{2}{3}\frac{\Delta N}{N_0} - \frac{1}{9}\left(\frac{\Delta N}{N_0}\right)^2 \right],$$

$$(E_p^F - \bar{U}) = E_0^F \left[1 - \frac{2}{3}\frac{\Delta N}{N_0} - \frac{1}{9}\left(\frac{\Delta N}{N_0}\right)^2 \right],$$

where $\Delta N = -\Delta Z = (N-Z)/2$.

(c) Hence show that the total kinetic energy of the nucleons in the nucleus is approximately

$$\tfrac{3}{5}E_0^F A + \tfrac{1}{3}E_0^F \frac{(N-Z)^2}{A}$$

$6.$ The total binding energies of the nuclei $^{15}_{8}O$, $^{16}_{8}O$ and $^{17}_{8}O$ are 111.95 MeV, 127.62 MeV and 131.76 MeV, respectively. Deduce the energies of the last occupied state and of the first unoccupied state of neutrons in $^{16}_{8}O$.

The total binding energy of $^{17}_{9}F$ is 128.22 MeV. Deduce the energy of the first unoccupied proton state in $^{16}_{8}O$. Why is it different from the energy of the corresponding neutron state? Account for the order of magnitude of the difference.

$7.$ Using the single-particle shell model predict the J^π of: $^{15}_{7}N$, $^{27}_{12}Mg$, $^{87}_{38}Sr$, $^{167}_{68}Er$ and $^{195}_{80}Hg$. The measured J^π are $\tfrac{1}{2}^-$, $\tfrac{1}{2}^+$, $\tfrac{9}{2}^+$, $\tfrac{7}{2}^+$ and $\tfrac{1}{2}^-$, respectively. Comment on any discrepancies.