Ph 406 FINAL EXAM SOLUTIONS

(1) 3 NUCLEON CHARGES: +, −, AND 0

$$A \equiv N_+ + N_0 + N_- \qquad Z \equiv N_+ - N_- = \text{TOTAL CHARGE}$$

a) MASS FORMULA: $M = A(1-a)$ — a = BINDING ENERGY PER NUCLEON IN EXTENDED NUCLEAR MEDIUM

$+ bA^{2/3}$ — LOSS OF BINDING ENERGY AT SURFACE, WHOSE AREA VARIES AS $A^{2/3}$

$+ d\dfrac{Z^2}{A^{1/3}}$ — ELECTROSTATIC ENERGY OF CHARGE Z IS SPHERE OF RADIUS $r \sim A^{1/3}$

$+ s\dfrac{(N_+-N_-)^2 + (N_+-N_0)^2 + (N_0-N_-)^2}{A}$ — SYMMETRY ENERGY IN FERMI-GAS MODEL OF NUCLEUS

$\left(+ \dfrac{\delta}{\sqrt{A}}\right)$ — [PAIRING ENERGY, WHICH WE IGNORE HERE]

OF THESE, ONLY THE SYMMETRY ENERGY MIGHT NOT BE 'OBVIOUS', COMPARED TO THE MASS FORMULA IN OUR UNIVERSE. (IN FACT, WE FIND A SLIGHTLY DIFFERENT FORM BELOW.)

YOU MAY WISH TO RECALL THE FERMI GAS ARGUMENT:

FOR EACH OF THE 3 NUCLEON TYPES $N_i \sim \mathrm{Vol}\, E_{f,i}^{3/2} \sim A\, E_{f,i}^{3/2}$

OR $E_{f,i} \sim \left(\dfrac{N_i}{A}\right)^{2/3}$ = FERMI ENERGY OF TYPE i

$U_i \approx N_i E_{f,i} \sim \dfrac{N_i^{5/3}}{A^{2/3}}$ = ENERGY (MASS) OF TYPE i IN THE NUCLEAR POTENTIAL WELL

NOW $N_i \approx A/3$ SO WRITE $N_i = \dfrac{A}{3} + \left(N_i - \dfrac{A}{3}\right) = \dfrac{A}{3}(1+\epsilon_i)$, $\quad \epsilon_i \equiv \dfrac{3N_i}{A} - 1$

$$U_i \sim \frac{N_i^{5/3}}{A^{2/3}} \approx \frac{A^{5/3}}{A^{2/3}}(1+\epsilon_i)^{5/3} \sim A\left(1 + \tfrac{5}{3}\epsilon_i + \tfrac{1}{2}\left(\tfrac{5}{3}\right)\left(\tfrac{2}{3}\right)\epsilon_i^2 + \dots\right)$$

$$\sum_i U_i \sim A + \frac{s}{A}\sum_i (3N_i - A)^2 \qquad \text{NOTING } \sum_i \epsilon_i = 0$$

FROM THE DEFINITIONS OF A & Z, $\quad N_+ = \dfrac{A + Z - N_0}{2}$, $\quad N_- = \dfrac{A - Z - N_0}{2}$

SO $\quad 3N_+ - A = \dfrac{A}{2} + \dfrac{3Z}{2} - \dfrac{3N_0}{2} = \dfrac{(A - 3N_0) + 3Z}{2}$

$\quad 3N_- - A = \dfrac{A}{2} - \dfrac{3Z}{2} - \dfrac{3N_0}{2} = \dfrac{(A - 3N_0) - 3Z}{2}$

SO $\sum (3N_i - A)^2 = \dfrac{2(A - 3N_0)^2}{4} + 2\cdot\dfrac{9}{4}Z^2 + (3N_0 - A)^2 = \dfrac{3}{2}(A - 3N_0)^2 + \dfrac{9}{2}Z^2$

SO THE SYMMETRY TERM CAN BE WRITTEN AS $\underline{\dfrac{s}{A}\left(Z^2 + \dfrac{(A - 3N_0)^2}{3}\right)}$

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① b) AT FIXED A, $M(Z, N_0) \sim M_0 + M_1 Z^2 + M_2 (A - 3N_0)^2$

THE MINIMUM IS CLEARLY AT $Z = 0$; $N_0 = A/3 \Rightarrow N_+ = N_- = N_0 = A/3$

I.e. β-STABLE NUCLEI HAVE EQUAL NUMBERS OF ALL 3 TYPES OF NUCLEI, AND ZERO (OR VERY SMALL) NET CHARGE

$Z \sim 0 \Rightarrow$ NO COULOMB BARRIER AGAINST FISSION, OR AGAINST 'α' DECAY WHERE $'\alpha' = (2N_+)(2N_-)(2N_0)$

BUT ARE THESE DECAYS ALLOWED BY ENERGY CONSERVATION?

FOR $Z=0$, $N_0 = A/3$, $M(A) = A(1-a) + bA^{2/3}$

FISSION: $A \to A/2 + A/2$ $\quad M(A/2) = \frac{A}{2}(1-a) + b\frac{A^{2/3}}{2^{2/3}}$

$$M(A) - M(A/2) - M(A/2) = bA^{2/3}\left(1 - \frac{2}{2^{2/3}}\right) < 0 \Rightarrow \text{FISSION IS FORBIDDEN!}$$

IF THE 'α' HAS BINDING ENERGY GIVEN BY $M_{'\alpha'} = 6(1-a) + b\,6^{2/3}$

AND $M(A-6) = (A-6)(1-a) + b(A-6)^{2/3}$

$$\sim A^{2/3}\left(1 - \frac{6}{A}\right)^{2/3} \approx A^{2/3}\left(1 - \frac{2}{3}\frac{6}{A}\right)$$

$$\approx (A-6)(1-a) + bA^{2/3} - \frac{4b}{A^{1/3}}$$

$$M(A) - M_{'\alpha'} - M(A-6) = b\left(\frac{4}{A^{1/3}} - 6^{2/3}\right) < 0 \quad \text{FOR } A > 1$$

$\Rightarrow$ 'α' DECAY FORBIDDEN ALSO

$\Rightarrow$ HIGH A NUCLEI ARE STABLE IN THIS UNIVERSE!

IN FACT IF TWO NUCLEI COLLIDE, IT IS ENERGETICALLY FAVORABLE THAT THEY COALESCE

$\Rightarrow$ 'STARS' CONSIST OF A SINGLE GIANT NUCLEUS, WITH $Z \sim 0$

$\Rightarrow$ THEY DON'T GLOW DUE TO ANY NUCLEAR PROCESS (A BIT LIKE A NEUTRON STAR)

SO LIFE IS BAD IN THIS UNIVERSE FOR VARIOUS REASONS:

NO STARS

NO CHEMISTRY AS $Z \lesssim 2$ ALWAYS

$\Rightarrow$ NO LIFE ....

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(2) a) FOR A ROTATIONAL EXCITATION, $E = \frac{1}{2} I \omega^2$ WHERE $\bar{J} = I \bar{\omega}$

SO $E = \frac{|\bar{J}|^2}{2I} = \frac{J(J+1)}{2I}$ NOTING EIGENVALUES OF $\bar{J}^2$

NOW A ROTATING SPHERE (OR EVEN ELLIPSOID) IS SYMMETRIC UNDER THE PARITY TRANSFORMATION $\bar{r} \to -\bar{r}$ $\Rightarrow$ $P = +$ ONLY. BUT $P = (-1)^J$ FOR ROTATIONS, SO HAVE EVEN J ONLY

HENCE $J^P = 0^+, 2^+, \dots (2i)^+$ IS EXPECTED SEQUENCE OF STATES

ALSO, EXPECT

J	0	2	4	6	8
$2I \cdot E = J(J+1)$	0	6	20	42	72

THIS SEQUENCE OF ENERGIES IS FAIRLY WELL MATCHED TO THE DATA, WITH $2I \cdot 100 \sim 6\,(\text{keV})^{-1}$ OR $I \sim 0.03\,(\text{keV})^{-1}$

COMPARE $I \sim \frac{2}{5} M R^2$ FOR $A = 170$

$M \sim 170 \cdot 930$ MeV

$R \sim 1.1\, A^{1/3}$ FERMI ~ 6 F

$I \sim (0.4)(170)(930)\, 36$ MeV - FERMI^2 NOW 1 FERMI $\sim \frac{1}{200\text{ MeV}}$

SO $I \sim \frac{(0.4)(170)(930)(36)}{(200)(200)\text{ MeV}} \sim \frac{50.0}{\text{MeV}} \sim \frac{50.0}{1000\text{ keV}} \sim 0.05\,(\text{keV})^{-1}$

IN FAIR AGREEMENT WITH THE DATA (THO $I_{DATA} < I_{MODEL}$ SUGGESTS THE WHOLE NUCLEUS IS NOT ROTATING AS A RIGID BODY....)

EQUIVALENTLY $I_{MODEL} \sim \frac{50}{1\text{ MeV}} \cdot \frac{200\text{ MeV} \cdot \text{Fm}}{1} \sim 10^4$ FERMI'S

$I_{DATA} \sim 5 \times 10^3$ FERMIS

② b) $^{15}_{7}N$ LEVELS :

IN THE SHELL MODEL THE GROUND STATE HAS NEUTRONS FILLING THE $1S_{1/2}$, $1P_{3/2}$ & $1P_{1/2}$ SHELLS,

PROTONS FILLING THE $1S_{1/2}$ & $1P_{3/2}$ SHELLS, BUT ONE VACANCY IN THE $1P_{1/2}$.

THE UNPAIRED PROTON DETERMINES THE GROUND STATE TO BE $\frac{1}{2}^-$

1ST EXCITED STATE IS $\frac{5}{2}^+$, CONSISTENT WITH THE UNPAIRED PROTON EXCITED TO THE $1d\,5/2$ SHELL

2ND EXCITED STATE IS $\frac{1}{2}^+$: NOW THE UNPAIRED PROTON IS IN THE NEXT HIGHER SHELL, $2S\,1/2$

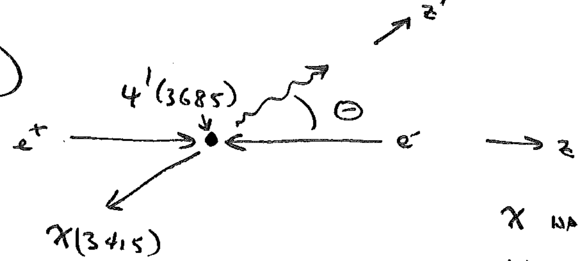
3RD EXCITED STATE IS $3/2^-$; IF THE UNPAIRED PROTON WERE IN THE NEXT HIGHER SHELL, $1d\,3/2$, WOULD EXPECT $3/2^+$!

INSTEAD, COULD BE THAT A PROTON FROM THE $1P_{3/2}$ SHELL IS EXCITED INTO THE $1P_{1/2}$ FILLING THE LATTER. THEN THE UNPAIRED $1P_{3/2}$ PROTON DETERMINES THE STATE AS $3/2^-$.

4TH EXCITED STATE IS $5/2^+$ COULD BE THAT A NEUTRON FROM THE FILLED $1P_{1/2}$ SHELL IS EXCITED TO THE $1d\,5/2$. THEN THE UNPAIRED PROTON & NEUTRON IN THE $1P_{1/2}$ TOGETHER GIVE $J^P = 0^+$, AND THE EXCITED NEUTRON DETERMINES THE STATE TO BE $5/2^+$

5TH EXCITED STATE IS $3/2^+$. THE UNPAIRED PROTON IN THE $1P_{1/2}$ IS EXCITED TO THE $1d\,3/2$ ....

③ a)



e^+e^- ANNIHILATION $\Rightarrow J_z = \pm 1$ ONLY FOR THE $\psi'(3685)$

χ HAS $J=0$ BY HYPOTHESIS

γ HAS $J_{z'} = \pm 1$ ONLY, AS IS REAL PHOTON

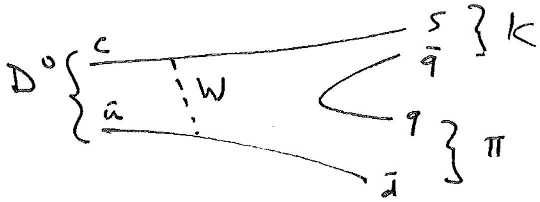
$\Rightarrow$ TOTAL $J_{z'} = \pm 1$ ONLY

SO ANGULAR DISTRIBUTION IS GIVEN BY PROJECTING $J_z = \pm 1$ ONTO $J_{z'} = \pm 1$

IE $\sigma(\theta) \sim |D^1_{11}|^2 + |D^1_{1-1}|^2 + |D^1_{-11}|^2 + |D^1_{-1-1}|^2$

$$\sim \left|\frac{1+\cos\theta}{2}\right|^2 + \left|\frac{1-\cos\theta}{2}\right|^2 \sim \underline{\underline{1+\cos^2\theta}}$$

③ b)



USE ISOSPIN ANALYSIS. BUT WITH SOME CAUTION. ISOSPIN IS NOT CONSERVED IN THE <u>WEAK</u> INTERACTION $c\bar{u} \to s\bar{d}$!

INDEED $c\bar{u}$ HAS $I_3 = -\frac{1}{2}$ (SINCE A $\bar{u}$ HAS $I_3 = -\frac{1}{2}$)

WHILE $s\bar{d}$ HAS $I_3 = +\frac{1}{2}$

BUT ISOSPIN IS CONSERVED IN THE <u>STRONG</u> INTERACTION $s\bar{d} \to s\bar{q}q\bar{d}$

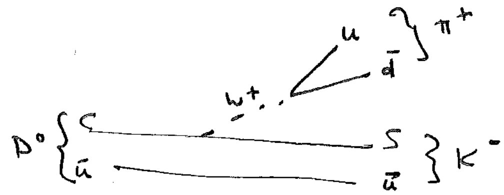
SO WE CAN WRITE $|s\bar{d}\rangle \to c_1 \underbrace{|s\bar{u}\rangle}_{K^-} \underbrace{|u\bar{d}\rangle}_{\pi^+} + c_2 \underbrace{|s\bar{d}\rangle}_{\bar{K}^0} \underbrace{|d\bar{d}\rangle}_{\left(-\frac{1}{\sqrt{2}}\pi^0\right)}$ ← NOTING $|\pi^0\rangle = \frac{|u\bar{u}\rangle - |d\bar{d}\rangle}{\sqrt{2}}$

OR $|\frac{1}{2},\frac{1}{2}\rangle \to \sqrt{\frac{2}{3}}\,|\frac{1}{2},-\frac{1}{2}\rangle|1,1\rangle + \left(-\sqrt{\frac{1}{3}}\right)\left(-\sqrt{\frac{1}{2}}\right)|\frac{1}{2},\frac{1}{2}\rangle|1,0\rangle$

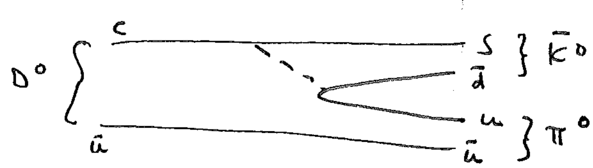
USING THE CLEBSCH-GORDAN CRIB SHEET FOR c_1 & c_2.

THUS $\dfrac{\Gamma(D^0 \to K^-\pi^+)}{\Gamma(D^0 \to \bar{K}^0\pi^0)} = \dfrac{|\sqrt{2/3}|^2}{(\sqrt{1/6})^2} = \underline{\underline{4}}$

ANOTHER CABIBBO-FAVORED DIAGRAM IS



AND ALSO



THESE MAY WELL HAVE COMPARABLE STRENGTHS, AND SO TEND TO REDUCE THE RATIO OF BRANCHING RATIOS TO LESS THAN 4....

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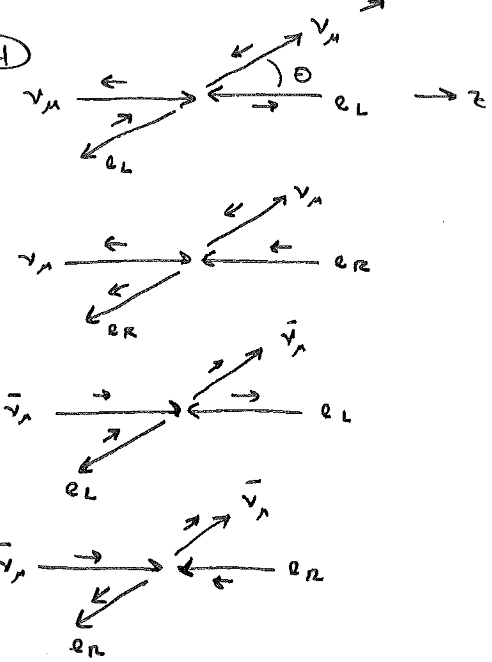
(4)



CASES 1 & 4 HAVE J_z & J_z' BOTH 0, SO EXPECT ISOTROPIC ANGULAR DISTRIBUTION

CASE 2 HAS $J_z = -1$, $J_z' = -1$

CASE 3 HAS $J_z = 1$, $J_z' = 1$

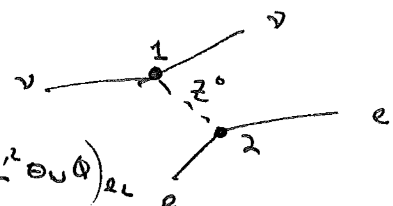
SO EXPECT ANGULAR DIST $\sim |D^1_{11}|^2 \sim \left|\frac{1+\cos\theta}{2}\right|^2$

INTEGRATING OVER ANGLE, $\int_{-1}^{1} d\cos\theta \left|\frac{1+\cos\theta}{2}\right|^2 = \frac{1}{4}\left(2+\frac{2}{3}\right) = \frac{2}{3}$

COMPARED TO $\int_{-1}^{1} d\cos\theta = 2$.

i.e. CASES 2 & 3 HAVE $\frac{1}{3}$ THE CROSS SECTION OF 1 & 4

TO GET THE VERTEX FACTORS, CONSIDER THE DIAGRAM



$$\text{AMPL} \sim \frac{g}{\cos\theta_W}\left(I_3 - \sin^2\theta_W Q\right)_{\nu \text{ or } \bar\nu} \left(\frac{1}{M_Z^2}\right) \frac{g}{\cos\theta_W}\left(I_3 - \sin^2\theta_W Q\right)_{e_L \text{ or } e_R}$$

$\uparrow$ PROPAGATOR

$$\sim \frac{G}{\sqrt{2}}\left(\frac{1}{2}\right)\cdot \begin{cases} -\frac{1}{2} - (\sin^2\theta_W)(-1) & \text{for } e_L \\ 0 - \sin^2\theta_W(-1) & \text{for } e_R \end{cases}$$

USING $I_3(\nu \text{ or } \bar\nu) = \frac{1}{2}$

$I_3(e_L) = -\frac{1}{2}$

$I_3(e_R) = 0$

$\uparrow$ WEAK ISOSPIN

$$\sigma \sim \sum_{e_L, e_R} (\text{AMPL})^2 \cdot (\text{ANGULAR FACTOR}) \cdot (\text{CM ENERGY})^2$$

$\uparrow$ BY DIMENSIONAL ANALYSIS, SINCE $\sigma \sim M^2$

$G \sim \frac{1}{M^2}$, AND $(\text{AMPL})^2 \sim G^2$

$$\text{SO } \sigma \sim G E_{cm}^2 \begin{cases} \left(\frac{1}{2} - \sin^2\theta_W\right)^2 + \frac{1}{3}\sin^2\theta_W & \text{FOR } \nu_\mu e \\ \frac{1}{3}\left(\frac{1}{2} - \sin^2\theta_W\right)^2 + \sin^2\theta_W & \text{FOR } \bar\nu_\mu e \end{cases}$$

$\uparrow$ e_L $\qquad$ $\uparrow$ e_R

[DETAILED CALCULATION SHOWS THERE IS AN OVERALL FACTOR OF $\frac{4}{\pi}$ MISSING FROM THE QUICK ESTIMATE.]