Princeton University

Ph205
Mechanics
Problem Set 5

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(1988)

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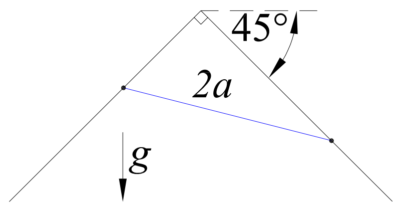
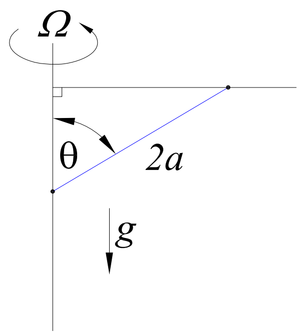
http://kirkmcd.princeton.edu/examples/

1. (a) A uniform rod of length $2a$ has its ends constrained to slide, without friction, on two wires that makes 45° angles as shown in the figure. Show that the length of a simple pendulum with the same frequency of oscillation is $l = 4a/3$.



(b) The gizmo of part (a) is rearranged so that it is forced to rotate at constant angular velocity Ω about the now-vertical wire.



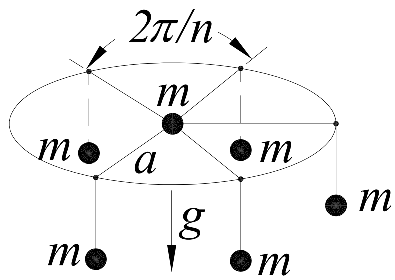
Construct an effective potential to show that the equilibrium angles are $\theta_0 = 0$ and $\cos\theta_0 = 3g/4a\Omega^2$, where θ is the angle of the sliding rod of length $2a$ to the vertical.

Also show that the frequency of small oscillations is,

$$\omega = \sqrt{\frac{3g}{4a} - \Omega^2}, \qquad \text{or} \qquad \omega = \Omega\sqrt{1 - \left(\frac{3g}{4a\,\Omega^2}\right)^2}, \tag{1}$$

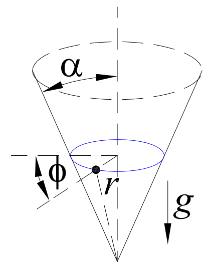
depending on which equilibrium is stable.

2. A mass m is connected to n strings and n other masses m. The strings hang over the edge of a circular table of radius a, and are constrained to pass over fixed points, equally spaced, on the circumference of the circle. There is no friction.



If the motion of the hanging masses is purely vertical, show that the frequency of small oscillations is $\omega = \sqrt{ng/(n+2)a}$ for $n > 2$, while for $n = 2$, $\omega = \sqrt{2g/a}$ of oscillations transverse to the strings on the table (and $\omega = 0$ for "oscillations" along the strings).

3. A cone of half angle α has its axis vertical and tip downwards. A particle slides freely on its interior surface.



Construct and sketch the effective potential to answer the following:

What is the equilibrium radius r_0 as a function of the angular momentum L_0 about the vertical axis?

What is the equilibrium angular velocity $\Omega = \dot{\phi}$?

What is the angular velocity of small oscillations about the equilibrium orbit?

Sketch the orbits of small oscillations for cones with $\sin\alpha = 1/2\sqrt{3}$ and $1/\sqrt{3}$.

4. Consider the central force problem (for a fixed force center) with a potential of the form,

$$V(r) = -\frac{C}{r^\lambda}, \qquad (\lambda \neq 0). \tag{2}$$

Show that circular orbits are stable only for $\lambda < 2$,[1] and that orbits which depart slightly from circularity can be written as,

$$r(t) = r_0(1 + \epsilon \cos \omega t), \qquad \theta(t) = \Omega\, t - \frac{2\epsilon}{\sqrt{2-\lambda}} \sin \omega t, \tag{3}$$

where $\epsilon \ll 1$, $\omega = \Omega\sqrt{2-\lambda}$, and Ω is the angular velocity of the circular orbit of radius r_0. These orbits are simple closed curves only if $\sqrt{2-\lambda}$ is an integer.[2]

Sketch orbits for $\lambda = 1$ (gravity), $\lambda = -2$ (springs), and $\lambda = -7$.

Note that the shape, but not the detailed time dependence, of the oscillating orbits is well described by a single epicycle, as first introduced by Apollonius ($\approx$ 200 BC), who also introduced the deferent to explain better the no-so-small oscillation of the orbit of the Moon about the Earth.[3]

[1]This result in known as Bertrands' Theorem, due to J. Bertrand, *Théorème relatif au mouvement d'un point attiré vers un centre fixe*, Comptes Rendus Acad. Sci. **77**, 849 (1873),
http://kirkmcd.princeton.edu/examples/mechanics/bertrand_cras_77_849_73.pdf
http://kirkmcd.princeton.edu/examples/mechanics/bertrand_cras_77_849_73_english.pdf

[2]More complicated closed curves exist for any rational ratio of ω/Ω.

[3]See, for example, J.L.E. Dreyer, *A History of Astronomy from Thales to Kepler* (Dover, 1953), http://kirkmcd.princeton.edu/examples/mechanics/dreyer_53.pdf, and G. Gallavotti, *Quasi periodic motions from Hipparchus to Kolmogorov*, Rend. Mat. Acc. Lincei **12**, 125 (2001), http://kirkmcd.princeton.edu/examples/mechanics/gallavotti_rmac_12_125_01.pdf.

5. (a) Show that the period of a speck of dust around the surface of a spherical boulder in outer space is the same at that of a low-altitude satellite around the Earth (*i.e.*, the period of an orbit just above the surface of a sphere depends only on the density of the sphere).

   (b) **Satellite Paradox.** A satellite is in a near-circular orbit about the Earth, which orbit is in Earth's upper atmosphere. The satellite experiences a drag force $\mathbf{F} = -\alpha \mathbf{v}$ with small α, and its orbit remains essentially circular at all times. Deduce $v(t)$ and the radius $r(t)$ of the orbit, for initial values of v_0 and r_0.

   As a first approximation, you may assume that α is constant, although it actually depends on the radius r of the orbit.

   Hint: "brute force" use of $\mathbf{F} = m\mathbf{a}$ *is insufficient here.*

6. A comet in a parabolic orbit about the Sun has perihelion at distance p.

   Recall that a parabola has eccentricity $\epsilon = 1$.

   (a) What is the total energy of the comet?

   (b) What is the angular momentum of the comet with respect to the Sun (neglecting any possible "spin" of the comet)?

   (c) What is the angle θ between the line from the Sun to the perihelion of the comet's orbit and the radius to the point where the orbits of the Earth and the comet intersect?

   (d) Show that the time spent by the comet inside the Earth's orbit (of radius a) is,

$$T = \frac{2}{3}\sqrt{\frac{2a^3}{GM}}\sqrt{1 - \frac{p}{a}}\left(1 + \frac{2p}{a}\right), \tag{4}$$

   where M is the mass of the Sun, and perturbations of the comet's orbit by the planets are neglected.

   Show also that $T_{\max} \approx 11$ weeks.

7. Consider a particle of mass m and angular momentum L about the origin, subject to the attractive central force,

$$F = -\frac{C}{r^3}\,, \tag{5}$$

with $C > 0$.

Discuss the character of circular orbits using the effective potential, and then discuss the forms of generals orbits using the orbit equation,

$$\frac{d^2u}{d\theta^2} + u = \cdots, \qquad u = \frac{1}{r}\,, \tag{6}$$

to show that there are 3 classes of orbits.

Sometimes a figure is worth a thousand words (although there are no figures in J.-L. Lagrange, *Mécanique Analytique* (1788),
http://kirkmcd.princeton.edu/examples/mechanics/lagrange_ma_v1_11.pdf
http://kirkmcd.princeton.edu/examples/mechanics/lagrange_ma_v2_15.pdf

8. **Precession of the Perihelion of Mercury.**

A famous problem is astronomy is the precession of the perihelion of the planet Mercury's orbit by 40″ per century in the direction of the motion of the orbit, beyond that due to the effect of other planets in the solar system.[4] The average radius of the orbit about the Sun is 6×10^{10} m, the eccentricity is $\epsilon = 0.20 - 6$, and the period is $T = 0.24$ Earth years.

Suppose that the force of gravity on a mass m is actually,

$$F = -\frac{Am}{r^2} - \frac{Bm}{r^3}. \tag{7}$$

Use the orbit equation to show that the form of the orbit is,

$$\frac{1}{r} = \frac{1 + \epsilon \cos \beta\theta}{a(1 - \epsilon^2)}, \qquad \text{where} \qquad \beta = \sqrt{1 - \frac{Bm}{L^2}}. \tag{8}$$

Supposing $\eta \equiv B/Aa$ is very small, show that the perihelion advances by,

$$\Delta\theta \approx \frac{\pi\eta}{1 - \epsilon^2}, \tag{9}$$

each revolution (invoking Kepler's 3rd law if necessary), and that $\eta \approx 1.4 \times 10^{-7}$ would be sufficient to explain the observed precession of Mercury's orbit.

For discussion of the precession of the perihelion in special relativity (without dust), see, http://kirkmcd.princeton.edu/examples/perihelion.pdf

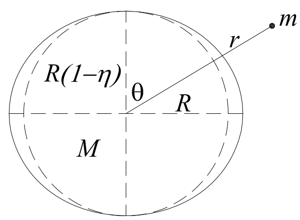
[4]For a review as of 1913, see W. De Sitter, *Some problems of astronomy (VII The secular variation of the elements of the four inner planets)*, The Observatory **36**, 296 (1913), http://kirkmcd.princeton.edu/examples/mechanics/desitter_obs_36_296_13.pdf

9. (a) **Oblate Sun.**

If the Sun has equatorial radius R that is larger than the polar radius $R(1-\eta)$, then its Newtonian gravitational potential for interaction with mass m at distance r becomes,

$$V = -\frac{GMm}{r} - \frac{GMm\eta R^2(1-3\cos^2\theta)}{5r^3} + \mathcal{O}(\eta^2), \tag{10}$$

where M is the mass of the Sun, and θ is the polar angle in a spherical coordinate system (r, θ, ϕ).



Use the effective-potential method to show that for orbits in the plane $\theta = 90°$ the equilibrium circular radius is related to the equilibrium angular velocity Ω by,

$$\frac{1}{r_0^3} = \frac{\Omega^2}{GM} - \frac{3\eta R^2}{5r_0^5}, \tag{11}$$

and that the angular frequency of small oscillations about this orbit is,

$$\omega = \Omega\sqrt{1 - \frac{6GM\eta R^2}{5\Omega^2 r_0^5}}. \tag{12}$$

For $M = 2 \times 10^{30}$ kg and $R = 7 \times 10^8$ m, what value of η is needed to explain the precession of the perihelion of Mercury?

(b) **General Relativity.**

Einstein's theory of general relativity as applied to planetary motion about the Sun modifies the Newtonian force law to,

$$F = -\frac{GMm}{r^2}\left(1 - \frac{3L^2}{(mrc)^2}\right), \tag{13}$$

where c is the speed of light. The dependence of F on r has the same form as that of an oblate Sun in Newtonian theory. Using part (a), you can quickly verify that Einstein's theory predicts an advance of the perihelion of,

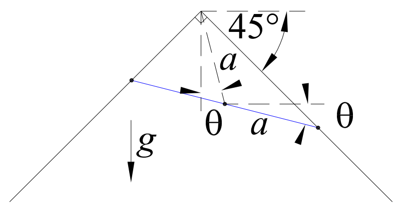
$$\Delta\phi = \frac{24\pi^3 r_0^2}{T^2c^2}, \tag{14}$$

per revolution, which for Mercury is very close to 40″ per century.

R.H. Dicke of Princeton spent a lot of effort measuring the oblateness of the Sun (with null results) to see if general relativity might be wrong.

Solutions

1. (a) We take the origin at the vertex of the two fixed wires, and angle θ to the vertical of the center of mass of the moving rod as the single coordinate in Lagrange's method.



The kinetic energy is,

$$T = \frac{ma^2\,\dot\theta^2}{2} + \frac{I_{\rm cm}\,\dot\theta^1}{2} = \frac{2ma^2\,\dot\theta^2}{3}, \qquad (15)$$

and the potential energy can be written as,

$$V = -mga\cos\theta. \qquad (16)$$

The equation of motion follows from this as,

$$\frac{d}{dt}\frac{\partial T}{\partial\dot\theta} = \frac{4ma^2\,\ddot\theta}{3} = -\frac{\partial V}{\partial\theta} = -mga\sin\theta. \qquad (17)$$

For small θ, this reduces to the springlike equation,

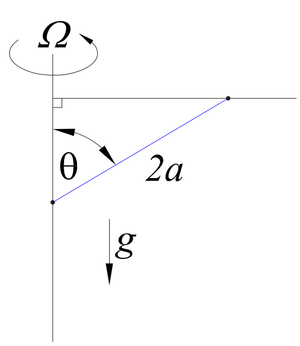
$$\ddot\theta = -\frac{3g}{4a}\,\theta, \qquad (18)$$

so the angular frequency of small oscillations is,

$$\omega = \sqrt{\frac{3g}{4a}}, \qquad (19)$$

which is the frequency of oscillation of a simple pendulum of length $4a/3$.

(b) This problem is Ex. 450, p. 371 of E.J. Routh, *The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 6th ed. (Macmillan, 1905),
http://kirkmcd.princeton.edu/examples/mechanics/routh_advanced_rigid_dynamics.pdf.



The kinetic energy is that of eq. (15) plus the kinetic energy $I\,\Omega^2/2$ of rotation of the rod about the vertical axis, where,

$$I = \frac{m}{2a}\int_0^{2a} dl\,(l\sin\theta)^2 = \frac{m(2a)^2\sin^2\theta}{3} = \frac{4ma^2\sin^2\theta}{3}\,. \tag{20}$$

Hence,

$$T = \frac{2ma^2\,\dot\theta^2}{3} + \frac{2ma^2\,\Omega^2\sin^2\theta}{3}\,. \tag{21}$$

The potential energy can again be written as,

$$V = -mga\cos\theta. \tag{22}$$

The equation of motion follows from the Lagrangian $\mathcal{L} = T - V$ as,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} = \frac{4ma^2\,\ddot\theta}{3} = \frac{\partial\mathcal{L}}{\partial\theta} = -mga\sin\theta + \frac{4ma^2\,\Omega^2\sin\theta\cos\theta}{3}\,. \tag{23}$$

At equilibrium, $\ddot\theta = 0$, such that the equilibrium angles are,

$$\theta_0 = 0 \qquad \textit{and} \qquad \cos\theta_0 = 3g/4a\,\Omega^2. \tag{24}$$

To describe the problem in terms of an effective potential, we note that the Lagrangian $\mathcal{L} = T - V$ is independent of time, so the Hamiltonian is constant (although the mechanical energy of the system varies with time),

$$H = \dot\theta\frac{\partial\mathcal{L}}{\partial\dot\theta} - \mathcal{L} = \frac{1}{2}\frac{4ma^2}{3}\,\dot\theta^2 - \frac{2ma^2\,\Omega^2\sin^2\theta}{3} - mga\cos\theta. \tag{25}$$

The system can be described as having an effective mass $m_{\rm eff} = 4ma^2/3$ and an effective potential,

$$V_{\rm eff} = -\frac{2ma^2\Omega^2\sin^2\theta}{3} - mga\cos\theta\,, \tag{26}$$

$$\frac{dV_{\rm eff}}{d\theta} = -\frac{4ma^2\,\Omega^2\sin\theta\cos\theta}{3} + mga\sin\theta, \tag{27}$$

$$\frac{d^2V_{\rm eff}}{d\theta^2} = -\frac{4ma^2\,\Omega^2\cos 2\theta}{3} + mga\cos\theta. \tag{28}$$

At equilibrium, $dV_{\rm eff}/d\theta = 0$, which yields that same equilibrium angles as found above (so we could have skipped finding the general equations of motion).
The equilibria are stable provided $d^2V_{\rm eff}(\theta_0)/d\theta^2 = k_{\rm eff} > 0$, in which case the angular frequency of small oscillations is $\omega = \sqrt{k_{\rm eff}/m_{\rm eff}}$.
For $\theta_0 = 0$,

$$k_{\rm eff} = \frac{d^2V_{\rm eff}}{d\theta^2} = mga - \frac{4ma^2\,\Omega^2}{3}\,, \tag{29}$$

$$\omega = \sqrt{\frac{mga - 4ma^2\,\Omega^2/3}{4ma/3}} = \sqrt{\frac{3g}{4a} - \Omega^2} \qquad \text{provided} \qquad \frac{3g}{4a\,\Omega^2} > 1. \tag{30}$$

For $\cos\theta_0 = 3g/4a\,\Omega^2$, $\cos 2\theta_0 = 2(3g/4a\,\Omega^2)^2 - 1$

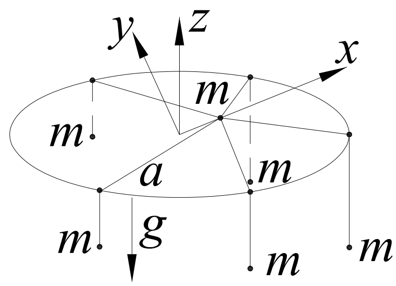
$$k_{\rm eff} = \frac{d^2V_{\rm eff}}{d\theta^2} = mga\frac{3g}{4a\,\Omega^2} - \frac{4ma^2\Omega^2}{3}\left[2\left(\frac{3g}{4a\,\Omega^2}\right)^2 - 1\right]$$

$$= \frac{3mg^2}{4\,\Omega^2} - \frac{3mg^2}{2\,\Omega^2} + \frac{4ma^2\Omega^2}{3} = \frac{4ma^2\Omega^2}{3} - \frac{3mg^2}{4\,\Omega^2}\,, \tag{31}$$

$$\omega = \sqrt{\frac{k_{\rm eff}}{4ma^2/3}} = \sqrt{\Omega^2 - \frac{9g^2}{16a^2\,\Omega^2}} = \Omega\sqrt{1 - \left(\frac{3g}{4a\,\Omega^2}\right)^2} \quad \text{provided} \quad \frac{3g}{4a\,\Omega^2} < 1. \tag{32}$$

2. This problem is Ex. 9, p. 401 of E.J. Routh, *The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 6th ed. (Macmillan, 1905), http://kirkmcd.princeton.edu/examples/mechanics/routh_advanced_rigid_dynamics.pdf.

   In an oscillatory mode, the mass m on the circular table of radius a moves in a straight line that passes through the center of the circle.[5] We take this line to be the x-axis, and the z-axis to be vertical.



   The lengths of the strings on the table, from mass m to the n equally spaced points on the circle, are,

$$d_j = \sqrt{a^2 - 2ax\cos\theta_j + x^2} \approx a\left(1 - \frac{x}{a}\cos\theta_j + \frac{x^2}{2a^2} - \frac{x^2}{2a^2}\cos^2\theta_j\right)$$
$$= a - x\cos\theta_j + \frac{x^2}{2a}\sin^2\theta_j. \qquad (33)$$

$$\dot{d}_j = -\dot{x}\cos\theta_j + \frac{x\dot{x}\sin^2\theta_j}{2a}, \qquad \dot{d}_j^2 \approx \dot{x}^2\cos^2\theta_j = \frac{\dot{x}^2}{2}(1+\cos 2\theta_j), \qquad (34)$$

   where θ_j is the angle to the x-axis of the string connected to hanging mass j, and in the approximations we keep terms to second order in the small quantities x and $\dot{x}$ in an analysis of small oscillations.

   Then, the z-coordinate of mass j is $z_j = -(l - d_j)$, where l is the length of the string.

   The potential energy can be written as,

$$V = \sum_i(-mgz_i) \approx -nmgl + mg\sum_i\left(a - 2x\cos\theta_i + \frac{x^2\sin^2\theta_i}{2a}\right). \qquad (35)$$

   Now,

$$\sum_j \cos\theta_j = Re\sum_j e^{i\theta_j} = Re\, e^{i\theta_1}\sum_j e^{(j-1)2\pi i/n} = Re e^{i\theta_1}\sum_j (e^{2\pi i/n})^{j-1}$$
$$= Re\, e^{i\theta_1}\frac{1-(e^{2\pi i/n})^n}{1-e^{2\pi i/n}} = 0, \qquad (36)$$

   and,

$$\sum_j \sin^2\theta_j = \sum_j \frac{1-\cos 2\theta_j}{2} = \frac{n}{2} - \sum_j \frac{\cos 2\theta_j}{2}. \qquad (37)$$

[5] There also exists an oscillatory mode in which mass m moves in a small circle, which is equivalent to the sum of two linear modes in directions 90° apart, and 90° out of phase. To find the angular frequency of these oscillations, it is sufficient to consider a single linear mode.

For $n \geq 3$, the sum is zero, as in eq. (36), but for $n = 2$ it is nonzero. So, for $n \geq 3$, the potential reduces to,

$$V \approx nmg(a - l) + \frac{mgnx^2}{2a}, \tag{38}$$

which is springlike with effective spring constant $k = mgn/a$.

The kinetic energy is,

$$T = \frac{m\dot{x}^2}{2} + \sum_j \frac{m\dot{d}_j^2}{2} \approx \frac{m\dot{x}^2}{2} + \frac{m\dot{x}^2}{4} \sum_j (1 + \cos 2\theta_j) \to \frac{m\dot{x}^2}{2}\left(1 + \frac{n}{2}\right) \quad \text{for } n \geq 3. \tag{39}$$

For $n \geq 3$ and small oscillations, the system reduces to one of effective mass $m(n+2)/2$ and spring constant $k = mgn/a$, for which the angular frequency of the oscillations is.

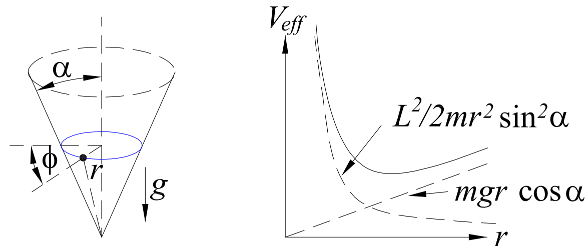
$$\omega = \sqrt{\frac{ng}{(n+2)a}}. \tag{40}$$

We now return to the special case of $n = 2$.

For motion along the line between the two hanging masses, we have $\theta_1 = 0$, $\theta_2 = \pi$, $\sum_j \sin^2\theta_j = 0$, and so the "oscillation" frequency is 0.

For motion transverse to the line between the two hanging masses, we have $\theta_1 = \pi/2$, $\theta_2 = -\pi/2$, $\sum_j \sin^2\theta_j = 2$ and $\sum_j \cos 2\theta_j = -2$, so the effective spring constant is mg/a, the effective mass is $m/2$, and the angular frequency of small oscillations is $\omega = \sqrt{2g/a}$.

3. For a particle sliding inside a cone of half angle α,



the kinetic energy is,

$$T = \frac{m\dot{r}^2}{2} + \frac{mr^2 \sin^2\alpha\, \dot{\phi}^2}{2}, \tag{41}$$

the potential energy can be written as,

$$V = mgr\cos\alpha. \tag{42}$$

and the (conserved) angular momentum about the vertical axis is,

$$L = mr^2 \sin^2\alpha\, \dot{\phi}, \qquad \dot{\phi} = \frac{L}{mr^2 \sin^2\alpha}. \tag{43}$$

The conserved energy of the system can then be written as,

$$E = T + V = \frac{m\dot{r}^2}{2} + \frac{L^2}{2mr^2 \sin^2\alpha} + mgr\cos\alpha \equiv \frac{m\dot{r}^2}{2} + V_{\rm eff}. \tag{44}$$

From the effective potential,

$$\frac{dV_{\rm eff}}{dr} = -\frac{L^2}{mr^3 \sin^2\alpha} + mg\cos\alpha, \qquad k_{\rm eff} = \frac{d^2V_{\rm eff}}{dr^2} = \frac{3L^2}{mr^4 \sin^2\alpha}\ (>0). \tag{45}$$

Given a value L_0 of the angular momentum, the equilibrium radius r_0 is for $dV_{\rm eff}/dr = 0$,

$$r_0 = \left(\frac{L_0^2}{m^2 g \sin^2\alpha\cos\alpha}\right)^{1/3}, \qquad L_0 = m\sin\alpha\sqrt{r_0^3 g\cos\alpha}. \tag{46}$$

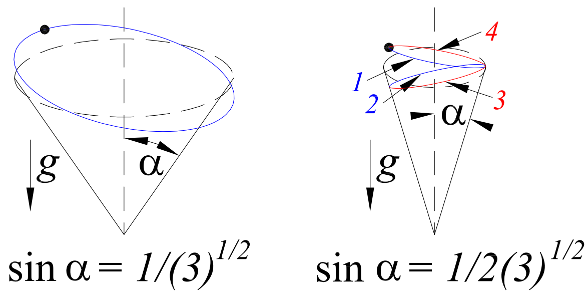
and the angular velocity of the particle in this equilibrium orbit is,

$$\Omega = \dot{\phi}_0 = \frac{L_0}{mr_0^2 \sin^2\alpha} = \frac{1}{\sin\alpha}\sqrt{\frac{g\cos\alpha}{r_0}}. \tag{47}$$

The angular frequency of small oscillations is,

$$\omega = \sqrt{\frac{k_{\rm eff}}{m}} = \sqrt{\frac{3L_0^2}{m^2 r_0^4 \sin^2\alpha}} = \sqrt{\frac{3g\cos\alpha}{r_0}} = \sqrt{3}\,\Omega\sin\alpha \tag{48}$$

For $\sin\alpha = 1/\sqrt{3}$ we have that $\omega = \Omega$, and the oscillating orbit is a tilted plane. For $\sin\alpha = 1/2\sqrt{3}$ we have that $\omega = \Omega/2$, and the oscillating orbit takes 2 revolutions to return to the same place, as shown in the figure below with the first revolution in blue, and the second in red.



For $\sin\alpha \geq 1/\sqrt{3}$, $\omega \geq \Omega$, the oscillating orbit is described as a "wobble" in Prob. 18, p. 399 of K.R. Symon, *Mechanics* (Addison-Wesley, 1971), http://kirkmcd.princeton.edu/examples/mechanics/symon_71.pdf, while for $\sin\alpha < 1/\sqrt{3}$, $\omega < \Omega$, the oscillating orbit is described as an "up-and-down spiraling motion".

For a more exotic example of motion of a particle on a cone, see http://kirkmcd.princeton.edu/examples/birkeland.pdf.

4. In the problem of a central force, from a fixed point, acting on mass m, the motion lies in a plane perpendicular to, say, the z-axis. The angular momentum $L = mr^2\,\dot\theta$, in a cylindrical coordinate system (r, θ, z), about the z axis is conserved, so we can write the (conserved) energy of the system as,

$$E = \frac{m}{2}(\dot r^2 + r^2\,\dot\theta^2) + V(r) = \frac{m\dot r^2}{2} + \frac{L^2}{2mr^2} + V(r) \equiv \frac{m\dot r^2}{2} + V_{\rm eff}\,, \tag{49}$$

where the introduction of the effective potential,

$$V_{\rm eff} = \frac{L^2}{2mr^2} + V(r), \tag{50}$$

renders the problem to be 1-dimensional.

For central potentials of the form,

$$V(r) = -\frac{C}{r^\lambda}\,, \tag{51}$$

for constants C and λ, we have,

$$\frac{dV_{\rm eff}}{dr} = -\frac{L^2}{mr^3} + \frac{\lambda C}{r^{\lambda+1}}\,, \qquad \frac{d^2V_{\rm eff}}{dr^2} = \frac{3L^2}{mr^4} - \frac{\lambda(\lambda+1)C}{r^{\lambda+2}}\,, \tag{52}$$

Equilibrium circular orbits for given angular momentum L_0 exist at radius r_0 such that $dV_{\rm eff}(r_0)/dr = 0$,

$$r_0^{2-\lambda} = \frac{L_0^2}{\lambda Cm}, \qquad \Omega = \dot\theta(r_0) = \frac{L_0}{mr_0^2}\,, \tag{53}$$

where Ω is the (constant) angular velocity of the stable orbit.

The effective spring constant for small oscillations about the equilibrium circular orbit is,

$$k_{\rm eff} = \frac{d^2V_{\rm eff}(r_0)}{dr^2} = \frac{L_0^2}{mr_0^4}\left(3 - \frac{\lambda(\lambda+1)Cmr_0^{2-\lambda}}{3L_0^2}\right) = \frac{L_0^2}{mr_0^4}(2-\lambda). \tag{54}$$

Stable orbits exists only if $k_{\rm eff} > 0$, *i.e.*,

$$\lambda < 2 \qquad \text{for stability.} \tag{55}$$

For small oscillations of a perturbed orbit, relative to a stable circular orbit, we write the radial motion as,

$$r(t) \approx r_0(1 + \epsilon\cos\omega t), \tag{56}$$

where $\epsilon \ll 1$ is a constant, and the angular frequency of the oscillation is,

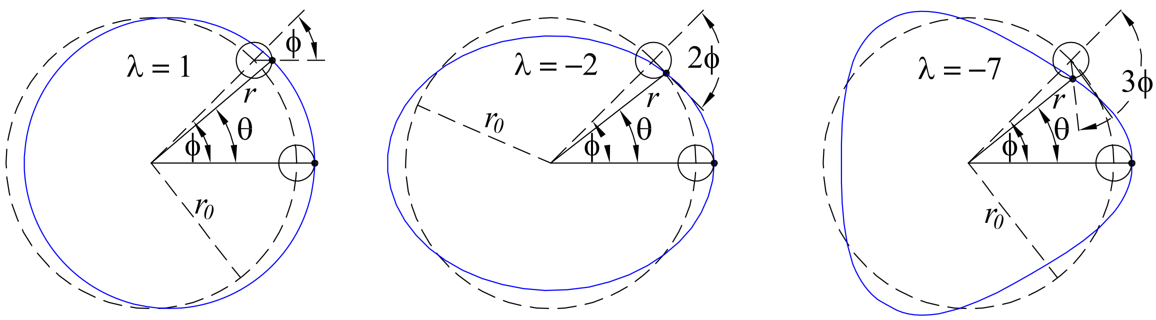
$$\omega = \sqrt{\frac{k_{\rm eff}}{m}} = \frac{L_0}{mr_0^2}\sqrt{2-\lambda} = \Omega\sqrt{2-\lambda}. \tag{57}$$

The corresponding motion in θ is then related by,

$$\dot{\theta} = \frac{L_0}{mr^2} \approx \frac{L_0}{mr_0^2}(1 - 2\epsilon\cos\omega t) = \Omega(1 - 2\epsilon\cos\omega t), \tag{58}$$

$$\theta(t) = \Omega(\left(t - \frac{2}{\omega}\epsilon\sin\omega t\right) = \Omega t - \frac{2\epsilon}{\sqrt{2-\lambda}}\sin\omega t. \tag{59}$$

The oscillatory motion is simply periodic only if ω is an integer multiple Ω, *i.e.*, for $\lambda = 1$ (gravity), -2 (springs) or -7, as illustrated in the figures below, where $\phi = \Omega t$.



Greek epicycles

Newtonian epicycles

For $\lambda = -2$,

$$r - r_0 = \epsilon r_0 \cos 2\Omega t, \qquad r_0(\theta - \Omega t) = -\epsilon r_0 \sin 2\Omega t, \tag{60}$$

which is uniform rotation with angular velocity -2Ω relative to the point $r_0, \Omega t$ in the rotating frame of the unperturbed orbit. That is, the motion can be described as motion in a circle (epicycle) about a point that moves in a circle.

This description in terms of an epicycle can also be applied to the cases of $\lambda = 1$ and -7, and gives the correct shape of the perturbed orbit. However, in these cases, the epicycle model does not correctly predict where the oscillating mass is at time t.[6] In particular, for $\lambda = 1$ (gravity), we have,

$$r - r_0 = \epsilon r_0 \cos\Omega t, \qquad r_0(\theta - \Omega t) = -2\epsilon r_0 \sin\Omega t, \tag{61}$$

which corresponds to motion in an ellipse with major axis twice the minor axis, in the rotating frame of the unperturbed mass. We could call this relative, elliptical motion a "Newtonian epicycle".

[6]To give a better description of gravitational orbits with large oscillations, Copernicus advocated use of epicycles upon epicycles (which can provide any desired accuracy. Kepler was the first to note that the a single ellipse suffices (for a single force center).

5. (a) The angular velocity ω of the orbit of a small mass m just above the surface of a spherical mass $M = 4\pi\rho r^3/3$ of radius r and mass density ρ is related by,

$$F = ma = m\omega^2 r = \frac{GmM}{r^2}, \qquad \omega^2 = \frac{GM}{r^3} = \frac{4\pi G\rho}{3}, \tag{62}$$

so the period $T = 2\pi/\omega$ is independent of r.

(b) **Satellite Paradox.**

We use an energy method.

The energy of mass m in a (near) circular orbit of radius r with velociity v about large mass M can be written as,

$$E = \frac{mv^2}{2} - \frac{GMm}{r} = -\frac{GMm}{2r} = -\frac{mv^2}{2}, \tag{63}$$

noting that $v^2 = \omega^2 r^2 = GM/r$, as follows from eq. (62),

Due to the friction of the atmosphere, $\mathbf{F} = -\alpha\mathbf{v}$, the satellite loses energy at rate,

$$\mathbf{F}\cdot\mathbf{v} = -\alpha v^2 = \frac{dE}{dt} = -mv\dot{v}, \tag{64}$$

$$\frac{\dot{v}}{v} = \frac{\alpha}{m}, \qquad v(t) = v_0\, e^{\alpha t/m}, \tag{65}$$

assuming that α is constant.

The velocity v increases as the satellites falls down!

The loss of kinetic energy due to friction is compensated by the conversion of gravitational potential energy into kinetic energy (at twice the rate of the loss of energy).[7]

For an approximately circular orbit,

$$r(t) = \frac{GM}{v^2} = \frac{GM}{v_0^2}\, e^{-2\alpha t/m} = r_0\, e^{-2\alpha t/m}. \tag{66}$$

A simple solution also follows from the assumption that $\mathbf{F} = -\alpha v^2\,\hat{\mathbf{v}}$, *i.e.*, $v(t) = v_0/(1 - \alpha v_0 t/m)$.[8] Again, the velocity increases as the satellite falls.

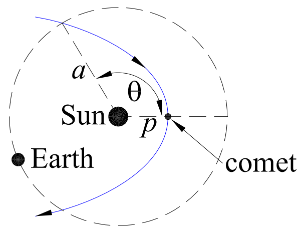
This problem was considered by Newton, as reviewed in D.G. King-Hele and D.C.M. Walker, *The Effect of Air Drag on Satellite Orbits: Advances in 1687 and 1987*, Vistas Astron. **30**, 269 (1967),
http://kirkmcd.princeton.edu/examples/mechanics/king-hele_va_30_269_87.pdf

[7]Among the relevant literature, see D.G. King-Hele, *The Descent of an Earth-Satellite Through the Upper Atmosphere*, J. Brit. Interplanetary Soc. **15**, 314 (1956),
http://kirkmcd.princeton.edu/examples/mechanics/king-hele_jbis_15_314_56.pdf,
L. Blitzer, *Satellite Paradox*, Am. J. Phys. **39**, 882 (1971),
http://kirkmcd.princeton.edu/examples/mechanics/blitzer_ajp_39_882_71.pdf.

[8]M. Morduchow and G. Volpe, *Exact Analytical Solutions for Orbits of Bodies with Atmospheric Drag* AIAA J. **11**, 381 (1973),
http://kirkmcd.princeton.edu/examples/mechanics/morduchow_aiaaj_11_381_73.pdf

6. *This is probs. 6.27-28, p. 271 of G.R. Fowles and G.L. Cassiday,* Analytical Mechanics, 7^{th} *ed. (Thomson Brooks/Cole, 2004),*
http://kirkmcd.princeton.edu/examples/mechanics/fowles_chap6.pdf

A comet in a parabolic orbit about the Sun has its perihelion at distance p.



(a) A parabolic orbit is the case between the bound elliptical orbits (with total energy $E < 0$), and the unbound hyperbolic orbits (with total energy $E > 0$. That is, the total energy of a mass in a parabolic orbit is zero.

We could also note that the energy is $E = mv^2/2 = GMm/r$, *so for large* r, $E \to mv_\infty^2/2$. *Also, angular momentum is conserved about the force center,* $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$, *so for large* r, *where* $\mathbf{v}$ *becomes parallel to the axis of the parabola, we must have* $v_\infty \to 0$, *and hence* $E = 0$.

This argument does not hold for a hyperbolic trajectory, for which $\mathbf{r}$ *becomes parallel to* $\mathbf{v}$, *so* v_∞ *can be/is* > 0, *and* $E > 0$.

(b) Recalling the analysis of the orbit equation, p. 105 of
http://kirkmcd.princeton.edu/examples/Ph205/ph205l10.pdf,
we have that for energy $E = 0$, the equation of the parabola with respect to the force center of mass M is,

$$\frac{1}{r} = \frac{GMm^2}{L^2}(1 + \cos\theta), \tag{67}$$

where L is the angular momentum of mass m with respect to the force center. The perihelion p is then related by,

$$\frac{1}{p} = \frac{2GMm^2}{L^2}, \qquad L = m\sqrt{2GMmp}. \tag{68}$$

(c) The equation (67) of the parabolic orbit can be rewritten as,

$$\frac{1}{r} = \frac{1 + \cos\theta}{2p}, \qquad \cos\theta = \frac{2p}{r} - 1. \tag{69}$$

Hence the angle of the intersection of the comet's and the Earth's orbit (of radius a) about the Sun is given by,

$$\cos\theta = \frac{2p}{a} - 1, \tag{70}$$

which exists only if $p < a$.

(d) We recall p. 106 of the above link, which showed that the time t for the mass to move from the perihelion to radius r is related by, using eq. (68),

$$t = \sqrt{\frac{m}{2}} \int_p^r \frac{r'\,dr'}{\sqrt{Er'^2 + GMmr' - L^2/2m}}\,. \tag{71}$$

$$\begin{aligned} t &= \sqrt{\frac{1}{2GM}} \int_p^r \frac{r'\,dr'}{\sqrt{r'-p}} = \sqrt{\frac{1}{2GM}} \int_0^{r-p} \frac{(x+p)}{\sqrt{x}}\,dx \\ &= \sqrt{\frac{1}{2GM}} \left(\frac{2}{3}(r-p)^{3/2} + 2p(r-p)^{1/2}\right) = \frac{1}{3}\sqrt{\frac{2r^3}{GM}}\sqrt{1-\frac{p}{r}}\left(1+\frac{2p}{r}\right). \end{aligned} \tag{72}$$

Hence, the time spent by the comet inside the Earth's orbit (of radius a) is ,

$$T = \frac{2}{3}\sqrt{\frac{2a^3}{GM}}\sqrt{1-\frac{p}{a}}\left(1+\frac{2p}{a}\right) \qquad (\text{for } p < a). \tag{73}$$

The maximum time is related by,

$$\frac{dT}{dp} = 0 = \frac{2}{3}\sqrt{\frac{2a^3}{GM}}\left[-\frac{1}{2a\sqrt{1-\frac{p}{a}}}\left(1+\frac{2p}{a}\right) + \frac{2}{a}\sqrt{1-\frac{p}{a}}\right], \qquad \frac{p}{a} = \frac{1}{2}, \tag{74}$$

$$T_{\max} = \frac{4}{3}\sqrt{\frac{a^3}{GM}} = \frac{2}{3\pi}\left(2\pi\sqrt{\frac{a^3}{GM}}\right) = 0.21\,\text{yr} \approx 11\,\text{weeks}. \tag{75}$$

7. This is Prob. 50, p. 154 of K.R. Symon, *Mechanics*, 3rd ed. (Addison-Wesley, 1971), http://kirkmcd.princeton.edu/examples/mechanics/symon_71.pdf.

The effective potential for a particle of mass m and angular momentum L about the origin, subject to the attractive central force,

$$F = -\frac{C}{r^3}, \qquad V = -\frac{C}{2r^2}, \tag{76}$$

with $C > 0$ is,

$$V_{\text{eff}} = \frac{L^2}{2mr^2} + V = \frac{L^2}{2mr^2} - \frac{C}{2r^2} = \frac{L^2 - 2mC}{2mr^2}, \tag{77}$$

$$\frac{dV_{\text{eff}}}{dr} = -\frac{L^2 - 2mC}{mr^3}, \tag{78}$$

$$\frac{d^2V_{\text{eff}}}{dr^2} = \frac{3(L^2 - 2mC)}{mr^4}. \tag{79}$$

Circular orbits (with $dV_{\text{eff}}(r_0)/dr = 0$) are possible with any radius r_0, but only for angular momentum $L = \sqrt{2mC}$. However, none of these circular orbits is stable, since the effective spring constant $k_{\text{eff}} = d^2V_{\text{eff}}(r_0)/dr^2$ is 0.

To discuss the form of the orbits, we consider the orbit equation,

$$\frac{d^2u}{d\theta^2} = -u - \frac{m}{L^2u^2}F(1/u) = u\left(\frac{Cm}{L^2} - 1\right) \equiv \beta u, \tag{80}$$

where L is the nonzero angular momentum about the force center. Hence, for orbits with initial conditions u_0 and $du/d\theta|_0 = -u^2\, dr/d\theta|_0 = u_0'$,

$$u = u_0 \cosh\beta\theta + \frac{u_0'}{\beta}\sinh\beta\theta = a\, e^{\beta\theta} + b\, e^{-\beta\theta}, \qquad (\beta > 0), \tag{81}$$

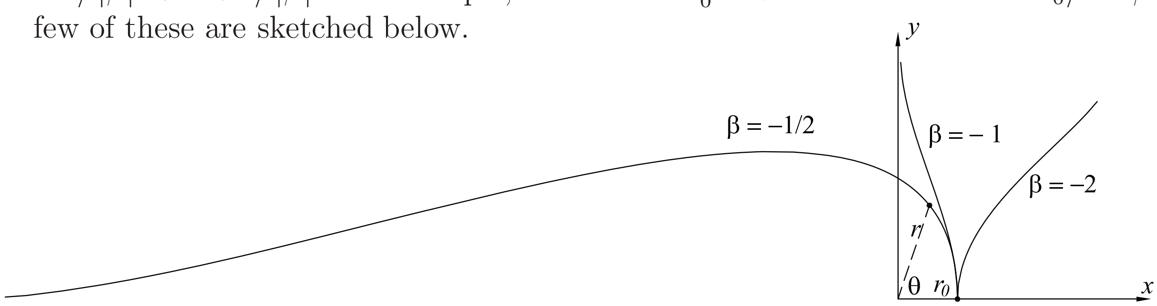
$$u = u_0 + u_0'\theta, \qquad (\beta = 0), \tag{82}$$

$$u = u_0\cos\beta\theta - \frac{u_0'}{\beta}\sin\beta\theta = u_0\cos(|\beta|\,\theta) + \frac{u_0'}{|\beta|}\sin(|\beta|\,\theta). \qquad (\beta < 0). \tag{83}$$

Circular orbits correspond to the form (82) with $u_0' = 0$ (and $L^2 = 2mC$). However, for any nonzero value of u_0', these orbits are spirals, either in to the origin or out to infinity.

Orbits of the form (81) are inward spirals for increasing θ if $a = 0$, but in general spiral outwards after possible inward behavior for small (positive) θ.

Orbits of the form (83) have $u = 0$ ($r = \infty$) for some value of θ in the interval $-\pi/|\beta| < \theta < \pi/|\beta|$. For example, orbits with $u_0' = 0$ have the form $r = r_0/\cos\beta\theta$; a few of these are sketched below.



8. The orbit equation for the force law,

$$F = -\frac{Am}{r^2} - \frac{Bm}{r^3}, \tag{84}$$

is,

$$\frac{d^2u}{d\theta^2} = -u - \frac{m}{L^2u^2}F(1/u) = \frac{Am^2}{L^2} - u\left(1 - \frac{Bm^2}{L^2}\right) \equiv \alpha - \beta^2 u, \tag{85}$$

where $\alpha = Am/L^2$, $\beta = \sqrt{1 - Bm^2/L^2}$ and L is the nonzero angular momentum of mass m about the force center.

Consequently, the orbit has the form,

$$u = \frac{1}{r} = \frac{\alpha}{\beta^2} + C\cos\beta\theta. \tag{86}$$

We are concerned with a possible, small correction, $-B/r^2$, to the Newtonian gravitational force, $-a/r^2$, so $\beta = \sqrt{1 - Bm^2/L^2}$ differs only slightly from unity, and the orbits are very similar to the elliptical orbits of Kepler and Newton. So, we infer that the form (86) can be written as,

$$u = \frac{1}{r} = \frac{1 + \epsilon\cos\beta\theta}{a(1 - \epsilon^2)}, \tag{87}$$

where a is the semimajor axis of the orbital ellipse, ϵ is the eccentricity, and the departure from the Newtonian form is in the factor $\cos\beta\theta$, which implies that the ellipse precesses slowly. The sense of the precession is in the direction of the orbital motion, since $\beta < 1$, such that θ must increase by greater than 2π from one perihelion to the next.

Writing $\beta = 1 - \delta$, we have that $\delta = Bm^2/2L^2$, and the perihelion advances by,

$$\Delta\theta = \frac{2\pi}{\beta} - 2\pi \approx 2\pi\delta = \frac{\pi Bm^2}{L^2}, \tag{88}$$

per orbital revolution.

From p. 96 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l9.pdf, we have that,

$$L = \frac{2\pi ma^2\sqrt{1 - \epsilon^2}}{T}, \tag{89}$$

while from Kepler's 3rd law,

$$T^2 = \frac{4\pi^2a^3}{A}. \tag{90}$$

Then,

$$L^2 = 4\pi^2m^2a^4(1 - \epsilon^2)\frac{A}{4\pi^2a^3} = Am^2a(1 - \epsilon^2), \tag{91}$$

and,

$$\Delta\theta = \frac{\pi B m^2}{L^2} = \frac{B}{2Aa(1-\epsilon^2)} = \frac{\pi\eta}{1-\epsilon^2}\,, \tag{92}$$

where $\eta = B/Aa \ll 1$.

The orbital period of Mercury is 0.24 Earth years, so in 100 Earth years, Mercury has $100/0.24 = 416.7$ orbital periods. During this time, the perihelion of Mercury advances by $40'' = 40/60 \cdot 60 \cdot (180/\pi) = 1.9 \times 10^{-4}$ rad, which is also $416.7\Delta\theta = 1360\eta$, for $\epsilon_{\rm Mercury} = 0.206$.

Altogether, the model of gravity with a $1/r^3$ correction could explain the advance of the perihelion of Mercury if,

$$\eta = \frac{B}{Aa} = \frac{1.9 \times 10^{-4}}{1360} = 1.4 \times 10^{-7}. \tag{93}$$

9. (a) For the gravitational potential,

$$V = -\frac{GMm}{r} - \frac{GMm\eta R^2(1 - 3\cos^2\theta)}{5r^3}, \tag{94}$$

for interaction in with mass m at distance r from the Sun, of mass M, the effective potential is,

$$V_{\text{eff}} = V + \frac{L^2}{2mr^2} = -\frac{GMm}{r} - \frac{GMm\eta R^2}{5r^3} + \frac{L^2}{2mr^2}, \tag{95}$$

$$\frac{dV_{\text{eff}}}{dr} = \frac{GMm}{r^2} + \frac{3GMm\eta R^2}{5r^4} - \frac{L^2}{mr^3}, \tag{96}$$

$$\frac{d^2V_{\text{eff}}}{dr^2} = -\frac{2GMm}{r^3} - \frac{12GMm\eta R^2}{5r^5} + \frac{3L^2}{mr^4}. \tag{97}$$

where M is the mass of the Sun, and the motion is in the plane $\theta = 90°$.

For the equilibrium circular orbit of radius r_0, the angular momentum is $L = mr_0^2\Omega$, where Ω is the equilibrium angular velocity, and the condition that eq. (96) be zero at equilibrium tells us that,

$$0 = \frac{GMm}{r_0^2} + \frac{3GMm\eta R^2}{5r_0^4} - m\Omega^2 r_0, \qquad \frac{1}{r_0^3} = \frac{\Omega^2}{GM} - \frac{3\eta R^2}{5r_0^5}, \tag{98}$$

which is a perturbed version of Kepler's 3rd law.

The effective spring constant for small oscillations about the equilibrium orbit is,

$$k_{\text{eff}}(r_0) = \frac{d^2V_{\text{eff}}}{dr^2} = -\frac{2GMm}{r_0^3} - \frac{12GMm\eta R^2}{5r_0^5} + 3m\Omega^2 = m\Omega^2 - \frac{6GMm\eta R^2}{5r_0^5}, \tag{99}$$

and the angular frequency of the small oscillations is,

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \Omega\sqrt{1 - \frac{6GM\eta R^2}{5\Omega^2 r_0^5}} \approx \Omega\left(1 - \frac{3GM\eta R^2}{5\Omega^2 r_0^5}\right) \equiv \Omega(1-\delta). \tag{100}$$

The period of a perturbed orbit is $2\pi/\omega$, which is slightly larger than that of a circular orbit. As such, the perihelion of a perturbed orbit advances, with angular velocity

$$\omega_{\text{precess}} \approx \Omega\delta = \frac{3GM\eta R^2}{5\Omega r_0^5}. \tag{101}$$

The observed precession of the perihelion of Mercury, 40″ per century, corresponds to an angular velocity of the precession,

$$\omega_{\text{precess}} = \frac{40}{60 \cdot 60 \cdot 180/\pi \cdot 100\pi \times 10^7} = 6.2 \times 10^{-14} \text{ rad/s}. \tag{102}$$

For this to be explained by eq. (101), due to oblateness of the Sun, we need,

$$\eta = \frac{5\Omega r_0^5 \omega_{\rm precess}}{3GMR^2} = \frac{10\pi r_0^5 \omega_{\rm precess}}{3GMR^2 T_{\rm Mercury}}$$
$$= \frac{10\pi \cdot (6 \times 10^{10})^5 \cdot 6.2 \times 10^{-14}}{3 \cdot 6.7 \times 10^{-11} \cdot 2 \times 10^{30} \cdot (7 \times 10^8)^2 \cdot 0.24\pi \times 10^7} \approx 10^{-3}, \qquad (103)$$

for $M = 2 \times 10^{30}$ kg, $R = 7 \times 10^8$ m, $r_0 = 6 \times 10^{10}$, $T_{\rm Mercury} = 0.24$ Earth year (of $\approx \pi \times 10^7$ s).

(b) The force law of an oblate Sun, follows from eq. (94) as,

$$F_{\rm oblate} = -\frac{GMm}{r^2}\left(1 - \frac{3\eta R^2}{5r^2}\right), \qquad (104)$$

for motion in the equatorial plane of the Sun, which is formally similar to the force law of general relativity,

$$F_{\rm Einstein} = -\frac{GMm}{r^2}\left(1 - \frac{3L^2}{(mrc)^2}\right), \qquad (105)$$

Hence, the results of part (a) apply for general relativity with the replacement,

$$\frac{\eta R^2}{5} \to \frac{L^2}{(mc)^2} = \frac{r_0^4 \Omega^2}{c^2}. \qquad (106)$$

Then, eq. (101) for the predicated angular velocity of the precession of the perihelion becomes,

$$\omega_{\rm precess} = \frac{3GM\eta R^2}{5\Omega r_0^5} \to \frac{3GM\Omega}{r_0 c^2} = \frac{6\pi GM}{r_0 c^2 T}, \qquad (107)$$

The advance of the perihelion per revolution is,

$$\delta\phi = \omega_{\rm precess} T = \frac{6\pi GM}{r_0 c^2} = \frac{24\pi^3 r_0^2}{T^2 c^2}, \qquad (108)$$

where we used Kepler's 3rd law, $GM/r_0^3 = \Omega^2 = 4\pi^2/T^2$ in the last step.

Strictly, r_0 is the semimajor axis a of the orbit, as noted by Einstein in eq. (14) of his calculation of the precession,
http://kirkmcd.princeton.edu/examples/GR/einstein_skpaw_831_15_english.pdf

$$\delta\phi = \frac{24\pi^3 a^2}{T^2 c^2 (1 - \epsilon^2)}, \qquad (109)$$

to order v^2/c^2, where ϵ is the eccentricity of the orbit.