Ph 406: Elementary Particle Physics
Problem Set 3

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1. Deduce the nonrelativistic form factors,

$$F(q^2) = \int \rho(r)\, e^{i\mathbf{q}\cdot\mathbf{r}}\, d^3\mathbf{r}, \tag{1}$$

for the spherically symmetric charge densities with characteristic radius R,

$$\rho_a(r) = \begin{cases} 3Q/4\pi R^3 & (r < R), \\ 0 & (r > R), \end{cases} \tag{2}$$

$$\rho_b(r) = \frac{Q}{4\pi R^2}\delta(r - R), \tag{3}$$

and

$$\rho_c(r) = \frac{Q}{2\pi\sqrt{2\pi}R^3} e^{-r^2/2R^2}, \tag{4}$$

all of which have total charge Q. Expand these form factors to order $(qR)^2$.

A neutral particle might have charge distributions ρ_+ and ρ_- with the above forms, but with different values of the characteristic radii R_+ and R_-.

The data are often fit to the form,[1]

$$F_n(q^2) = \frac{Q}{[1 + (qR)^2]^n}, \tag{5}$$

with $n = 2$. What are the corresponding forms of the charge distributions $\rho_n(r)$ for $n = 1$, 2 and 3?

2. **Arbitrary 2×2 Unitary Matrices and Pauli Spin Matrices**

This problem concerns operators that act on 2-component spinors. Such operators can be expressed as 2×2 matrices. Operators that preserve the normalization of a state are called unitary.

Two of the simplest unitary operators on 2-component spinors are the identify matrix $\mathbf{I}_2 = \mathbf{I}$, and the spin-flip operator X (called the NOT operator in quantum computation),

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathsf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

[1]For a review of nucleon form factors, see C.F. Perdrisat *et al.*, *Nucleon electromagnetic form factors*, Prog. Part. Nucl. Phys. **59**, 694 (2007), http://kirkmcd.princeton.edu/examples/EP/perdrisat_ppnp_59_694_07.pdf.

An arbitrary 2×2 unitary matrix U can be written as

$$\mathsf{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (7)$$

where a, b, c and d are complex numbers such that $\mathsf{U}\mathsf{U}^\dagger = \mathbf{I}$. The decomposition (7) is somewhat trivial. Express the general unitary matrix U as the sum of four unitary matrices, times complex coefficients, of which two are the classical unitary matrices $\mathbf{I}$ and X given above. Denote the "partner" of $\mathbf{I}$ by Z and the "partner" of X by Y such that

$$\mathsf{XY} = i\mathsf{Z}, \qquad \mathsf{YZ} = i\mathsf{X}, \qquad \mathsf{ZX} = i\mathsf{Y}. \qquad (8)$$

You have, of course, rediscovered the so-called Pauli spin matrices,[2,3]

$$\boldsymbol{\sigma}_x\ (= \boldsymbol{\sigma}_1) = \mathsf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_y\ (= \boldsymbol{\sigma}_2) = \mathsf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_z\ (= \boldsymbol{\sigma}_3) = \mathsf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (9)$$

As usual, we define the Pauli "vector" $\boldsymbol{\sigma}$ as the triplet of matrices

$$\boldsymbol{\sigma} = (\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z). \qquad (10)$$

Show that for ordinary 3-vectors $\mathbf{a}$ and $\mathbf{b}$,

$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) = (\mathbf{a} \cdot \mathbf{b})\ \mathbf{I} + i\ \boldsymbol{\sigma} \cdot \mathbf{a} \times \mathbf{b}. \qquad (11)$$

With this, show that a general 2×2 unitary matrix can be written as

$$\mathsf{U} = e^{i\delta} \left(\cos\frac{\theta}{2}\ \mathbf{I} + i \sin\frac{\theta}{2}\ \hat{\mathbf{u}} \cdot \boldsymbol{\sigma} \right) = e^{i\delta} e^{i\frac{\theta}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}}, \qquad (12)$$

where δ and θ are real numbers and $\hat{\mathbf{u}}$ is a real unit vector.[4] By the exponential e^{O} of an operator O we mean the Taylor series $\sum_n \mathsf{O}^n/n!$ where $\mathsf{O}^0 = \mathbf{I}$.

What is the determinant of the matrix representation of U? The subset of 2×2 unitary matrices with unit determinant is called the special unitary group SU(2). What is the version of eq. (12) that describes 2×2 special unitary operators?

You may wish to convince yourself of a factoid related to eq. (12), namely that if A *is a square matrix of any order such that* $\mathsf{A}^2 = \mathbf{I}$*, then* $e^{i\theta\mathsf{A}} = \cos\theta\ \mathbf{I} + i \sin\theta\ \mathsf{A}$*, provided that* θ *is a real number. It follows that* A *can also be written in the exponential form*

$$\mathsf{A} = e^{i\pi/2} e^{-i\frac{\pi}{2}\mathsf{A}} = e^{-i\pi/2} e^{i\frac{\pi}{2}\mathsf{A}}. \qquad (13)$$

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[2] W. Pauli, *Zur Quantenmechanik des magnetischen Elektrons*, Z. Phys. **43**, 601 (1927), http://kirkmcd.princeton.edu/examples/QM/pauli_zp_43_601_27.pdf.

[3] The Pauli spin matrices (and the unit matrix $\mathbf{I}$) are not only unitary, they are also hermitian, meaning that they are identical to their adjoints: $\boldsymbol{\sigma}_j^\dagger = \boldsymbol{\sigma}_j$.

[4] Note that if make the replacements $\theta \to -\theta$ and $\hat{\mathbf{u}} \to -\hat{\mathbf{u}}$ we obtain another valid representation of U, since the physical operation of a rotation by angle θ about an axis $\hat{\mathbf{u}}$ is identical to a rotation by $-\theta$ about the axis $-\hat{\mathbf{u}}$.

There are several unitary operators of interest, such as the Pauli matrices, that are their own inverse. If we call such an operator V, *then its exponential representation of* V *can be written in multiple ways,*

$$\mathsf{V} = e^{i\delta} e^{i\frac{\theta}{2}\hat{\mathbf{v}}\cdot\boldsymbol{\sigma}} = \mathsf{V}^{-1} = e^{-i\delta} e^{-i\frac{\theta}{2}\hat{\mathbf{v}}\cdot\boldsymbol{\sigma}}. \tag{14}$$

3. Give the explicit 4×4 matrix form of the four Dirac matrices γ_μ,[5] as well as that for $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, in their representation via the 2×2 Pauli matrices $\mathbf{I}$ and $\boldsymbol{\sigma}_i$, $i = 1, 2, 3$,

$$\gamma_0 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ -\boldsymbol{\sigma}_i & 0 \end{pmatrix}, \tag{15}$$

It should be then evident that $\mathsf{tr}(\gamma_\mu) = 0 = \mathsf{tr}(\gamma_5)$, where tr is the trace operator. Then, it immediately follows that $\mathsf{tr}(\not{a}) = 0$, where $\not{a} \equiv a^\mu \gamma_\mu$ and a_μ is an arbitrary 4-vector.

Show that

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\mathbf{I}_4, \tag{16}$$

where $\eta_{\mu\nu}$ has diagonal elements $1, -1, -1, -1$ and $\mathbf{I}_4$ is the 4×4 unit matrix,[6] and hence that

$$\mathsf{tr}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}, \qquad \text{and} \qquad \mathsf{tr}(\not{a}\not{b}) = 4a_\mu b^\mu \equiv 4ab. \tag{17}$$

Show also that

$$\mathsf{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}), \tag{18}$$

and hence that

$$\mathsf{tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(ab)(cd) - (ac)(bd) + (ad)(bc)]. \tag{19}$$

A factoid which you need not demonstrate is that the Dirac equivalent of eq. (11) is

$$\not{a}\not{b} = ab\mathbf{I}_4 + \frac{a^\mu b^\nu}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \tag{20}$$

If you think that maxtrix manipulation is the key to physics, then you might enjoy my course, **Physics of Quantum Computation**,
http://kirkmcd.princeton.edu/examples/ph410problems.pdf.

[5]The matrices γ_μ were introduced by Dirac in the form used here, but with his γ_4 being our γ_0, in sec. 3 of *The Quantum Theory of the Electron*, Proc. Roy. Soc. London A **117**, 610 (1928),
http://kirkmcd.princeton.edu/examples/QED/dirac_prsla_117_610_28.pdf.

[6]The matrix $\mathbf{I}_4$ is typically denoted by 1.