PRINCETON UNIVERSITY

Ph304 Midterm Examination
Electrodynamics

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Do all work you wish graded in the exam booklets provided.

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Please do all work in the exam booklets provided.

You may use either Gaussian or SI units on this exam.

1. (10 pts.) Find the **approximate** radius of a grounded conducting sphere such that when placed between like charges whose separation is $2a$, the force on these charges vanishes.

2. (10 pts.) Find the magnetic force (magnitude and direction) on a loop of radius a that carries current I when the loop is distance b away from an infinite sheet of a classical superconductor = a material inside of which the fields $\mathbf{E}$ and $\mathbf{B}$ are zero, but which can have a layer of charges and/or currents on its surface. The plane of the loop is parallel to the surface of the sheet.

   You may restrict your discussion to the two limiting cases $b \ll a$, and $a \ll b$.

3. (20 pts.) A conducting sphere of radius a, relative dielectric constant $\epsilon = 1$ (*i.e.*, $\mathbf{D} = \mathbf{E}$), and relative permeability $\mu = 1$ rotates with constant angular velocity ω about a diameter. A constant, uniform, external magnetic field $\mathbf{B}$ is applied parallel to the axis of rotation. The total charge on the sphere is zero. Assuming that you can ignore any magnetic field due to the rotating sphere, calculate the following steady-state quantities (in any order):

   a) The electric field $\mathbf{E}$ everywhere.

   b) The volume charge density ρ inside the sphere and the charge density σ on its surface.

   c) The electric potential ϕ everywhere, defining the potential at infinity to be zero.

   Comment briefly on how the solution would differ if the sphere were superconducting.

   Reminder: in spherical coordinates (r, θ, φ),

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial r^2 E_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial \sin\theta E_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial \varphi}, \tag{1}$$

$$\nabla \phi = \hat{\mathbf{r}}\frac{\partial \phi}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial \phi}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial \phi}{\partial \varphi}. \tag{2}$$

   Hint: a possible sequence is to calculate the electric field inside the sphere, the charge density inside the sphere, the potential inside the sphere, the potential outside the sphere, the electric field outside the sphere, and finally the surface charge density. Check that the total charge is zero. Note that in the steady state, charges are at rest with respect to an ordinary conductor, unless there is an electromotive force present – which there is not in the present problem.

Solutions

1. [Prob. 1, Chap. 5 of Smythe, *Static and Dynamic Electricity.*]

   The solution is readily found using the image method for a single charge and a grounded conducting sphere, since any number of charges and their images can be superimposed and the conducting sphere remains at zero potential.

   Recall that the image of charge q at distance a from a grounded conducting sphere of radius b is $q' = -qb/a$, and the image charge is located at distance $c = b^2/a$ from the center of the sphere, on the line joining the center of the sphere and charge q.

   In the present problem, image charges q' are located at positions c on both sides of the center of the sphere. Hence, the force on one of the charges q is

$$\begin{aligned} F &= \frac{q^2}{(2a)^2} + \frac{qq'}{(a-c)^2} + \frac{qq'}{(a+c)^2} = q^2\left[\frac{1}{4a^2} - \frac{2b}{a}\frac{a^2+c^2}{(a^2-c^2)^2}\right] \\ &= q^2\left[\frac{1}{4a^2} - 2ab\frac{a^4+c^4}{(a^4-b^4)^2}\right] \end{aligned} \tag{3}$$

   Requiring this force to vanish, we find

$$a^8 - 2a^4b^4 + b^8 = 8a^7b + 8a^3b^5. \tag{4}$$

   We suppose that $b \ll a$ and neglect higher powers of b in eq. (4) to find the approximate result $a^8 \approx 8a^7b$, or $b \approx a/8$.

   In the next approximation, we could write $b \approx a(1+\epsilon)/8$ in the term $8a^7b$ and $b \approx a/8$ elsewhere to find

$$a^8 - 2a^4\left(\frac{a}{8}\right)^4 + \left(\frac{a}{8}\right)^8 \approx a^8(1+\epsilon) + 8a^3\left(\frac{a}{8}\right)^5. \tag{5}$$

   and

$$\epsilon \approx -\frac{1}{2048} - \frac{1}{4096} + \frac{1}{16777216} \approx -\frac{3}{4096} \approx -0.0007. \tag{6}$$

2. [Similar to Griffiths' prob. 7.43.]

   This problem is also suitable for solution via the image method – for magnetic materials. The magnetic field outside the sheet can be thought of as due to the original loop plus an imaginary loop at distance b below the surface of the sheet. The force on the original loop is that due to its interaction with the magnetic field of the image loop. The image current "clearly" has magnitude I, but what is its direction?

   To determine the direction of the image current, we consider the boundary conditions.

   Since $\nabla \cdot \mathbf{B} = 0$ everywhere, the perpendicular component of $\mathbf{B}$ is continuous across the boundary. Since $\mathbf{B} = 0$ inside the superconductor, we must have $B_\perp = 0$ at its surface.

   At the surface of the sheet there can free currents, so $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J}$ which tells us that there can be a tangential component of $\mathbf{B}$ next to the surface of the superconductor.

Hence, the image current must produce a magnetic field that cancels the normal component of the field from the original loop at the surface of the sheet. Therefore, the image current has the opposite sense to that of the original current. And since antiparallel currents repel, the loop is repelled by the superconducting sheet. [A magnet can be levitated above a superconductor.]

We can confirm this result by considering the loop as a magnetic dipole made from a set of + and − (fictitious) magnetic monopoles. We see that the image prescription for magnetic poles above a classical superconductor is that the image pole has the SAME SIGN as the original pole, so that the sum of their fields is parallel to the sheet (not perpendicular as for electric charges). Since the loop is parallel to the sheet, the dipole axis is perpendicular to the sheet. Taking the original dipole to have its + pole farther from the sheet, the image dipole has its + pole farther from the sheet. Hence the image dipole has the opposite orientation as the original dipole, and so the currents flow in the opposite direction.

In the limit that radius a is small compared to height b the loops can be approximated as point dipoles. We take the z axis along the line of centers of the two dipoles, of strength

$$\pm\mathbf{p} = \pm\frac{\pi a^2 I}{c}\hat{\mathbf{z}}. \tag{7}$$

The force on the dipole due to the magnetic field of the image dipole is given by

$$F_z = \frac{\partial(\mathbf{p}\cdot\mathbf{B}_{\rm image})}{\partial z} = p\frac{\partial B_z(z=b)}{\partial z}. \tag{8}$$

The magnetic field along the axis of the image dipole $-\mathbf{p}$ is given by

$$B_z(z) = \frac{3(-\mathbf{p}\cdot\hat{\mathbf{z}}) - (-p)}{r^3} = -\frac{2p}{(z+b)^3}. \tag{9}$$

Hence, eqs. (7)-(9) combine to give

$$F = 3\frac{2p^2}{(2b)^4} = \frac{3\pi^2 a^4 I^2}{8c^2 b^4} \qquad (a \ll b). \tag{10}$$

Here, the positive force implies the loop is repelled by the superconductor.

When $b \ll a$, the radius of the loop is large compared to its distance from the plate, and the loops are not simply equivalent to point dipoles of strength $\pi a^2 I/c$. Rather, we note that in this case the two loops are very close together, so the magnetic field from one loop at the position of the other is essentially the same as that due to a straight wire carrying current I, namely

$$B = \frac{2I}{c(2b)} = \frac{I}{cb}. \tag{11}$$

The magnetic field is perpendicular to the loop, so the total force on the original loop is therefore

$$F = \frac{2\pi a I B}{c} = \frac{2\pi a I^2}{c^2 b} \qquad (b \ll a). \tag{12}$$

As above, this force repels the loop from the sheet.

3. [Sec. 48 of Mason and Weaver, *The Electromagnetic Field* (Dover, 1929). Uses Heaviside-Lorentz units $\Rightarrow$ factors of 4π different from Gaussian units!]

In the steady state, charges cannot be in motion relative to a sphere of finite conductivity unless there is a driving electromotive force – which is absent in the present problem. Otherwise, Joule losses would quickly reduce the relative velocity of the charges to zero. Hence, if a nonzero charge density ρ arises, the charges are rotating with angular velocity ω.

This contrasts with the case of a superconductor, in which currents can flow without an electromotive force. In this case, surface currents develop so as to cancel the external magnetic field in the interior of the sphere (for any angular velocity ω). The surface current would vary as $\sin\theta$, as discussed in Ph501 Problem Set 4, prob. 9a. Outside the sphere these currents would add a dipole magnetic field to the uniform external magnetic field. The surface current is not due to the rotation of a net surface charge density, as this would require the interior of the superconductor to have a nonzero charge density, and hence a nonzero electric field. Rather, the surface of the sphere remains neutral, and the electric field is everywhere zero. The surface currents are thus unrelated to the angular velocity of the sphere, which can have any value without changing the magnetic fields. [For another variant, see Griffiths' prob. 7.45.]

Returning to the case of finite conductivity, the key argument is that there can be no net force on charges inside the sphere due to the macroscopic $\mathbf{E}$ and $\mathbf{B}$ fields there.

One way to argue is via the full version of Ohm's law, Griffiths' eq. (7.2), applied in the rest frame of the rotating sphere, where $\mathbf{J}^\star = 0$ according to the first paragraph of this solution. We therefore conclude that $0 = \sigma\mathbf{E}^\star = \gamma\sigma(\mathbf{E} + \mathbf{v}/c \times \mathbf{B})$, noting that $\mathbf{B}$ is transverse to $\mathbf{v}$, and hence any needed electric field $\mathbf{E}$ will be also. Thus, the electric field inside the sphere is related by

$$\mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{B}. \tag{13}$$

This leaves unresolved the question as to what force provides the centripetal acceleration $\omega^2 r_\perp$ of the electrons and ions at distance $r_\perp$ from the axis of the sphere. Consider first the case of zero magnetic field. Whenever a conductor spins about an axis, internal forces must be generated to provide the centripetal force, or the conductor would fly apart. There must be microscopic forces that act on the conduction electron as well as on the positive ion lattice, or all the conduction electrons would accumulate the surface leaving the interior positively charged and hence unstable against breakup. Since conductors do not typically fall apart when spun, we infer that microscopic internal forces, presumably due to electromagnetic fields, will provide the centripetal force $-m\omega^2\mathbf{r}_\perp$ for both electron and ions so that the bulk material remains neutral.

The result that the electric field $\mathbf{E}^\star$ must vanish in the rest frame of the rotating sphere implies that the atoms cannot have taken on a dipole deformation proportional to $\mathbf{r}_\perp$, as might have appealed to one's intuition. Otherwise, the the sphere would have a bulk polarization $\mathbf{P}^\star$ proportional to $\mathbf{r}_\perp$, a uniform bound charge density $\rho_b^\star = -\nabla \cdot \mathbf{P}^\star$ (looking ahead to eq. (17)), and hence a nonzero electric field $\mathbf{E}^\star$.

In any case, when we add external fields their effect is in addition to the microscopic forces that provide the centripetal force. In the steady-state of a rotating conductor the interior charges must rotate as for a rigid body, so there must be no net additional force on these charges. Since the rotating free charges experience a $\mathbf{v} \times \mathbf{B}$ force, there must be some additional force that cancels this. The free charge distribution rearranges itself until it generates an electric field that cancels the magnetic force. That is, the resulting volume charge distribution ρ obeys

$$\mathbf{F}_{\text{macroscopic}} = 0 = \rho \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right), \tag{14}$$

so far as the macroscopic fields $\mathbf{E}$ and $\mathbf{B}$ are concerned. Hence, the electric field in the interior of the sphere is

$$\begin{aligned} \mathbf{E} &= -\frac{\mathbf{v}}{c} \times \mathbf{B} = -\frac{\omega \hat{\mathbf{z}} \times \mathbf{r}}{c} \times B\hat{\mathbf{z}} = \frac{\omega B}{c}[(\mathbf{r} \cdot \hat{\mathbf{z}})\hat{\mathbf{z}} - \mathbf{r}] = -\frac{\omega B r}{c}(\hat{\mathbf{r}} \sin^2\theta + \hat{\theta} \sin\theta\cos\theta) \\ &= -\frac{\omega B r_\perp}{c}\hat{\mathbf{r}}_\perp, \end{aligned} \tag{15}$$

noting that

$$\hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\theta}\sin\theta, \qquad \text{and} \qquad \hat{\mathbf{r}}_\perp = \hat{\mathbf{r}}\sin\theta + \hat{\theta}\cos\theta, \qquad \text{and} \qquad r_\perp = r\sin\theta. \tag{16}$$

The charge distribution can now be obtained via the first Maxwell equation,

$$\rho = \frac{\nabla \cdot \mathbf{E}}{4\pi} = -\frac{\omega B}{4\pi c}\left(\frac{1}{r^2}\frac{\partial r^3 \sin^2\theta}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial r\sin^2\theta\cos\theta}{\partial\theta} \right) = -\frac{\omega B}{2\pi c}, \tag{17}$$

recalling eq. (1). The total charge in the interior of the sphere is

$$Q = \frac{4\pi a^3 \rho}{3} = -\frac{2\omega B a^3}{3c}. \tag{18}$$

It is noteworthy that the charge distribution in the interior is uniform, but the electric field is not spherically symmetric. This can happen if the surface charge distribution (required since the sphere is neutral overall) is not spherically symmetric.

The strategy for the remainder of the problem is as follows. Use $\mathbf{E} = -\nabla\phi$ to deduce the form of the electric potential in the interior of the sphere. Then extrapolate the potential to the exterior by matching at the boundary $r = a$. Then, we calculate the electric field outside the sphere, and finally we can calculate the surface charge distribution.

This problem is azimuthally symmetric, so the potential depends only on r and θ. Using eq. (1), we have

$$\mathbf{E} = -\nabla\phi = -\hat{\mathbf{r}}\frac{\partial\phi}{\partial r} - \frac{\hat{\theta}}{r}\frac{\partial\phi}{\partial\theta}, \tag{19}$$

Hence, for $r < a$ eq. (15) tells us that

$$\frac{\partial\phi}{\partial r} = \frac{\omega B r \sin^2\theta}{c}, \qquad \text{and so} \qquad \phi(r < a) = \phi_0 + \frac{\omega B r^2 \sin^2\theta}{2c}. \tag{20}$$

As a check, eqs. (15) and (19) also tell us that

$$\frac{\partial \phi}{\partial \theta} = \frac{\omega B r^2 \sin\theta \cos\theta}{c}, \qquad \text{and likewise} \qquad \phi = \phi_0 + \frac{\omega B r^2 \sin^2\theta}{2c}. \tag{21}$$

For $r > a$, the charge density vanishes so $\nabla^2 \phi = 0$ there, and we can expand the potential in terms of Legendre functions as

$$\phi(r > a) = \sum \frac{A_n}{r^{n+1}} P_n(\cos\theta), \tag{22}$$

choosing $\phi(r = \infty) = 0$. To match this to eq.(20) at $r = a$, we note that

$$P_0 = 1, \qquad P_2 = \frac{3\cos^2\theta - 1}{2} = \frac{2 - 3\sin^2\theta}{2}, \qquad \text{so} \qquad \sin^2\theta = \frac{2}{3}(P_0 - P_2), \tag{23}$$

and

$$\phi(r < a) = \left(\phi_0 + \frac{\omega B r^2}{3c}\right) P_0 - \frac{\omega B r^2}{3c} P_2. \tag{24}$$

Matching eqs. (22) and (24) at $r = a$, we see that all the A_i vanish except A_0 and A_2, which obey

$$A_0 = a\phi_0 + \frac{\omega B a^3}{3c}, \qquad \text{and} \qquad A_2 = -\frac{\omega B a^5}{3c}. \tag{25}$$

The potential is then

$$\phi(r < a) = \phi_0 + \frac{\omega B r^2}{3c}(1 - P_2), \tag{26}$$

$$\phi(r > a) = \phi_0 \frac{a}{r} + \frac{\omega B a^2}{3c}\left(\frac{a}{r} - \frac{a^3}{r^3} P_2\right). \tag{27}$$

We can now calculate the electric field outside the sphere to be

$$E_r(r > a) = -\frac{\partial \phi}{\partial r} = \phi_0 \frac{a}{r^2} + \frac{\omega B a^2}{3c}\left(\frac{a}{r^2} - 3\frac{a^3}{r^4} P_2\right), \tag{28}$$

$$E_\theta(r > a) = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\omega B a^5 \sin\theta \cos\theta}{c r^4}. \tag{29}$$

For comparison, we rewrite the electric field (15) in the interior as

$$E_r(r < a) = -\frac{2\omega B r}{3c}(1 - P_2), \tag{30}$$

$$E_\theta(r < a) = -\frac{\omega B r \sin\theta \cos\theta}{c}. \tag{31}$$

The tangential electric field E_θ must be continuous at the boundary $r = a$, which is satisfied by eqs. (29) and (31).

The surface charge density σ can now be found via a Gaussian pillbox surrounding a segment of the surface $r = a$:

$$\sigma = \frac{E_r(r = a^+) - E_r(r = a^-)}{4\pi} = \frac{\phi_0}{4\pi a} + \frac{\omega B a}{12\pi c}(3 - 5P_2), \tag{32}$$

The total charge on the surface of the sphere is

$$Q = 4\pi a^2 \left(\frac{\phi_0}{4\pi a} + \frac{\omega B a}{4\pi c} \right) = \phi_0 a + \frac{\omega B a^3}{c}, \tag{33}$$

which must be the negative of the total charge (18) inside the sphere, since the total charge is zero. Hence, at great length we determine the constant ϕ_0 to be

$$\phi_0 = -\frac{\omega B a^2}{3c}. \tag{34}$$

We can now go back and tidy up the quantities that contain ϕ_0:

$$\begin{aligned}
\phi(r < a) &= \frac{\omega B}{3c}(r^2 - a^2 - r^2 P_2), && (35)\\
\phi(r > a) &= -\frac{\omega B a^5 P_2}{3cr^3}, && (36)\\
E_r(r > a) &= -\frac{\omega B a^5 P_2}{cr^4}, && (37)\\
\sigma &= \frac{\omega B a}{12\pi c}(2 - 5P_2). && (38)
\end{aligned}$$

How big is the charge density σ, in terms of electrons/cm^2? Suppose, for example that $\omega = 1$ rad/s, $B = 1$ tesla $= 10^4$ gauss, and $a = 1$ cm. Then, $\sigma \approx 10^4/10^{12} = 10^{-8}$ esu/cm^2. Since the charge of the electron is $e \approx 5 \times 10^{-10}$ esu, the surface charge density would be about 20 electrons/cm^2.

Returning to the issue of the centripetal force, we consider its magnitude compared to that of the $\mathbf{v} \times \mathbf{B}$ force.

$$\frac{m_e \omega^2 r}{evB/c} \approx \frac{m_e c}{e} \frac{\omega}{B} \approx \frac{10^{-27} \cdot 10^{10}}{10^{-10}} \frac{\omega}{B} \approx 10^{-7} \frac{\omega}{B}. \tag{39}$$

For the example of $\omega = 1$ rad/s and $B = 10^4$ gauss, the ratio is negligible. Even in the Earth's magnetic field, ≈ 1 gauss, the ratio would not be appreciable until $\omega \approx 10^7$ rad/s! Hence, the issue of the microscopic origin of the centripetal force in a spinning conductor is more of pedagogic than practical interest.

Note. This example is abstracted from the larger topic of **unipolar (or homopolar) induction** (Faraday, 1831). From eq. (13), and also eq. (30), we see that the radial electric field in the equatorial plane inside the conducting sphere is

$$E_r(r < a, \theta = \pi/2) = -\frac{\omega r B}{c}. \tag{40}$$

Hence there is a voltage difference $\Delta V = \omega B a^2/2$ between the axis and the equator of the sphere. If a load resistor R is connected via wires with **sliding** contacts at the pole and the equator of the sphere, a current $I = \Delta V/R$ will flow, and power can be extracted from the system. In this case, there is a torque exerted on the radial current by the magnetic field,

$$N = \int_0^a r F_\theta \, dr = \frac{IB}{c} \int_0^a r \, dr = \frac{IBa^2}{2c}, \tag{41}$$

and an external source must provide input power

$$P = N\omega = \frac{\omega I B a^2}{2c} = I \Delta V, \tag{42}$$

which exactly equals the power dissipated in the load resistor.

This is very reassuring, except we recall the basic consequence of the Lorentz force law, that **magnetic fields do no work**. On reflection, we realize that the torque described by eq. (41) is on the conductions electrons, and not on the lattice of positive ions, which is is what the outside source makes mechanical contact with. But, because the currents must flow essentially radially, the lattice must set up azimuthal electric fields to counteract the azimuthal magnetic force. These electric fields are what do the work....

Note 2. Contemporary interest in this problem is because of its possible relevance to the difficult question of the magnetism of planets and stars. Two web pages on this intriguing topic are
http://www-istp.gsfc.nasa.gov/earthmag/dmglist.htm
http://www.psc.edu/science/glatzmaier.html
The planetary dynamo problem is an aspect of magnetohydrodynamics. See chap. 18 of http://www.pma.caltech.edu/Courses/ph136/ph136.html for an up-to-date introduction to this field.

Note 3. The solution presented here is based on the conventional wisdom that no charge separation occurs in a spinning object that is in zero external electric and magnetic fields. The difficulty in explaining planetary magnetism has led to the conjecture, particularly by P.M. Blackett in 1947, that spinning neutral objects can generate magnetic fields without the presence of internal convective flow. This idea is generally believed to be incorrect, but still has its enthusiasts (somewhat on the fringes of science). See, for example,
http://www.stardrive.org/Jack/sirag-vigier3.pdf

Note 4. In a paramagnetic medium there is a small effect whereby rotation induces a small magnetic field even in the absence of external electric and magnetic fields, as noted by S.J. Barnett, Phys. Rev. **6**, 239 (1915). Paramagnetic materials contain permanent moments whose average alignment is zero in the absence of an external magnetic field. If a paramagnetic material is placed in a rotating frame, the Coriolis effect on the molecular currents induces a precession of the magnetic moments that has the same sense no matter what the orientation of the magnetic moment. This results in a net effective current in the same sense as the angular velocity of the body, and hence a net magnetic field. There is an even smaller effect of the same sign due to the centrifugal force.

Since iron is paramagnetic, this might have some relation to the Earth's magnetic field, but Barnett calculated that it would imply a field of about 10^{-10} gauss. Nonetheless, Barnett was able to detect the effect in spinning iron rods, finding a value close to 1/2 that predicted by classical theory. Since this work was done the same year as the Einstein-de Haas experiment, which is conceptually related, and the latter claimed to agree with classical theory, Barnett's work never attracted much attention. But Einstein was wrong and Barnett was right. We now know that paramagnetism in iron is a nonclassical effect, and Barnett should be credited as having the first experimental result that showed the electron gyromagnetic ratio to be 2, not 1.