Ph 406: Elementary Particle Physics
Problem Set 10

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1. The combined symmetry CP of charge conjugation (C) and space inversion (parity, P) is expected to be violated in the conjugate decay modes $B_d^0 \to \pi^+\pi^-$ and $\bar{B}_d^0 \to \pi^+\pi^-$. As in the K^0-$\bar{K}^0$ system, there is "mixing" in the B_d^0-$\bar{B}_d^0$ system such that one state can oscillate into the other. This makes possible an observable effect of CP violation in the time dependence of the decay of an initially pure B_d^0 meson state, because of interference between CP -violating effects in the mixing and in the decay.

   The time evolution of an initially pure B_d^0 meson is (from Prob. 6, Set 9, with a change of notation)

$$|B^0_{d,\rm phys}(t)\rangle = e^{-imt} e^{-\Gamma t/2} \left(\cos \frac{\Delta m\, t}{2} |B_d^0\rangle + i \sin \frac{\Delta m\, t}{2} \sqrt{\frac{\langle \bar{B}_d^0 | B_d^0 \rangle}{\langle B_d^0 | \bar{B}_d^0 \rangle}} |\bar{B}_d^0\rangle \right), \qquad (1)$$

   where m and Δm are the average mass of and the mass difference between the mass eigenstates of the B_d^0-$\bar{B}_d^0$ system, and Γ is the total decay rate of a B_d^0 meson.

   The Lagrangian for the charged-current (CC) interaction of quarks with the W bosons has the form

$$\mathcal{L}^{\rm CC} = \frac{g}{\sqrt{2}} \left(\, \bar{u} \; \bar{c} \; \bar{t} \, \right)_L \gamma^\mu W_\mu^+ V_{\rm CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L \; + \; \text{h.c.}, \qquad (2)$$

   where the Cabibbo-Kobayashi-Maskawa matrix, $V_{\rm CKM}$, is a 3×3 unitary matrix of coupling constants,

$$V_{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}, \qquad (3)$$

   that you may approximate as being real numbers with the exception of $V_{td} = |V_{td}|\, e^{i\phi_{td}}$ and $V_{ub} = |V_{ub}|\, e^{i\phi_{ub}}$, which have imaginary terms.[1]

   Facts: The quark content of the mesons are $B_d^0 = (\bar{b}d)$, $\pi^+ = (\bar{d}u)$, and $\pi^- = (\bar{u}d)$.

   a) Sketch the quark-level box diagrams for the mixing processes $\langle \bar{B}_d^0 | B_d^0 \rangle$ and $\langle B_d^0 | \bar{B}_d^0 \rangle$, and determine the dependence of these matrix elements on the $V_{\rm CKM}$ coupling constants. If nature were CP invariant, what would be the relation between the matrix elements?

      Assume for simplicity that only t-quarks participate in internal quark lines.

[1] The gist of the Lagrangian is that a weak-interaction vertex $q_1 \to q_2 W^-$ (or $q_1 \overline{q}_2 \to W^-$ or $W^+ q_1 \to q_2$ or $W^+ \to \overline{q}_1 q_2$) involves a factor $V_{q_1 q_2}$, while vertices $\overline{q}_1 \to \overline{q}_2 W^+$, $\overline{q}_1 q_2 \to W^+$, $W^- \overline{q}_1 \to \overline{q}_2$, $W^- \to q_1 \overline{q}_2$ involve a factor $V^\star_{q_1 q_2}$.

b) Draw the Feynman diagrams in the tree-level[2] spectator model for the conjugate decay processes $B_d^0 \to \pi^+\pi^-$ and $\bar{B}_d^0 \to \pi^+\pi^-$, and extract the dependence of the matrix elements for these processes on the $V_{\rm CKM}$ coupling constants.

There also exist "penguin" diagrams for these decays, which need not be considered here.

c) Use the results of parts a) and b) to evaluate the time dependence of the decay rate of an initially pure B_d^0 meson to $\pi^+\pi^-$. Identify the effect of CP violation in this.

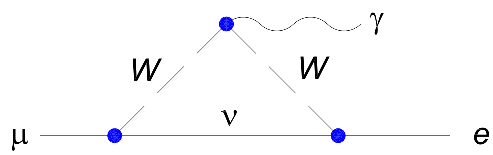
d) Use the unitarity constraint on $V_{\rm CKM}$ and your knowledge of the charged-current interaction to numerically approximate the ratio of the imaginary parts of V_{ub} and V_{td},

$$\frac{Im(V_{ub})}{Im(V_{td})}. \tag{4}$$

2. **$\mu \to e\gamma$?**

Although the particle now called the muon was discovered in 1937[3] its character was little understood for the next 10 years. Yukawa[4] had predicted a particle that would decay into an electron and neutrino, but by 1940 there was weak evidence that the muon/mesotron did not decay according to $\mu \to e\nu$ or $\mu \to e\gamma$, and Nordheim[5] speculated that the decay might be $\mu \to e\nu\nu$. Only after more-convincing experiments in 1947-48 did it become generally accepted that this was indeed the dominant decay mode.[6]

In 1958 Feinberg[7] made a model of the decay $\mu \to e\gamma$ according to the diagram,



and noted that a quick estimate of the decay rate is only a factor $\alpha \approx 10^{-2}$ times that for $\mu \to e\nu\nu$. However, even at that time it was known that the branching fraction of $\mu \to e\gamma$ is less than 10^{-5}. In this context, Pontecorvo's suggestion[8] began to gain

[2] Tree level means only two vertices in the language of Feynman diagrams.

[3] C.D. Anderson and S.H. Neddermeyer, *Cloud Chamber Observations of Cosmic Rays at 4300 Meters Elevation and Near Sea-Level*, Phys. Rev. **50**, 263 (1937), http://kirkmcd.princeton.edu/examples/EP/anderson_pr_50_263_37.pdf.

[4] H. Yukawa, *On the Interaction of Elementary Particles. I*, Proc. Math. Phys. Soc. Japan **17**, 48 (1935), http://kirkmcd.princeton.edu/examples/EP/yukawa_ppmsj_17_48_35.pdf.

[5] L.W. Nordheim, *On the Nature of the Meson Decay*, Phys. Rev. **59**, 554 (1941), http://kirkmcd.princeton.edu/examples/EP/nordheim_pr_59_554_41.pdf.

[6] L. Michel, *Energy Spectrum of Secondary Electrons from μ-Meson Decay*, Nature **163**, 959 (1949), http://kirkmcd.princeton.edu/examples/EP/michel_nature_163_959_49.pdf.

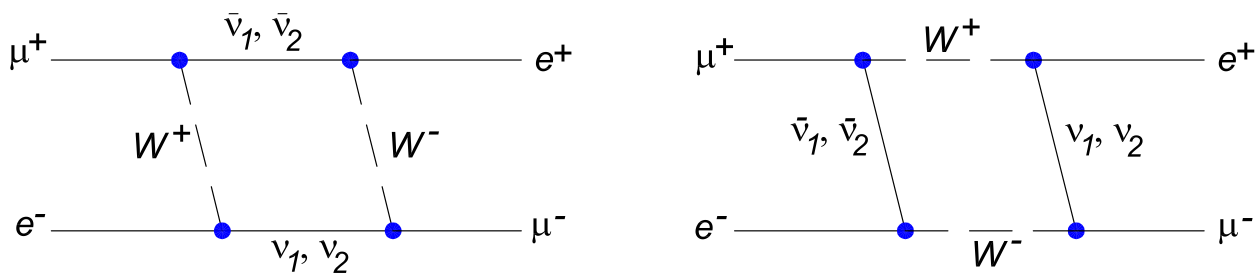
[7] G. Feinberg, *Decays of the μ Meson in the Intermediate-Meson Theory*, Phys. Rev. **110**, 1482 (1958), http://kirkmcd.princeton.edu/examples/EP/feinberg_pr_110_1482_58.pdf. This may be the first appearance of the W boson (called I by Feinberg) in a Feynman diagram.

[8] B. Pontecorvo, *Mesonium and Antimesonium*, Sov. Phys. JETP **6**, 429 (1957), http://kirkmcd.princeton.edu/examples/EP/pontecorvo_spjetp_6_429_57.pdf.

favor, that the dominant decay is $\mu^- \to e^-\nu_\mu\overline{\nu}_e$ where (e, ν_e) and (μ, ν_μ) have different, conserved lepton numbers. But, as we now know that the neutrinos have mass and there are oscillations between ν_e and $\nu_m u$, and we might again expect a significant branching fraction for $\mu \to e\gamma$ (present limit[9] $< 10^{-12}$).

Make a simple estimate of the muon branching fraction to $e\gamma$ based on the above diagram, including the possibility of neutrino oscillations. *Recall the spirit of the quick estimate of Prob. 1, set 9.*

Pontecorvo (1957) was nominally concerned that unless the μ and e (and ν_μ and ν_e) had different lepton number, it would be possible for a μ^+e^- "atom" (mesonium) to oscillate into μ^-e^+ (antimesonium). This effect is forbidden in case of different lepton numbers and massless neutrinos, but it might be possible in case of massive neutrinos (with mass eigenstates $\nu_{1,2}$) that can transform into one another, as sketched in the diagrams below.



Show that the total amplitude for these diagrams actually vanishes, by considering the couplings at the various vertices (in a two-neutrino model).

3. **Polarization States of Gravitational Waves**

Discuss how the concept of gauge invariance leads to an understanding that gravitational waves have only two independent polarization states, and briefly illustrate the effect of plane gravitational waves of these two polarizations that are incident on a massive sphere of radius small compared to a wavelength.

An argument for gravitation is an extension of that for electromagnetism.

Electromagnetic Waves

A familiar argument considers a unit, plane electromagnetic wave in free space with electric field given by the real part of

$$\mathbf{E} = \hat{\mathbf{E}}_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{5}$$

where $\hat{\mathbf{E}}_0$ is a unit vector (possibly complex), $\mathbf{k}$ is the wave vector and $\omega = kc$ is the angular frequency, with c being the speed of light in vacuum. This field obeys the first Maxwell equation, $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ (in empty space), which implies that

$$\hat{\mathbf{E}}_0 \cdot \mathbf{k} = 0, \tag{6}$$

[9]J. Adam *et al.*, *New Constraint on the Existence of the $\mu^+ \to e^+\gamma$ Decay*, Phys. Rev. Lett. **110**, 201801 (2013), http://kirkmcd.princeton.edu/examples/EP/adam_prl_110_201801_13.pdf.

and hence there are only two independent possibilities for the unit vector $\hat{\mathbf{E}}_0$, which are correspond to two independent "polarization" states of the wave, both of which have electric field transverse to the wave vector. Whereas, for waves inside matter, in general $\boldsymbol{\nabla} \cdot \mathbf{E} \neq 0$, so there can be waves with longitudinal, as well as transverse, polarization.

Here, we consider a longer argument based on the scalar and vector potentials ϕ and $\mathbf{A}$, which has the merit of being extendable to the case of gravitational waves.

The potentials can be considered as components of a 4-vector potential,

$$\phi_\mu = (\phi, \mathbf{A}). \tag{7}$$

Plane waves of the potentials have the form

$$\phi_\mu = \epsilon_\mu \, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \tag{8}$$

where ϵ_μ is a constant 4-vector. In principle, there are 4 independent types of polarization for waves of the 4-potential.

Because the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be deduced from derivatives of the potentials, the latter have some degree of arbitrariness, which fact has come to be discussed under the theme of gauge invariance.[10] One consequence of gauge invariance is that one can choose to enforce one relation among the derivatives of the potentials, now called a gauge condition, or choice of gauge. When electromagnetic waves are concerned, a particularly useful choice is the Lorenz condition,[11]

$$\partial_\mu \phi^\mu = 0 = \frac{1}{c}\frac{\partial \phi}{\partial t} - \boldsymbol{\nabla} \cdot \mathbf{A} \qquad \text{(Lorenz)}, \tag{9}$$

in Gaussian units. Applying the Lorenz-gauge condition (9) to the 4-potential wave (8), we have that

$$k_\mu \phi^\mu = 0, \tag{10}$$

where

$$k_\mu = (\omega, \mathbf{k}c), \tag{11}$$

is the wave 4-vector.

We can say that the Lorenz condition (9) has eliminated one of the four possible polarization states, leaving three. In the rest of this note, we suppose that the wave

[10]For a historical review, see J.D. Jackson and L.B. Okun, *Historical roots of gauge invariance*, Rev. Mod. Phys. **73**, 663 (2001), http://kirkmcd.princeton.edu/examples/EM/jackson_rmp_73_663_01.pdf.

[11]L. Lorenz, *On the Identity of the Vibrations of Light with Electrical Currents*, Phil. Mag. **34**, 287 (1867), http://kirkmcd.princeton.edu/examples/EM/lorenz_pm_34_287_67.pdf.
For a survey of several gauge conditions, see J.D. Jackson, *From Lorenz to Coulomb and other explicit gauge transformations*, Am. J. Phys. **70**, 917 (2002),
http://kirkmcd.princeton.edu/examples/EM/jackson_ajp_70_917_02.pdf.

vector **k** is in the z-direction. Then we can write one basis for the three remaining polarization states ϵ_μ of the 4-potential as

$$\epsilon_\mu^{(1)} = (0, 1, 0, 0), \tag{12}$$

$$\epsilon_\mu^{(2)} = (0, 0, 1, 0), \tag{13}$$

$$\epsilon_\mu^{(3)} = (kc/\omega, 0, 0, 1). \tag{14}$$

We now show that for waves with $\omega = kc$, as holds in vacuum, the longitudinal polarization state $\epsilon_\mu^{(3)}$ can be eliminated by a gauge transformation (while staying within the Lorenz gauge).[12]

The (gauge) transformation,

$$\phi_\mu \to \phi_\mu + \partial_\mu \Omega, \qquad \phi \to \phi + \frac{1}{c}\frac{\partial \Omega}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} - \boldsymbol{\nabla}\Omega, \tag{15}$$

does not change the electromagnetic fields

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{16}$$

and the revised potentials (15) still satisfy the Lorenz condition (9) provided that

$$\partial_\mu \partial^\mu \Omega = 0 = \frac{1}{c^2}\frac{\partial^2 \Omega}{\partial t^2} - \nabla^2 \Omega. \tag{17}$$

For example, we can choose

$$\Omega = \frac{1}{ik} e^{ik(z-ct)}, \qquad \partial_\mu \Omega = -(1, 0, 0, 1)\, e^{ik(z-ct)}. \tag{18}$$

Then, for waves with $\omega = kc$, the revised polarization state 3 vanishes,

$$\epsilon_\mu^{(3)} = (1, 0, 0, 1) \to \epsilon_\mu^{(3)} + \partial_\mu \Omega = (0, 0, 0, 0). \tag{19}$$

This confirms that familiar result that for electromagnetic waves which obey the free-space relation that $\omega = kc$ there is no longitudinal polarization state. Loosely speaking, we can gauge away the longitudinal polarization of waves in free space, but not for way in matter. *The transverse polarizations states (13)-(14) are altered by the gauge transformation (15), but it is usual to redefine the transverse polarization states after the gauge transformation to have their original forms again.*

Gravitational Waves

In Einstein's theory of gravitation one considers waves as weak perturbations of the metric tensor,

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi_{\mu\nu}, \tag{20}$$

[12]The choice of a gauge condition is not sufficient for the potentials to be unique. For an example of two rather different sets of potentials in the Lorenz gauge for waves inside a rectangular metallic cavity, see sec. 2.2.3 of K.T. McDonald, *Potentials for a Rectangular Electromagnetic Cavity* (Mar. 4, 2011), http://kirkmcd.princeton.edu/examples/cavity.pdf.

where $\eta_{\mu\nu}$ is the (Euclidean) metric for empty space,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{21}$$

For isotropic spacetime, we infer that the "potential" tensor $\phi_{\mu\nu}$ is symmetric, with only 10 independent components. We can enforce a Lorenz-like gauge condition on the derivatives of $\phi_{\mu\nu}$ that

$$\partial_\mu \eta^{\mu\lambda} \phi_{\lambda\nu} = \partial^\mu \phi_{\mu\nu} 0. \tag{22}$$

The four conditions (22) reduce the number of independent components of $\phi_{\mu\nu}$ to six.

Show that further use of gauge transformations, within the Lorenz-like gauge, reduces the number of independent components of $\phi_{\mu\nu}$ to two.

However, arguments based on consideration of waves $\phi_{\mu\nu}$ in otherwise empty space miss a noteworthy issue: that weak gravitational waves inside low-density matter can have five independent polarization states. This is clearer in a quantum view in which gravitational waves are associated with spin-2 quanta, which have five independent spin components, in general. Hence, we infer that there exists one more condition on the $\phi_{\mu\nu}$ which holds even for waves inside low-density matter. Here, we simply state this condition to be that $\phi_{\mu\nu}$ is traceless.

Consider gravitational waves in free space that propagate in the z-direction,

$$\phi_{\mu\nu} = \epsilon_{\mu\nu}\, e^{ik(z-ct)}, \tag{23}$$

where $\epsilon_{\mu\nu}$ is the (constant) polarization tensor....

4. **Classical Aspects of the Aharonov-Bohm Effect**

In 1926 Fock noted[13] that Schrödinger's equation for an electric charge e of mass m in electromagnetic fields described by potentials $A_\mu = (\phi, \mathbf{A})$ can be written

$$\frac{(-i\mathbf{D})^2}{2m}\psi = iD_0\psi, \qquad \text{using the "altered" derivative} \qquad D_\mu = \partial_\mu + ieA_\mu, \tag{24}$$

which is gauge invariant only if the gauge transformation of the potentials, $A_\mu(x_\nu) \to A_\mu + \partial_\mu\Omega(x_\nu)$, is accompanied by a phase change of the wavefunction, $\psi(x_\nu) \to$

[13] V. Fock, *Über die invariante Form der Wellen- und der Bewegungsgleichungen für einen geladenen Massenpunkt*, Z. Phys. **39**, 226 (1926),
http://kirkmcd.princeton.edu/examples/QM/fock_zp_39_226_26.pdf.
See also, F. London, *Quantenmechanische Deutung der Theorie von Weyl*, Z. Phys. **42**, 375 (1927),
http://kirkmcd.princeton.edu/examples/QM/london_zp_42_375_27.pdf.
See also, p. 206 of W. Pauli, *Relativistic Field Theories of Elementary Particles*, Rev. Mod. Phys. **13**, 203 (1941), http://kirkmcd.princeton.edu/examples/EP/pauli_rmp_13_203_41.pdf.

$e^{-ie\Omega(x_\nu)}\psi$. Yang and Mills (1954)[14] may have been the first to point out that Fock's argument can be inverted such that a requirement of local phase invariance of the form $\psi(x_\nu) \to e^{-ie\Omega(x_\nu)}\psi$ implies the existence of an interaction described by a potential A_μ (and charge e) which satisfies gauge invariance and modifies Schrödinger's equation via the altered derivative D_μ. This led to a greater appreciation of the significance of potential in the quantum realm.

Separately, consideration of possible interference effects in electron microscopy[15] led Aharonov and Bohm[16] to consider an electron that moves only outside a long solenoid magnet (where $\mathbf{B} = 0$ to a good approximation), and which accumulates a different phase, related to the vector potential $\mathbf{A}$, in its wavefunction depending on which side of the magnet it passes. The resulting interference pattern, which depends on the (gauge-invariant) magnetic flux in the solenoid, has been observed in subsequent experiments.[17] *This result is often misinterpreted as evidence that the vector potential* $\mathbf{A}$ *is "observable" in the quantum realm. A better statement is that there exist quantum-electrodynamic effects on the behavior of an electron which moves only in a region of zero external electric and magnetic field, but where the vector potential (in any choice of gauge) is nonzero. Note that the observed result relates directly to the magnetic field* $\mathbf{B}$ *rather than to the vector potential* $\mathbf{A}$*; the paradox is more that the observed quantum effect seems to be action-at-a-distance (as bothered Einstein, Podolsky and Rosen in another context) between the magnetic field and the electron.*

The quantum interference effect in the Aharonov-Bohm experiment is impressive, but there are already disconcerting issues in purely classical considerations thereof. It is often remarked that there is no classical effect on an electron that passes outside a long solenoid magnet, where $\mathbf{B}_{\text{solenoid}} = 0$. However, the current density that generates the solenoid field is affected by the magnetic field of the moving electron (even assuming that the electric charge density associated with the current density is zero).

Problem: Deduce the force on a solenoid of radius a about the z-axis that carries azimuthal surface current density $K_\theta = I$ per unit length, when an electron of velocity $\mathbf{v} = v\,\hat{\mathbf{y}}$ is at position $(x, y, z) = (b, vt, 0)$, where $v \ll c$ and $|b| \gg a$.[18]

That is, Newton's third law is not obeyed by this configuration!

Issues like this were noted by Ampère in the 1820's and led him to doubt the existence of isolated, moving electric charges, which view put particle physics on hold for 60

[14]C.N. Yang and R.L. Mills, *Conservation of Isotopic Spin and Isotopic Gauge Invariance*, Phys. Rev. **96**, 191 (1954), http://kirkmcd.princeton.edu/examples/EP/yang_pr_96_191_54.pdf.

[15]See. p. 21 of W. Ehrenberg and R.E. Siday, *The Refractive Index in Electron Optics and the Principles of Dynamics*, Proc. Phys. Soc. London B **62**, 8 (1949), http://kirkmcd.princeton.edu/examples/EM/ehrenberg_ppsl_b62_8_49.pdf.

[16]Y. Aharonov and D. Bohm, *Significance of Electromagnetic Potentials in Quantum Theory*, Phys. Rev. **115**, 485 (1959), http://kirkmcd.princeton.edu/examples/QM/aharonov_pr_115_485_59.pdf.

[17]R.G. Chambers, *Shift of an Electron Interference Pattern by an Enclosed Magnetic Flux*, Phys. Rev. Lett. **5**, 3 (1960), http://kirkmcd.princeton.edu/examples/QM/chambers_prl_5_3_60.pdf.

[18]Assume that the magnetic field of the electron is not "shielded" by the solenoid, which shielding would imply additional currents that create additional magnetic field external to the solenoid that lead to a force on the moving electron.

years. Only after Poynting (1884)[19] developed the notion that electromagnetic fields can support a flux of energy (and hence also contain momentum), did physicists have the confidence to reconsider the concept of elementary charged particles.

In retrospect we note that the issue of apparent violation of Newton's third law could have been resolved earlier, based on Faraday's insight that what we now call the vector potential $\mathbf{A}$ (called the "electrotonic state" by Faraday[20]) can be associated with "electromagnetic momentum," as formulated mathematically by Maxwell.[21] In Gaussian units, the electromagnetic momentum associated with a charge distribution ϱ that is immersed in a vector potential $\mathbf{A}$ (in the Coulomb gauge, strictly speaking) is given (for quasistatic motion) by[22,23]

$$\mathbf{P}_{\rm EM} = \int \frac{\varrho \mathbf{A}}{c}\, d\text{Vol}. \tag{25}$$

Problem: Use eq. (25) to deduce the electromagnetic momentum of the electron + solenoid when the electron is at $(x, y, z) = (b, vt, 0)$, and from this show that $d\mathbf{P}_{\rm EM}/dt$ is equal and opposite to the force on the solenoid found previously.

This seems to be a satisfactory resolution to the issue of momentum conservation, but a disconcerting result remains. Suppose the electric charge were at rest; then the electromagnetic momentum (25) is nonzero, while the solenoid is at rest also and seems to contain no net momentum. Hence, we have an example of a system at rest which seems to contain nonzero total momentum!

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[19] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trains. Roy. Soc. London **175**, 343 (1884), http://kirkmcd.princeton.edu/examples/EM/Poynting_ptrsl_175_343_84.pdf.

[20] See Arts. 60, 1661, 1729 and 1733 of M. Faraday, *Experimental Researches in Electricity* (London, 1839), http://kirkmcd.princeton.edu/examples/EM/faraday_exp_res_v1.pdf.

[21] Secs. 22-24 and 57 of J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865), http://kirkmcd.princeton.edu/examples/EM/maxwell_ptrsl_155_459_65.pdf.

[22] The Faraday-Maxwell form (25) is a classical effect of the solenoid on the electron, but it does not imply that the vector potential is observable in classical electrodynamics. Rather, we note that it is equivalent to the Poynting-Poincaré form, $\mathbf{P}_{\rm EM} = \int \mathbf{E} \times \mathbf{B}\, d\text{Vol}/4\pi c$, as shown, for example, in K.T. McDonald, *Four Expressions for Electromagnetic Field Momentum* (April 10, 2006), http://kirkmcd.princeton.edu/examples/pem_forms.pdf. While the Faraday-Maxwell form for the electromagnetic momentum suggests that this resides with the electron, the Poynting-Poincaré form suggests that it resides in the solenoid. This classical ambiguity is a preview of the Aharonov-Bohm effect that an electron can be affected by an electromagnetic field even if the latter is zero at the electron.

[23] The Coulomb-gauge vector potential $\mathbf{A}^{(C)}$ is "rotational," meaning that $\boldsymbol{\nabla} \cdot \mathbf{A}^{(C)} = 0$. In a general gauge, the vector potential can be written (using Helmholtz' theorem) as $\mathbf{A} = \mathbf{A}_{\rm irr} + \mathbf{A}_{\rm rot}$ where $\boldsymbol{\nabla} \times \mathbf{A}_{\rm irr} = 0$ and $\boldsymbol{\nabla} \cdot \mathbf{A}_{\rm rot} = 0$. Then, a gauge transformation $\mathbf{A} \to \mathbf{A} - \boldsymbol{\nabla}\Omega$, $\phi \to \phi + \partial\Omega/\partial ct$ implies that $\mathbf{A}_{\rm irr} + \mathbf{A}_{\rm rot} \to (\mathbf{A}_{\rm irr} - \boldsymbol{\nabla}\Omega) + \mathbf{A}_{\rm rot}$, such that $\mathbf{A}_{\rm rot}$ is actually a gauge-invariant quantity. Note that $\mathbf{A}_{\rm rot} = \mathbf{A}^{(C)}$, *i.e.*, the rotational part of the vector potential in any gauge is the Coulomb-gauge vector potential.

It is sometimes said that gauge-invariant quantities are "observable." However, the Coulomb-gauge vector potential $\mathbf{A}^{(C)} = \mathbf{A}_{\rm rot}$ in general involves instanteous effects of distant currents, so claims that the Coulomb-gauge vector potential is "observable" are associated with claims that instantaneous action at a distance is also "observable." This author takes the view that instantaneous action at a distance is not "observable" (even in the quantum realm) and that the Coulomb-gauge vector potential is not "observable."

See also K.T. McDonald, *Orbital and Spin Angular Momentum of Electromagnetic Fields* (Mar. 12, 2009), http://kirkmcd.princeton.edu/examples/spin.pdf.

Peculiarities of this sort were first noticed by Shockley in 1967,[24] and remain an arcane aspect of classical physics, where some systems contain "hidden" momentum (such that systems "at rest" indeed have zero total momentum). One can give a semiplausible classical model of the so-called "hidden" momentum in the present example,[25] whose main significance for the Aharonov-Bohm effect is to remind us that even in a "classical" view, the electron is "entangled" with the magnetic field of the solenoid, although that field happens to be zero at the location of the electron. While the field of the solenoid has no "classical" effect on the electron, the electron does have a "classical" effect on the solenoid, so the two objects should not be regarded as independent entities. In this context, it should be pleasing, rather than disturbing, that in the quantum realm the solenoid has an effect on the electron.[26]

In the author's view, the Aharonov-Bohm effect (and the related debate about the "observability" of potentials[27]) misses the point that the role of the potentials (which must obey gauge invariance) combined with the notion of local phase invariance is to determine the form of the interactions of elementary particles. It is the nonobservability of the potentials, because they are subject to gauge transformations, which leads the potentials to be included in the altered derivatives D_μ, that makes them so important in the development of the theory of elementary particles and fields.

[24] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on "Magnetic Currents,"* Phys. Rev. Lett. **18**, 876 (1967),
http://kirkmcd.princeton.edu/examples/EM/shockley_prl_18_876_67.pdf.

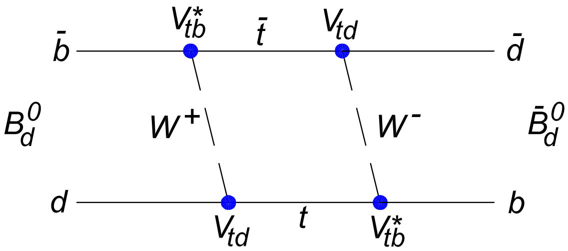
[25] See, for example, K.T. McDonald, *"Hidden" Momentum in a Current Loop* (June 30, 2012),
http://kirkmcd.princeton.edu/examples/penfield.pdf.

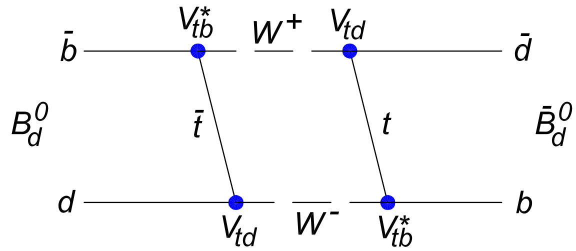
[26] This theme was developed by Aharonov for the "dual" example in which a loop of current (magnetic dipole) interacts with a line charge parallel to the axis of the loop: Y. Aharonov, P. Pearle and L. Vaidman, *Comment on "Proposed Aharonov-Casher effect: Another example of an Aharonov-Bohm effect arising from a classical lag,"* Phys. Rev. A **37**, 4052 (1988),
http://kirkmcd.princeton.edu/examples/QM/aharonov_pra_37_4052_88.pdf.
See also L. Vaidman, *Role of potentials in the Aharonov-Bohm effect*, Phys. Rev. A **86**, 040101 (2012),
http://kirkmcd.princeton.edu/examples/QM/vaidman_pra_86_040101_12.pdf.

[27] Y. Aharonov and D. Bohm, *Further Discussion of the Role of Electromagnetic Potentials in the Quantum Theory*, Phys. Rev. **130**, 1625 (1963), and references therein,
http://kirkmcd.princeton.edu/examples/QM/aharonov_pr_130_1625_63.pdf.

Solutions

1. a) $B_d^0 \to \bar{B}_d^0$ mixing is described by the two box diagrams:

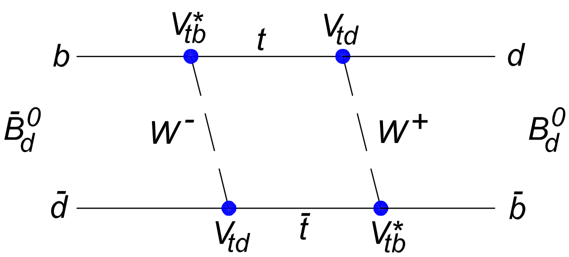


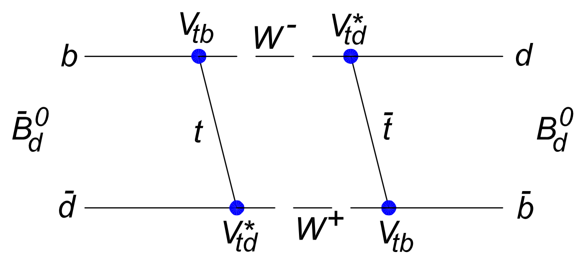


The Lagrangian (2) tells us the vertex involving the d quark in the initial state, and that with the $\bar{d}$ quark in the final state, have coupling factor V_{td} of the V_{CKM} matrix, while the vertex with the $\bar{b}$ quark in the initial state, and that with the b quark in the final state have coupling factor $V_{tb}^\star$ of the Hermitian conjugate matrix $V_{\text{CKM}}^\dagger$. Hence, the matrix element is

$$\langle \bar{B}^0 | B^0 \rangle \propto (V_{td} V_{tb}^*)^2 = V_{tb}^2 \, |V_{td}|^2 \, e^{2i\phi_{td}}. \tag{26}$$

The diagrams for the conjugate process $\bar{B}_d^0 \to B_d^0$ are



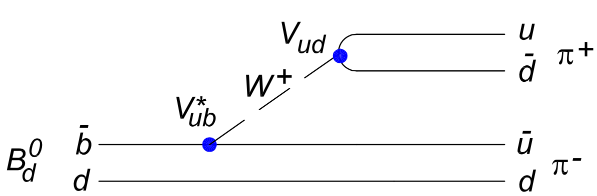


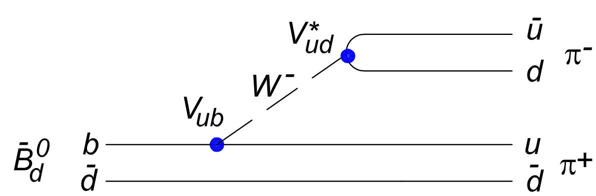
Hence, the matrix element for the conjugate process is

$$\langle B^0 | \bar{B}^0 \rangle \propto (V_{td}^* V_{tb})^2 = V_{tb}^2 \, |V_{td}|^2 \, e^{-2i\phi_{td}}. \tag{27}$$

If CP were conserved, the matrix elements for the conjugate processes should have equal values, and hence ϕ_{td} would be zero.

b) The tree diagrams for the decays $B^0(\bar{B}^0) \to \pi^+\pi^-$ are





The matrix element for $B^0 \to \pi^+\pi^-$ is read off of the first diagram as

$$\langle \pi^+\pi^- | B^0 \rangle \propto V_{ud} V_{ub}^* = V_{ud} \, |V_{ub}| \, e^{-i\phi_{ub}}. \tag{28}$$

while the matrix element for $\bar{B}^0 \to \pi^+\pi^-$ is read off of the second diagram as

$$\langle \pi^+\pi^- | \bar{B}^0 \rangle \propto V_{ud}^* V_{ub} = V_{ud} V_{ub} = V_{ud} \, |V_{ub}| \, e^{i\phi_{ub}}. \tag{29}$$

Again, if CP were conserved, we expect that these two matrix elements would have equal values (since $CP\ |\pi^+\pi^-\rangle = +|\pi^+\pi^-\rangle$ for a spin-0 state), so that ϕ_{ub} would be zero.

The simplicity of the results (28)-(29) is marred by the existence of so-called "penguin" (loop) diagrams for the decay $B^0 \to \pi^+\pi^-$, which diagrams have different weak phases, and whose amplitudes relative to that of the tree diagrams are difficult to calculate. In principle, the relative amplitudes of the penguin and tree diagrams can be resolved by a combined analysis of the decays $B_d^\pm \to \pi^\pm\pi^0$, $B_d^0(\bar{B}_d^0) \to \pi^0\pi^0$, and $B_d^0(\bar{B}_d^0) \to \pi^+\pi^-$, although this will be quite difficult in practice. See, for example, http://kirkmcd.princeton.edu/examples/cp_primer.pdf.

c) Using the results of part a), the time evolution (1) an initially pure B^0 can now be written

$$B^0 \to \cos\frac{\Delta m\, t}{2} B^0 + i\sin\frac{\Delta m\, t}{2}\, e^{2i\phi_{td}}\,\bar{B}^0. \tag{30}$$

Using the result of part b) the amplitude for the decay of $B^0(t)$ to $\pi^+\pi^-$ can now be written

$$\begin{aligned}\langle\pi^+\pi^-|B^0(t)\rangle &\propto \cos\frac{\Delta m\, t}{2}\, e^{-i\phi_{ub}} B^0 + i\sin\frac{\Delta m\, t}{2}\, e^{2i\phi_{td}} e^{i\phi_{ub}}\,\bar{B}^0 \\ &\propto \cos\frac{\Delta m\, t}{2} B^0 + i\sin\frac{\Delta m\, t}{2}\, e^{2i(\phi_{td}+\phi_{ub})}\,\bar{B}^0,\end{aligned} \tag{31}$$

and the decay rate is

$$\Gamma_{B^0_{\rm phys}(t)\to\pi^+\pi^-} \propto \left|\cos\frac{\Delta m\, t}{2} + i\sin\frac{\Delta m\, t}{2}\, e^{2i(\phi_{td}+\phi_{ub})}\right|^2. \tag{32}$$

The interference terms in the rate are

$$\begin{aligned}& i\sin\frac{\Delta m\, t}{2}\cos\frac{\Delta m\, t}{2}\, e^{2i(\phi_{td}+\phi_{ub})} - i\sin\frac{\Delta m\, t}{2}\cos\frac{\Delta m\, t}{2}\, e^{-2i(\phi_{td}+\phi_{ub})} \\ \propto\ & -\sin 2(\phi_{td}+\phi_{ub})\sin\Delta m\, t,\end{aligned} \tag{33}$$

which causes a directly measurable change in the time-dependent decay rate due to the CP violating phases ϕ_{td} and ϕ_{ub}.

If we had begun at $t = 0$ with a pure $\bar{B}^0$ state, the sign of the interference term would be reversed. Hence, the CP -violating interference term (33) leads to a nonzero asymmetry of time-dependent rate of decay of initially pure B^0 and $\bar{B}^0$ states.

To observe the effect (33), we must know that at $t = 0$ the B was in fact a pure B^0. But in general, B mesons are produced in particle-antiparticle pairs, and both the B and $\bar{B}$ immediately begin to evolve into mixed states. Decays of the second B to final states that are not self conjugate are often used to determine whether it was a B or $\bar{B}$ at the time of its decay. If the B-$\bar{B}$ pair was produced in a pure 2-particle quantum state, then we would know that the first B was the anti of the second B, but only at the moment of the decay of the second particle – even though the two particles are separated in space (EPR paradox, Prob. 6, Set 9).

d) There are 6 unitary constraints from the orthogonality of the rows and columns of the matrix $V_{\rm CKM}$. Choosing columns 1 and 3,

$$0 = V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = V_{ud}V_{ub}^* + V_{cd}V_{cb} + V_{td}V_{tb}. \tag{34}$$

The imaginary part of this relation is

$$-V_{ud}Im(V_{ub}) + Im(V_{td})V_{tb} = 0. \tag{35}$$

Hence,

$$\frac{Im(V_{ub})}{Im(V_{td})} = \frac{V_{tb}}{V_{ud}} = \frac{1}{\cos\theta_C} = \frac{1}{0.97} \approx 1.03. \tag{36}$$

Here we have used the facts that $u \to d + X$ is a dominant transition with $V_{ud} = \cos\theta_C$, where $\theta_C \approx 0.22$ is the Cabibbo angle, and the process $t \to b + X$ is dominant transition with $V_{tb} \approx 1$.

2. The vertex factor for the photon radiated in the process $\mu \to e\gamma$ is the charge e, so the decay rate will include an additional factor of $e^2 = \alpha$ compared to that for $\mu \to e\overline{\nu}_e\nu_m u$ (which is the dominant decay mode).

   While the μ can convert into ν_μ with emission of a W boson, the ν_μ cannot convert to an electron if the muon and electron have different lepton number. However, a ν_μ oscillates into a ν_e over distance L with probability (given on p. 357, Lecture 19 of the Notes)

$$P_{\nu_\mu\to\nu_e}(L, E) = \sin^2 2\theta_{12}\sin^2\frac{\Delta m_{12}^2 L}{4E}\,. \tag{37}$$

   In the decay $\mu \to e\gamma$ the neutrino energy is of order m_μ, and the distance L is c times the characteristic time scale for a weak interaction, $\tau \approx 1/m_W$. Since this L is very small, the decay rate for $\mu \to e\gamma$ also includes an additional factor of $\sin^2 2\theta_{12}(\Delta m_{12}^2/m_\mu m_W)^2$.

   Altogether, we estimate the branching fraction to be

$$B_{\mu\to e\gamma} \approx \alpha\sin^2 2\theta_{12}\left(\frac{\Delta m_{12}^2}{m_\mu m_W}\right)^2. \tag{38}$$

   The present best value of $\sin^2 2\theta_{12}$ is 0.86, while $\Delta m_{12}^2 \approx 8\times10^{-5}$ eV^2, so eq. (38) predicts the branching fraction to be $\approx 10^{-32}$.

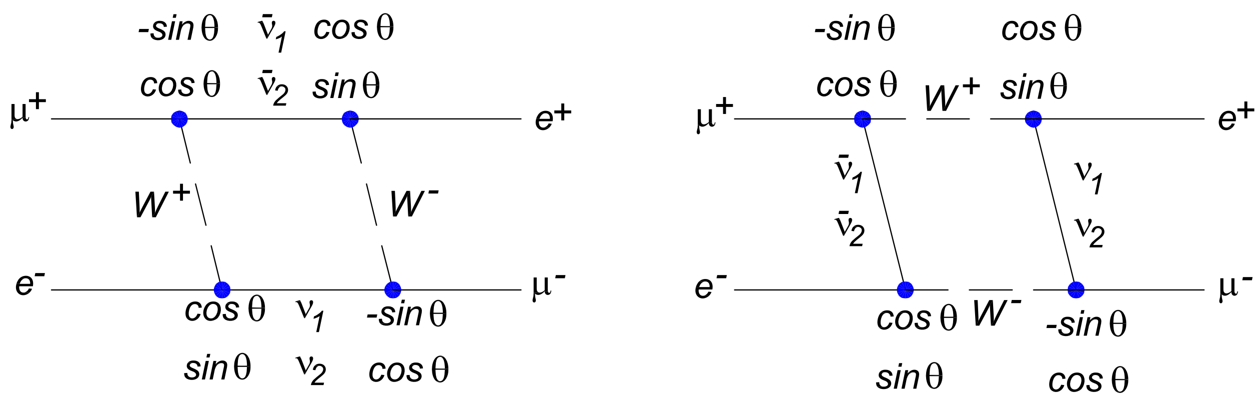
   Detailed computation[28] *indicates that the above diagram leads to a branching fraction of roughly* $\alpha\sin^2 2\theta_{12}\,(\Delta m_{12}^2/M_W^2)^2 \approx 10^{-38}$. *That is, the quick estimate present above misses a factor* m_μ^2/m_w^2 *associated (I believe) with details of the loop diagram.*

[28] S. Petcov, *The processes* $\mu \to e + \gamma$, $\mu \to e + e + \bar{e}$, $\nu' \to \nu + \gamma$ *in the Weinberg-Salam model with neutrino mixing*, Sov. J. Nucl. Phys. **25**, 340 (1977),
http://kirkmcd.princeton.edu/examples/EP/petcov_sjnp_25_340_77.pdf.
B.W. Lee and R.E. Schrock, *Natural suppression of symmetry violation in gauge theories: Muon- anti electron-lepton-number nonconservation*, Phys. Rev. D **16**, 1444 (1977),
http://kirkmcd.princeton.edu/examples/EP/lee_prd_16_1444_77.pdf.

In a two-neutrino model the neutrino flavor states ν_e, ν_μ are related to the mass eigenstates ν_1, ν_2 by the unitary transformations

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \qquad \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}, \tag{39}$$

where the mixing angle θ is called θ_{12} in the three-neutrino scenario. In the mesonium-antimesonium oscillation diagrams, a μ couples to ν_μ, and hence to ν_1 with strength $-\sin\theta$ and to ν_2 with strength $\cos\theta$, *etc.* The relative coupling strengths at the four vertices of the diagrams are illustrated in the figure below.



Each of the two diagrams has four variants, and the sum of the relative amplitudes of these four variants is

$$\begin{aligned} &[(-\sin\theta)\cos\theta][\cos\theta(-\sin\theta)] + [(-\sin\theta)\cos\theta][\sin\theta\cos\theta)] \\ &+[\cos\theta\sin\theta)][\cos\theta(-\sin\theta)] + [\cos\theta\sin\theta)][\sin\theta\cos\theta)] = 0. \end{aligned} \tag{40}$$

Thus, there is no amplitude for mesonium-antimesonium oscillations in the massive two-neutrino scenario even though there are $\nu_1 \leftrightarrow \nu_2$ oscillations. *This is an illustration of the so-called GIM mechanism for the lepton sector.*[29]

3. We consider gravitational waves in free space that propagate in the z-direction, characterized by the tensor potential,

$$\phi_{\mu\nu} = \epsilon_{\mu\nu}\, e^{ik(z-ct)}, \tag{41}$$

where $\epsilon_{\mu\nu}$ is the (constant) polarization tensor.

The Lorenz-like condition (22) tells us that

$$k^\mu \epsilon_{\mu\nu} = 0, \qquad k_\mu = kc(1,0,0,1), \qquad \Rightarrow \epsilon_{0\nu} = \epsilon_{3\nu}. \tag{42}$$

The requirement of gauge invariance of the potentials $\phi_{\mu\nu}$ tells us that the transformation

$$\phi_{\mu\nu} \to \phi'_{\mu\nu} = \phi_{\mu\nu} + \partial_\mu \Omega_\nu + \partial_\nu \Omega_\mu \tag{43}$$

[29]S.L. Glashow, J. Iliopoulos, and L. Miani, *Weak Interactions with Lepton-Hadron Symmetry*, Phys. Rev. D **2**, 1285 (1970), http://kirkmcd.princeton.edu/examples/EP/glashow_prd_2_1285_70.pdf.

does not change the physics provided the 4-vector Ω_μ satisfies the free-space wave equation,

$$\partial_\nu \partial^\nu \Omega_\mu = 0 = \frac{1}{c^2}\frac{\partial^2 \Omega_\mu}{\partial t^2} - \nabla^2 \Omega_\mu. \tag{44}$$

Hence, we can consider

$$\Omega_\mu = \chi_\mu \, e^{ik(z-ct)}, \tag{45}$$

for any constant 4-vector χ_μ. Applying the gauge transformation (45) to the wave potentials (44), the transformed polarization states are

$$\epsilon'_{\mu\nu} = \epsilon_{\mu\nu} + k_\mu \chi_\nu + k_\nu \chi_\mu. \tag{46}$$

We choose the four constants χ_μ to eliminate the $\epsilon'_{0\nu}$:

$$\epsilon'_{00} = \epsilon_{00} + 2k_0\chi_0 = \epsilon_{00} + 2kc\chi_0, \qquad \Rightarrow \chi_0 = -\epsilon_{00}/2kc, \tag{47}$$

$$\epsilon'_{01} = \epsilon_{01} + k_0\chi_1 + k_1\chi_0 = \epsilon_{01} + kc\chi_1, \qquad \Rightarrow \chi_1 = -\epsilon_{01}/kc, \tag{48}$$

$$\epsilon'_{02} = \epsilon_{02} + k_0\chi_2 + k_2\chi_0 = \epsilon_{02} + kc\chi_2, \qquad \Rightarrow \chi_2 = -\epsilon_{02}/kc, \tag{49}$$

$$\epsilon'_{03} = \epsilon_{03} + k_0\chi_3 + k_3\chi_0 = \epsilon_{03} + kc\chi_3 - \epsilon_{00}/2, \qquad \Rightarrow \chi_3 = (\epsilon_{00}/2 - \epsilon_{03})/kc. \tag{50}$$

So far,

$$\epsilon'_{00} = \epsilon'_{01} = \epsilon'_{02} = \epsilon'_{03} = 0. \tag{51}$$

The Lorenz-like condition (42), applied to $\epsilon'_{\mu\nu}$, tells us that

$$\epsilon'_{30} = \epsilon'_{31} = \epsilon'_{32} = \epsilon'_{33} = 0. \tag{52}$$

Since $\epsilon'_{\mu\nu}$ is symmetric, we also have that

$$\epsilon'_{10} = \epsilon'_{20} = \epsilon'_{13} = \epsilon'_{23} = 0. \tag{53}$$

The remaining nonzero components are ϵ'_{11}, ϵ'_{22} and $\epsilon'_{12} = \epsilon'_{21}$. Since $\epsilon'_{\mu\nu}$ is traceless, $\epsilon'_{22} = -\epsilon'_{11}$, and the polarization tensor $\epsilon'_{\mu\nu}$ has only two independent (transverse) components, ϵ'_{11} and ϵ'_{12},[30]

$$\epsilon'_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon'_{11} & \epsilon'_{12} & 0 \\ 0 & \epsilon'_{12} & -\epsilon'_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{54}$$

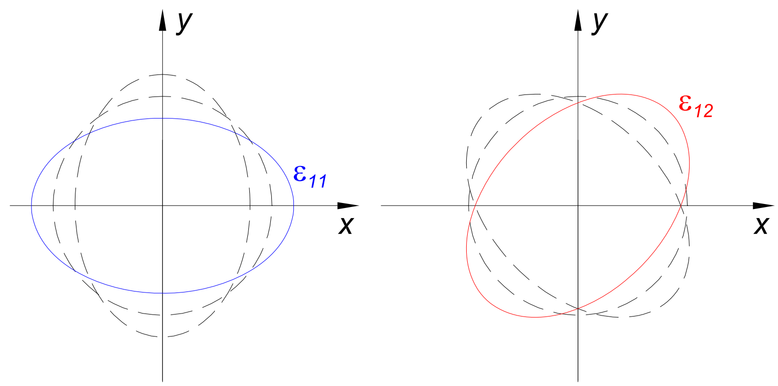
[30]The fact that a massless particle with spin S can have only two spin (polarization) states $S_z = \pm S$ when propagating in the z direction is discussed from the perspective of relativistic invariance in E. Wigner, *Relativistic Invariance and Quantum Phenomena*, Rev. Mod. Phys. **29**, 255 (1957), http://kirkmcd.princeton.edu/examples/QED/wigner_rmp_29_255_57.pdf.

Physical Significance of the Two Polarizations

The gravitational waves parameterized by ϵ_{11} and ϵ_{12} (dropping the $'$s in eq. (54)) perturb the weak-field metric tensor according to eq. (20), and so affect the invariant interval between two events,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu + \phi_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu + \epsilon_{\mu\nu}\, e^{ik(z-ct)} dx^\mu dx^\nu. \qquad (55)$$

If a gravitational plane wave of polarization ϵ_{11} is incident on a massive sphere, the x-separation between pairs of points increases, while the y-separation decreases. As a result, the sphere is (slightly) deformed into an ellipsoid, as sketched on the left below. *Of course, half a wave period later, the x-separation has decreased and the y-separation has increased.*



In contrast, a wave of polarization ϵ_{12} increases the separation between points for which $dx = dy$, and decreases the separation when $dx = -dy$, as shown on the right in the sketch above.

These oscillatory deformations have a quadrupole character, with the two polarization states rotated by 45° with respect to one another (compared to the 90° rotation between x and y linear polarizations of electromagnetic waves). *Likewise, the lowest multipole of gravitational waves emitted by an oscillating mass is the quadrupole.*

4. The magnetic field $\mathbf{B}_e$ at position $\mathbf{x}$ of an electron of charge $-e$ and velocity $\mathbf{v}$ at position $\mathbf{x}_e$ is (in Gaussian units, and for $v \ll c$)

$$\mathbf{B}_e(\mathbf{x}, \mathbf{x}_e) = \frac{\mathbf{v}}{c} \times \mathbf{E}_e(\mathbf{x}, \mathbf{x}_e) = -\frac{e\mathbf{v} \times \mathbf{R}}{cR^3}, \qquad \text{where} \qquad \mathbf{R} = \mathbf{x} - \mathbf{x}_e. \qquad (56)$$

The force of this magnetic field from an electron at $(x, y, z) = (b, vt, 0)$ on a solenoidal (surface) current density $K_\theta(r = a, \theta, z) = I$ per unit length is

$$\begin{aligned}
\mathbf{F} &= \int \frac{\mathbf{K} \times \mathbf{B}_e}{c}\, d\text{Area} \\
&= -\frac{Iev}{c^2} \int_{-\infty}^{\infty} dz \int_0^{2\pi} a\, d\theta\, \frac{(-\sin\theta, \cos\theta, 0) \times \{\hat{\mathbf{y}} \times [(a\cos\theta, a\sin\theta, z) - (b, vt, 0)\}}{[(a\cos\theta - b)^2 + (a\sin\theta - vt)^2 + z^2]^{3/2}} \\
&= -\frac{aIev}{c^2} \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\theta\, \frac{(-\sin\theta, \cos\theta, 0) \times (z, 0, b - a\cos\theta)}{(a^2 + b^2 + v^2t^2 - 2ab\cos\theta - 2avt\sin\theta + z^2)^{3/2}}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{aIev}{c^2}\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz\, \frac{[\cos\theta(b-a\cos\theta), \sin\theta(b-a\cos\theta), -z\cos\theta]}{(a^2+b^2+v^2t^2-2ab\cos\theta-2avt\sin\theta+z^2)^{3/2}} \\
&= -\frac{2aIev}{c^2}\int_0^{2\pi} d\theta\, \frac{[\cos\theta(b-a\cos\theta), \sin\theta(b-a\cos\theta), 0]}{b^2+v^2t^2-2ab\cos\theta-2avt\sin\theta+a^2} \\
&\approx -\frac{2aIev}{c^2(b^2+v^2t^2)}\int_0^{2\pi} d\theta\, [\cos\theta\,(b-a\cos\theta), \sin\theta\,(b-a\cos\theta), 0] \\
&\qquad\qquad \left(1+\frac{2ab}{b^2+v^2t^2}\cos\theta+\frac{2avt}{b^2+v^2t^2}\sin\theta\right) \qquad (57) \\
&= -\frac{2\pi a^2 Iev}{c^2(b^2+v^2t^2)}\left[-1+\frac{2b^2}{b^2+v^2t^2}, \frac{2bvt}{b^2+v^2t^2}, 0\right] = -\frac{2\pi a^2 Iev}{c^2(b^2+v^2t^2)}\left[\frac{b^2-v^2t^2}{b^2+v^2t^2}, \frac{2bvt}{b^2+v^2t^2}, 0\right].
\end{aligned}
$$

This force is very small, being of order $1/c^2$, and clarification of its possible effect on the system is more of "academic" than practical interest. Note that $\pi a^2 I/c$ is the magnetic moment per unit length along the solenoid

The uniform magnetic field $\mathbf{B}_{\text{solenoid}} = B\,\hat{\mathbf{z}}$ inside the solenoid has magnitude $B = 4\pi I/c$, as follows from Ampère's law. This field can also be deduced from a vector potential $\mathbf{A}$ whose only nonzero component in a cylindrical coordinate system (r, θ, z) is $A_\theta(r)$, where $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ implies for a loop of radius r that

$$
\oint \mathbf{A}\cdot d\mathbf{l} = 2\pi r A_\theta = \int \boldsymbol{\nabla}\times\mathbf{A}\cdot d\mathbf{Area} = \mathbf{B}\cdot d\mathbf{Area} = \frac{4\pi^2 I}{c}\begin{cases} r^2 & (r<a), \\ a^2 & (r>a). \end{cases} \qquad (58)
$$

Outside the solenoid, the magnetic field is zero while the vector potential is

$$
A_\theta(r>a) = \frac{2\pi a^2 I}{cr}, \qquad \mathbf{A} = \frac{2\pi a^2 I}{cr^2}(-y, x, 0). \qquad (59)
$$

According to eq. (25) of Faraday and Maxwell, the system of electron plus solenoid has electromagnetic momentum

$$
\mathbf{P}_{\text{EM}} = -\frac{e\mathbf{A}(b, vt, 0)}{c} = -\frac{2\pi a^2 Ie}{c^2(b^2+v^2t^2)}(-vt, b, 0)\,. \qquad (60)
$$

The time derivative of this is

$$
\begin{aligned}
\frac{d\mathbf{P}_{\text{EM}}}{dt} &= -\frac{2\pi a^2 Iev}{c^2(b^2+v^2t^2)}\left[-1+\frac{2v^2t^2}{b^2+v^2t^2}, \frac{-2bvt}{b^2+v^2t^2}, 0\right] \\
&= \frac{2\pi a^2 Iev}{c^2(b^2+v^2t^2)}\left[\frac{b^2-v^2t^2}{b^2+v^2t^2}, \frac{2bvt}{b^2+v^2t^2}, 0\right] = -\mathbf{F} = -\frac{d\mathbf{P}_{\text{mech}}}{dt}\,, \qquad (61)
\end{aligned}
$$

on comparison with eq. (57). Thus,

$$
\mathbf{P}_{\text{EM}} = -\frac{e\mathbf{A}(b, vt, 0)}{c} = -\frac{2\pi a^2 Ie}{c^2(b^2+v^2t^2)}(-vt, b, 0)\,. \qquad (62)
$$

The time derivative of this is

$$\frac{d\mathbf{P}_{\rm total}}{dt} = \frac{d\mathbf{P}_{\rm EM}}{dt} + \frac{d\mathbf{P}_{\rm mech}}{dt} = 0, \tag{63}$$

and the total momentum of the system is constant in time. The electrical current in the solenoid carries momentum, but naïvely we expect that the total mechanical momentum of a current loop would be zero; however, this is not the case if the current loop is subject to an external electric field, as in the present example.

The unbalanced force of the moving electron on the solenoid serves to change its "hidden" internal mechanical momentum, while the bulk of the solenoid remains at rest as the electron passes by.[31,32]

For completeness, we evaluate the electromagnetic momentum according to the Poynting-Poincaré prescription (ignoring the self-field-momentum of the moving electron),

$$\begin{aligned}\mathbf{P}_{\rm EM} &= \int \frac{\mathbf{E}_e \times \mathbf{B}_{\rm solenoid}}{4\pi c}\, d{\rm Vol} \approx \frac{1}{4\pi c}\int_{-\infty}^{\infty} \pi a^2\, dz\, \frac{-e(-b,-vt,z)}{(b^2+v^2t^2+z^2)^{3/2}} \times \frac{4\pi I(0,0,1)}{c} \\ &= -\frac{\pi a^2 Ie(-vt,b,0)}{c^2}\int_{-\infty}^{\infty}\frac{dz}{(b^2+v^2t^2+z^2)^{3/2}} = -\frac{2\pi a^2 Ie(-vt,b,0)}{c^2(b^2+v^2t^2)}\,, \end{aligned} \tag{64}$$

as in eq. (60).

It turns out there is a third way that the electromagnetic momentum can be computed (for quasistatic examples)[33] based on the electric scalar potential ϕ and the current density,

$$\begin{aligned}\mathbf{P}_{\rm EM} &= \int \frac{\phi\mathbf{J}}{c^2}\, d{\rm Vol} = \int \frac{\phi_e\mathbf{K}}{c^2}\, d{\rm Area} \\ &= -\frac{Ie}{c^2}\int_{-\infty}^{\infty} dz \int_0^{2\pi} a\, d\theta\, \frac{(-\sin\theta,\cos\theta,0)}{[(a\cos\theta-b)^2+(a\sin\theta-vt)^2+z^2]^{1/2}} \\ &= -\frac{aIe}{c^2}\int_{-\infty}^{\infty} dz \int_0^{2\pi} d\theta\, \frac{(-\sin\theta,\cos\theta,0)}{(z^2+b^2+v^2t^2-2ab\cos\theta-2avt\sin\theta+a^2)^{1/2}} \\ &\approx -\frac{aIe}{c^2}\int_{-\infty}^{\infty} \frac{dz}{(z^2+b^2+v^2t^2)^{1/2}} \int_0^{2\pi} d\theta\, (-\sin\theta,\cos\theta,0)\left(1+\frac{ab\cos\theta+avt\sin\theta}{z^2+b^2+v^2t^2}\right) \\ &= -\frac{\pi a^2 Ie(-vt,b,0)}{c^2}\int_{-\infty}^{\infty}\frac{dz}{(z^2+b^2+v^2t^2)^{3/2}} = -\frac{2\pi a^2 Ie(-vt,b,0)}{c^2(b^2+v^2t^2)}\,. \end{aligned} \tag{65}$$

[31] All this is rather subtle, and apparently not well known, as a paper based on this example was recently published in Phys. Rev. Lett. claiming that the Lorentz force law must be wrong. For discussion by the author of this dismal issue, see K.T. McDonald, *Mansuripur's Paradox* (May 2, 2012), http://kirkmcd.princeton.edu/examples/mansuripur.pdf.

[32] For a discussion of the character of the "hidden" mechanical momentum in a current loop, see footnote 15, and example 12.12 of D.J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice Hall, 1999), http://kirkmcd.princeton.edu/examples/EM/griffiths_em3.pdf

[33] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621 (1969), http://kirkmcd.princeton.edu/examples/EM/furry_ajp_37_621_69.pdf.

The fact that electromagnetic momentum can be computed several different ways reminds us that even in "classical" systems the components should be regarded as "entangled" rather than "independent."