Princeton University

Ph205
Mechanics
Problem Set 4

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(1988)

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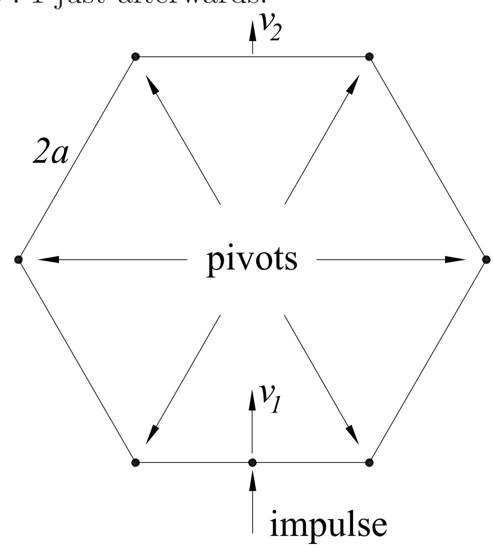
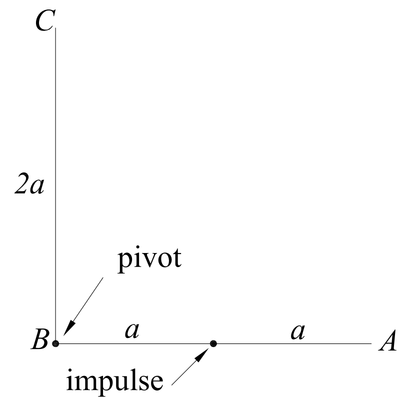
http://kirkmcd.princeton.edu/examples/

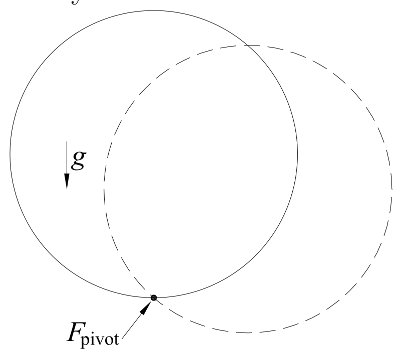
1. A regular hexagon has edges of length $2a$. The edges are rigid rods of mass m that are joined together with frictionless pivots. If a blow is struck perpendicular to the midpoint of the lower edge, show that the ratio of the velocities v_1 and v_2 of the lower and upper edges is $10 : 1$ just afterwards.



2. Rods AB and BC, shown in the figure below, each have mass m and length $2a$. They are joined at B by a frictionless pivot. Initially the rods are at rest, with angle $ABC = 90°$. A blow is struck at the midpoint of AB such that the system moves as a rigid body in the next instant (*i.e.*, the angle ABC remains 90° just after the impulse). Show that the impulse makes angle 45° to rod AB, and that velocity $v_A = \sqrt{13}\, v_C$ just after the impulse.



3. A uniform disk of mass m rotates without friction in a vertical plane about a point on its circumference. Initially the disk is balanced above the pivot point – then it falls. Show that the force on the pivot is $\sqrt{17}\,mg/3$ when the disk has rotated by $90°$, and $11\,mg/3$ when it has rotated by $180°$.

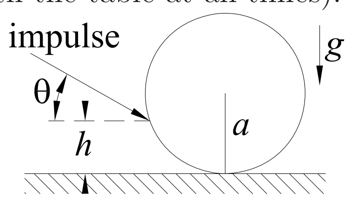


It is instructive to consider the force component along both x-y and r-θ axes.

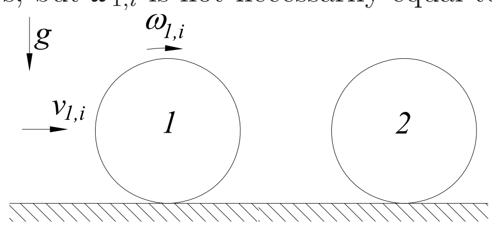
4. **Billiards**

   (a) **Returning Ball.** If the (uniform) cue ball, of radius a is struck with the cue horizontal, the ball can never return. But if the cue is tilted it is possible.

   Suppose the impulse and the center of the ball lie in a vertical plane, and that the impulse is applied at height h above the table. Show that the angle θ of the cue to the horizontal must obey $\tan\theta > 1/\sqrt{2a/h - 1}$ for the ball to return (assuming that friction at the table during the impulse can be ignored, and that the ball remains in contact with the table at all times).



   (b) **Follow Shots and Draw Shots.** The cue ball, 1, strikes another ball, 2, of the same mass and radius such that the initial (horizontal) velocity $\mathbf{v}_{1,i}$ of ball 1 is along the line of centers of the balls 1 and 2, with ball 2 initially at rest. The initial angular velocity $\boldsymbol{\omega}_{1,i}$ is perpendicular to the vertical plane that contains the line of centers, but $\omega_{1,i}$ is not necessarily equal to $v_{1,i}/a$.



   Assume no friction between the balls, and that the collision is elastic.

   Show that when ball 2 finally rolls without slipping, $v_{2,f} = 5v_{1,i}/7$, independent of μ the coefficient of sliding friction of the ball with the table.

   Also show that when ball 1 finally rolls without slipping, $v_{1,f} = 2\omega_{1,i}/7$.

   Hence, ball 1 follows ball 2 unless $\omega_{1,i} \leq 0$.

   (c) **English** (optional).

   If the cue is not horizontal, and not pointing at the center of the ball, the latter acquires “english”, and does not move in a straight line. We wish to deduce the path of the center of mass of the ball.

   If the ball is not to move in straight line, there must be some friction perpendicular to $\mathbf{v}_{\rm cm}$ of the ball. This requires there to be some rotation about an axis parallel to $\mathbf{v}_{\rm cm}$, which rotation is called “english”.

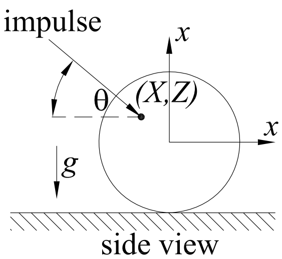
   The concept of “english” also includes rotation of the ball about the vertical axis, but this does not influence the motion of the ball between collisions. If the area of contact of the ball with the table is small, there is negligible torque about the vertical (z) axis, so ω_z never changes, and does not affect $\mathbf{v}_{\rm cm}$.

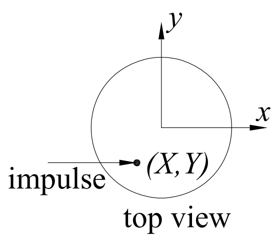
   A uniform sphere, of moment of inertia $I = ma^2/5$ about its center, has angular momentum $\mathbf{L} = I\boldsymbol{\omega}$ for any direction of angular velocity $\boldsymbol{\omega}$.

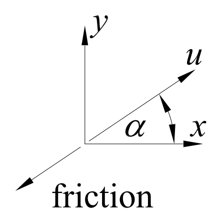
To discuss the motion of the ball, first deduce its initial motion after being struck by the cue, and then deduce the subsequent motion (changes in which are caused by friction of the table against the ball.

Some suggestions:

To find the initial motion, use a coordinate system with z vertical, and the cue in the x-x plane, with the origin at the center of the ball (initially at rest). Let (X, Y, Z) be the coordinates of the point of contact of the cue with the ball, $X^2 + Y^2 + Z^2 = a^2$.







If P is the magnitude of the impulse, and the cue makes angle θ to the horizontal, then $\mathbf{P} = (P\cos\theta, 0, -P\sin\theta)$.

Calculate $v_{x,i}$, $\omega_{x,i}$ and ω_{y_i}.

To discuss the subsequent motion of the ball, define $\mathbf{u}$ to be the velocity of the point on the ball in instantaneous contact with the table ($\mathbf{u} = 0$ for rolling without slipping). Also define α as the angle of $\mathbf{u}$ to the x-axis: $u_x = u\cos\alpha$, $u_y = u\sin\alpha$.

The force of sliding friction is then $\mathbf{F} = -\mu mg\,\hat{\mathbf{u}}$.

The equations of motion are then $m\dot{\mathbf{v}} = \mathbf{F}$ and $I\dot{\boldsymbol{\omega}} = \mathbf{r}\times\mathbf{F}$, where $\mathbf{r} = (0, 0, -a)$.

Relate $\mathbf{u}$ to $\mathbf{v}$ and $\boldsymbol{\omega}$, and find $\dot{u}_x$ and $\dot{u}_y$, and then $\dot{\mathbf{u}}$ and $\dot{\alpha}$.

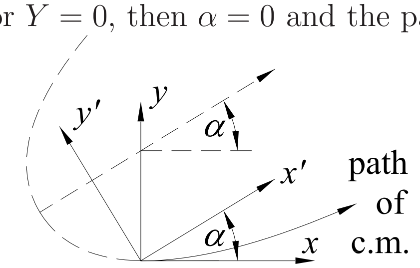
Show that the ball begins to roll without slipping at time $t = 2u_i/7\mu g$ after being struck by the cue, and that α is constant.

Hence, in a coordinate system with x' along $\mathbf{u}$, $v_{y'}$ and $\dot{v}_{x'}$ are constant. The path of the ball is a parabola whose axis makes angle α to the x-axis.

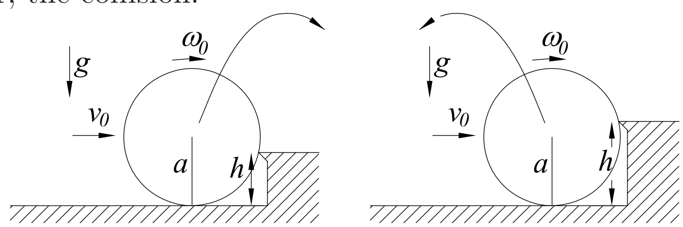
Verify that,

$$\tan\alpha = \frac{-\frac{5}{2}\frac{Y}{a}\sin\theta}{\left(1 + \frac{5}{2}\frac{Z}{a}\right)\cos\theta + \frac{5}{2}\frac{X}{a}\sin\theta}. \qquad (1)$$

If either $\sin\theta = 0$ or $Y = 0$, then $\alpha = 0$ and the path reduces to a straight line.



5. A ball of radius a collides with a bumper of height $h < a$. Just before the collision the center of the ball has velocity $\mathbf{v}_0$ perpendicular to the bumper, and angular velocity $\boldsymbol{\omega}_0$ directed along $\mathbf{v}_0 \times \mathbf{g}$, as in the figure. Suppose there is no slipping at the bumper during, or after, the collision.



(a) Show that the ball can jump up onto the cushion/step if,

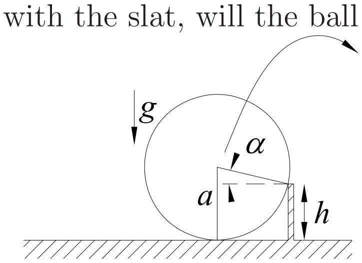
$$v_0 \geq \frac{a}{a-h}\sqrt{\frac{14gh}{5}} \quad \text{and} \quad \omega_0 = 0, \tag{2}$$

$$\text{or} \quad v_0 \geq \frac{a}{7a-5h}\sqrt{\frac{70gh}{5}} \quad \text{and} \quad \omega_0 = \frac{v_0}{a}. \tag{3}$$

Show also that if $\omega_0 = v_0/a$ and $a < h < 7a/5$ and condition (3) holds, the ball jumps upward and to the left, and then falls onto the table.

(b) If v_0 and h are great enough the ball will lose contact with the cushion/step and fly into the air. For $v_0 = v_{0,\min}$ found in eqs. (2) and (3), show that the ball flies if $h \geq 7a/17$.

(c) (Optional.) Suppose the step is just a narrow slat of height $h < a$. Again suppose there is no slipping at the bumper during, or after, the collision. If the ball flies up and loses contact with the slat, will the ball hit the slat again?



Show that the ball will hit the slat at time t after the initial collision given by a real positive root (if this exists) of the equation,

$$\frac{1}{4}g^2t^2 - gvt\cos\alpha + v^2 - ag\sin\alpha = 0, \tag{4}$$

where v is the velocity of the center of the ball just after the collision, and *alpha* is the angle of the center of the ball to the top of the slat, as shown in the figure.

6. **Relativistic Rocket.** We don't use much special relativity in Ph205, but you may wish to do this problem to keep limber.

   In the nonrelativistic rocket problem we used conservation of mass and momentum to deduce the equation of motion.[1] In the relativistic version, we replace conservation of momentum by conservation of total energy, include the rest energy/mass.

   Try it yourself. If you find you need help, consider the following suggestions:

   Consider a typical relativity trick. First discuss the equation of motion in the rest frame in of the rocket, and then transform to the lab frame in which the rocket is moving.

   In the rest frame of the rocket, let $m^\star$ be the rest mass of the rocket (plus fuel) at some time $t^\star$. After a short time, the rocket has mass $m_1^\star$, and velocity $dv^\star$ as a result of spewing out exhaust of mass $dm^\star < 0$ at velocity $u > 0$ relative to the rocket.

   Consider conservation of mass/energy and momentum in this frame to show that for small $dv^\star$ and $|dm^\star| \ll m^\star$,

$$dv^\star \approx -u\frac{dm^\star}{m^\star} \tag{5}$$

   In the lab frame, the velocity of the rocket changes from v to $v + dv$ during the above process. Use the relativistic velocity transformation of $dv^\star$ to the frame where the rocket has velocity v to show that,

$$dv^\star \approx \gamma^2 dv = c\, d\left(\tanh^{-1}\frac{v}{c}\right), \tag{6}$$

   where $\gamma = 1/\sqrt{1 - v^2/c^2}$ and c is the speed of light.[2] Finally, show that,

$$\frac{v}{c} = \frac{\left(\frac{m_0^\star}{m^\star}\right)^{2u/c} - 1}{\left(\frac{m_0^\star}{m^\star}\right)^{2u/c} + 1}, \tag{7}$$

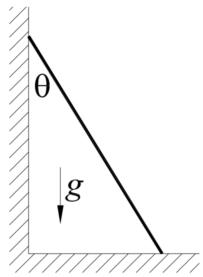
   where $m^\star = m_0^\star$ when $v = 0$.

[1]See p. 80 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l7.pdf

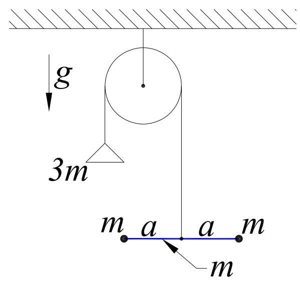
[2]For what it's worth, $\gamma^2\, dv$ is a relativistic invariant even for finite dv.

7. A ladder rests at angle θ against a frictionless wall with its feet on a frictionless floor. If the ladder starts from rest at angle θ_0, show that it loses contact with the wall when,

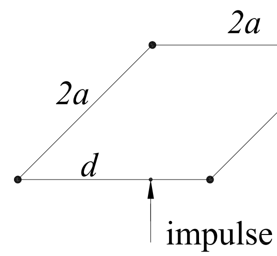
$$\cos\theta = \frac{2}{3}\cos\theta_0. \tag{8}$$



8. A thin rod of length $2a$ and mass m has point masses m attached at both ends. A string tied to the middle of the rod goes over a massless pulley and is attached to a mass of $3m$. At a time when the system is at rest, one of the masses at the ends of the rods falls off. Show that the just after this, the angular acceleration of the rod is $\ddot{\theta} = 18g/17a$, and the tension in the string is $30mg/17$, neglecting the radius of the pulley.



9. A rhombus with rigid sides of length $2a$ and frictionless pivots at its corners lies on a table without friction. At what distance d from an acute corner should an impulse be applied, perpendicular to the side, such that the rhombus rotates and translates but does not change shape?



At what distance should the (perpendicular) impulse be applied such that the long diagonal of the rhombus translates but does not rotate, while the rhombus collapses to a straight line (along the diagonal)?

The problem shows the importance of a good choice of coordinates in Lagrange's method. Here, there are 4 degrees of freedom: rotation of the diagonal, deformation, and x-y translations of the center.

We desire coordinates such that each corresponds to the motion in only one of these degrees of freedom. That is, the kinetic energy depends only on the four $\dot{q}_j^2$ and not on any $\dot{q}_j\dot{q}_l$. Then, if the generalized impulse associated with coordinate j vanishes, there is no impulsive motion in that coordinate.

10. **Super-Ball Bounces.** A Super-Ball can be idealized as a rigid sphere whose collisions with another idealized, rigid object conserve mechanical energy.

For collisions in which the motion of relevant mass centers lies in a plane, the Super-Ball can be described by 5 variable coordinates, x and y of its center, and its "spin" angular velocity $\boldsymbol{\omega}$.

During the collision, angular momentum is conserved about the point of contact relevant to the collision.

Thus, far, we have 2 relations among 5 variables of the Super-Ball.

It seems common to suppose that there is no slipping of the Super-Ball at the point of contact during the collision. This has the implication that ω_x and ω_y are reduced to zero during the collision, as well as the velocity of the point of contact of the ball during the collision. With these assumptions, there are enough conditions to determine the motion of the Super-Ball after the collision.

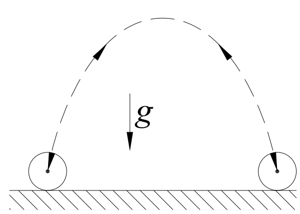
However, the literature seems to favor imposition of an additional condition, that kinetic energy for motion perpendicular to the plane of the collision is separately conserved.

With all these conditions, the problem is overconstrained.

Consider two cases in which a Super-Ball moves in the x-y plane, with the y-axis vertical, and initial angular velocity $\omega_x = \omega_y = 0$ but ω_z nonzero, and eventually collides with a hard floor at $y = 0$. For each case, discuss the motion supposing that either energy of the y-motion is conserved, or that there is no slippage at the point of contact of the collision with the floor.

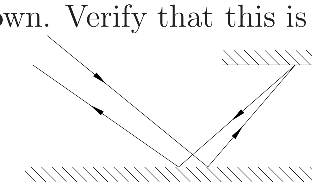
(a) A Super-Ball is dropped vertically with spin $\boldsymbol{\omega} = (0, 0, \omega)$ onto a hard floor. Discuss the subsequent motion of the ball through several bounces.

(b) Under what conditions does the Super-Ball bounce back and forth continuously between two fixed points on a hard floor?

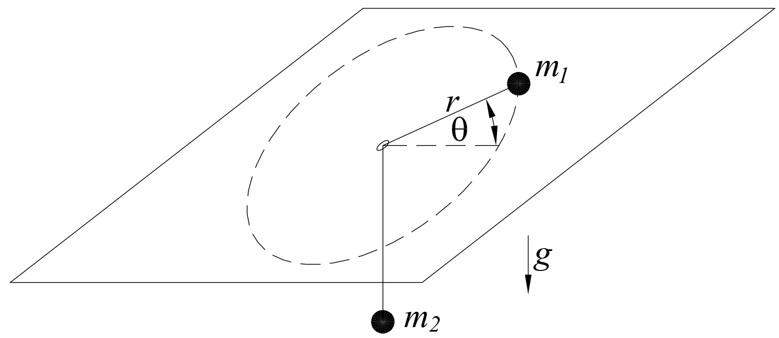


(c) (Optional.)

Another surprising behavior of a Super-Ball is that when thrown hard (so that its trajectory consists essentially of straight-line segments) with no initial spin onto the floor such that it bounces up and under a table, after the bounce on the underside of the table, and another bounce on the floor, the ball returns close to from where it was thrown. Verify that this is predicted by our model.



11. Mass m_1 is connected to mass m_2 by a string that passes through a hole in a frictionless, horizontal table.



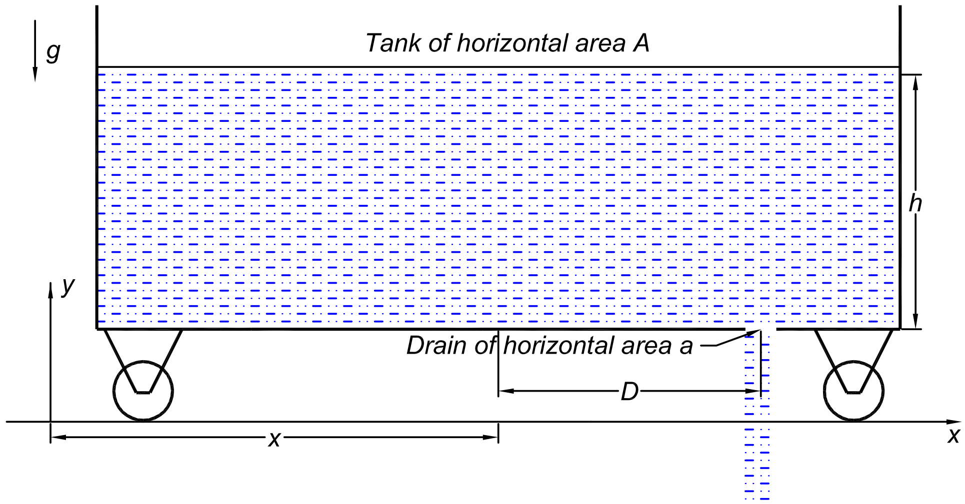
(a) Derive the equations of motion, and construct an effective potential for it.
What is the equilibrium radius r_0 for angular momentum, L_0?
Show that the angular frequency of small oscillations about r_0 is,

$$\omega = \sqrt{\frac{3m_2 g}{(m_1 + m_2)r_0}}. \tag{9}$$

(b) Suppose that m_1 initially moves is a circle of radius r_0, when some dust is sprinkled on the table, resulting in a small coefficient μ of sliding friction. How far does m_1 travel, and for what time, supposing also that the motion remains approximately circular at all times?

12. (Optional challenge problem:) **Motion of a Leaky Tank Car**

Discuss the motion of a tank car, intially at rest on frictionless, horizontal tracks, after a drain is opened, supposing the water flows out of the drain vertically in the rest frame of the car.



This problem is nontrivial if the drain is off center.

A qualitative discussion can be given with no equations, independent of the form $\dot{m}(t)$ of the flow of water.

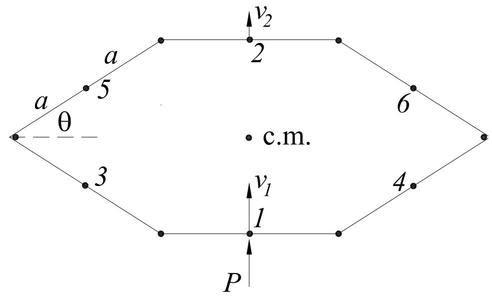
For greater detail, suppose that the flow obeys the usual Torricelli approximation for that out of a small hole in a tank.

A solution is at http://kirkmcd.princeton.edu/examples/tankcar.pdf.

Solutions

1. *This problem is from Ex. 2, p. 332 of* E.J. Routh, *The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 7^{th} ed. (Macmillan, 1905),
http://kirkmcd.princeton.edu/examples/mechanics/routh_elementary_rigid_dynamics.pdf

The impulsive motion of the hexagonal structure, consisting of six rods of mass m and length $2a$, can be described via coordinates, $x = x_{\text{cm}}$, $y = y_{\text{cm}}$ and θ, with y upwards in the figure below.



The centers of the six rods are at,

$$x_{1,2} = x \qquad x_{3,5} = x - a(1 + \cos\theta), \qquad x_{4,6} = x + a(1 + \cos\theta), \tag{10}$$

$$y_{1,2} = y \mp 2a\sin\theta, \qquad y_{3,4} = y - a\sin\theta, \qquad y_{5,6} = y + a\sin\theta. \tag{11}$$

The kinetic energy is, for $\dot{x} = 0$,

$$T = \sum_i \frac{m(\dot{x}_i^2 + \dot{y}_i^2)}{2} + 4\frac{I\dot{\theta}^2}{2} = 2ma^2\sin^2\theta\,\dot{\theta}^2 + 3m\dot{y}^2 + 6ma^2\cos^2\theta\,\dot{\theta}^2 + \frac{2ma^2\dot{\theta}^2}{3}$$

$$= 3m\dot{y}^2 + 4ma^2\cos^2\theta\,\dot{\theta}^2 + \frac{8ma^2\dot{\theta}^2}{3}\,, \tag{12}$$

noting that the moment of inertia of each rod about its center is $I = ma^2/3$.

The generalized forces associated with an upward force $\mathbf{F}$ at point 1 are,

$$Q_y = \mathbf{F}\cdot\frac{\partial\mathbf{r}_1}{\partial y} = F, \qquad Q_\theta = \mathbf{F}\cdot\frac{\partial\mathbf{r}_1}{\partial\theta} = -2aF\cos\theta, \tag{13}$$

and the generalized impulses associated with the impulse $P = \int F\,dt$ (at time $t = 0$) are,

$$P_y = P, \qquad P_\theta = -2aP\cos\theta. \tag{14}$$

Lagrange's equations for the motion just after the impulse (with $\dot{y} = 0 = \dot{\theta}$ just before it, and $\cos\theta = 1/2$) are,

$$P_y = P = \frac{\partial T}{\partial\dot{y}} = 6m\dot{y}, \qquad \dot{y} = \frac{P}{6m}\,, \tag{15}$$

$$P_\theta = -2aP\cos\theta = \frac{\partial T}{\partial\dot{\theta}} = 8ma^2\cos^2\theta\,\dot{\theta}^2 + \frac{16ma^2\dot{\theta}}{3}\,. \qquad \dot{\theta} = -\frac{3P}{22ma}. \tag{16}$$

The velocities of points 1 and 2 just after the impulse, when $\cos\theta = 1/2$, are,

$$v_1 = \dot{y} - 2a\cos\theta\,\dot{\theta} = \frac{P}{6m} + \frac{3P}{22m} = \frac{40P}{132m}\,, \tag{17}$$

$$v_2 = \dot{y} + 2a\cos\theta\,\dot{\theta} = \frac{P}{6m} - \frac{6P\cos\theta}{22m} = \frac{4P}{132m}\,, \tag{18}$$

and,

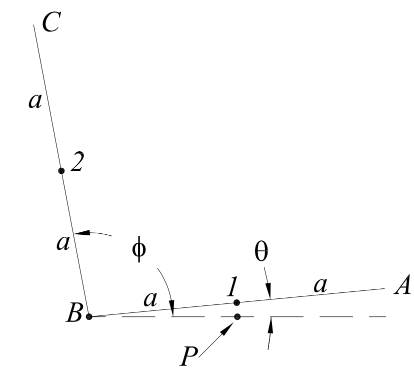
$$\frac{v_1}{v_2} = 10. \tag{19}$$

2. *This problem is from Ex. 1, p. 163 of* E.T. Whittaker, *A Treatise on Analytical Dynamics of Particles and Rigid Bodies* (Cambridge U. Press, 1904, 1917, 1927, 1937), http://kirkmcd.princeton.edu/examples/mechanics/whittaker_dynamics_17.pdf

To use a Lagrangian method, we adopt coordinates (x, y) of point B and angles θ and ϕ of rods AB and BC to a fixed direction, where initially,

$$x = y = \theta = 0, \tag{20}$$

$$\phi = 90^\circ, \qquad \dot{x} = \dot{y} = \dot{\theta} = \dot{\phi} = 0. \tag{21}$$



The centers of rods AB and BC, at points 1 and 2, are,

$$x_1 = x + a\cos\theta, \qquad y_1 = y + a\sin\theta, \qquad x_2 = x + a\cos\phi, \qquad y_2 = y + a\sin\phi, \tag{22}$$

and the kinetic energy is,

$$T = m\left(\dot{x}^2 - a\dot{x}(\sin\theta\,\dot{\theta} + \sin\phi\,\dot{\phi}) + \dot{y}^2 + a\dot{y}(\cos\theta\,\dot{\theta} + \cos\phi\,\dot{\phi}) + \frac{2}{3}a^2\dot{\theta}^2 + \frac{2}{3}a^2\dot{\phi}^2\right), \tag{23}$$

recalling that the moment inertia of each rod about its center is $I = ma^2/3$.

The generalized impulses associated with $\mathbf{P} = (P_x, P_y)$, which is applied at point 1, and Lagrange's equation of motion just after the impulse when the initial conditions (20), but not (21), still hold, are,

$$\mathcal{P}_x = \mathbf{P}\cdot\frac{\partial\mathbf{r}_1}{\partial x} = P_x = \frac{\partial T}{\partial\dot{x}} = m[2\dot{x} - a(\sin\theta\,\dot{\theta} + \sin\phi\,\dot{\phi})] = 2m\dot{x} - ma\dot{\phi}, \tag{24}$$

$$\mathcal{P}_y = \mathbf{P}\cdot\frac{\partial\mathbf{r}_1}{\partial y} = P_y = \frac{\partial T}{\partial\dot{y}} = m[2\dot{y} + a(\cos\theta\,\dot{\theta} + \cos\phi\,\dot{\phi})] = 2m\dot{y} + ma\dot{\theta}, \tag{25}$$

$$\begin{aligned}\mathcal{P}_\theta = \mathbf{P}\cdot\frac{\partial\mathbf{r}_1}{\partial\theta} &= -aP_x\sin\theta + aP_y\cos\theta = aP_y \\ = \frac{\partial T}{\partial\dot{\theta}} &= m\left[-a\dot{x}\sin\theta + a\dot{y}\cos\theta + \frac{8}{3}a^2\dot{\theta}\right] = ma\dot{y} + \frac{4}{3}ma^2\dot{\theta},\end{aligned} \tag{26}$$

$$\mathcal{P}_\phi = \mathbf{P}\cdot\frac{\partial\mathbf{r}_1}{\partial\phi} = 0 = \frac{\partial T}{\partial\dot{\phi}} = m\left[-a\dot{x}\cos\phi - a\dot{y}\sin\phi + \frac{8}{3}a^2\dot{\phi}\right] = -ma\dot{x} + \frac{4}{3}ma^2\dot{\phi}. \tag{27}$$

We desire that the initial motion be like that of a rigid body, so $\dot{\phi} = \dot{\theta}$, and hence eq. (27) tells us that (just after the impulse),

$$\dot{x} = \frac{4}{3}a\dot{\theta}. \tag{28}$$

Then, eq.(24) implies,

$$P_x = \frac{8}{3} ma\dot{\theta} - ma\dot{\theta} = \frac{5}{3} ma\dot{\theta}, \tag{29}$$

and eqs. (25)-(26) imply,

$$P_y = 2m\dot{y} + am\dot{\theta} = m\dot{y} + \frac{4}{3} ma\dot{\theta}, \qquad \dot{y} = \frac{ma\dot{\theta}}{3}, \qquad P_y = \frac{5}{3} ma\dot{\theta} = P_x. \tag{30}$$

Thus, the impulse **P** should be applied at 45° to the center of rod AB for the initial motion be to that of a rigid body.

Just after the impulse,

$$v_{\mathrm{A}} = (\dot{x}^2, \dot{y} + 2a\dot{\theta}) = \frac{a\dot{\theta}}{3}(4, 7). \qquad v_{\mathrm{A}}^2 = \frac{65a^2\dot{\theta}^2}{9}, \tag{31}$$

while

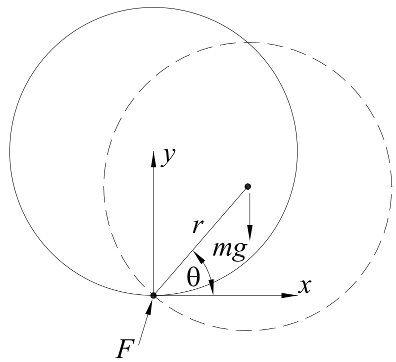
$$v_{\mathrm{C}} = (\dot{x}^2 - 2a\dot{\theta}, \dot{y}) = \frac{a\dot{\theta}}{3}(-2, 1). \qquad v_{\mathrm{C}}^2 = \frac{5a^2\dot{\theta}^2}{9}, \tag{32}$$

and

$$v_{\mathrm{C}} = \sqrt{13}\, v_{\mathrm{A}}. \tag{33}$$

3. *This problem is from Ex. 1, p. 84 of* E.J. Routh, *The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 7th ed. (Macmillan, 1905), http://kirkmcd.princeton.edu/examples/mechanics/routh_elementary_rigid_dynamics.pdf

We can use Newtonian methods, with both rectangular coordinates (x, y) and cylindrical coordinates (r, θ).



The constraint force $\mathbf{F}$ on the pivot does no work, so mechanical energy is conserved,

$$E = \frac{I\dot\theta^2}{2} + mgy = \frac{3mr^2\dot\theta^2}{4} + mgr\sin\theta = mgr, \qquad \dot\theta^2 = \frac{4g}{3r}(1-\sin\theta), \qquad \ddot\theta = -\frac{2g\cos\theta}{3r}, \tag{34}$$

noting that the moment of inertial of the uniform disk of mass m and radius r about this point is $I = I_{\rm cm} + mr^2 = 3mr^2/2$ by the parallel-axis theorem, and that the system is initially at rest when $\theta = 90^\circ$.[3]

The force equation for the disk, using cylindrical coordinates (r, θ),

$$\mathbf{F} - mg\,\hat{\mathbf{y}} = m\mathbf{a}_{\rm cm} = m(\ddot r - r\dot\theta^2)\,\hat{\mathbf{r}} + m(r\ddot\theta + 2\dot r\dot\theta)\,\hat{\boldsymbol{\theta}} = -mr\dot\theta^2\,\hat{\mathbf{r}} + mr\ddot\theta\,\hat{\boldsymbol{\theta}}. \tag{36}$$

For $\theta = 0$, $\hat{\mathbf{r}} = \hat{\mathbf{x}}$, $\hat{\boldsymbol{\theta}} = -\hat{\mathbf{y}}$, and,

$$\mathbf{F}(\theta = 0) = -mr\dot\theta^2\,\hat{\mathbf{r}} + mr\ddot\theta\,\hat{\boldsymbol{\theta}} + mg\,\hat{\mathbf{y}} = -\frac{4mg}{3}\,\hat{\mathbf{x}} - \frac{mg}{3}\,\hat{\mathbf{y}}, \qquad F(\theta = 0) = \frac{\sqrt{17}\,mg}{3}. \tag{37}$$

For $\theta = -90^\circ$, $\hat{\mathbf{r}} = -\hat{\mathbf{y}}$, $\hat{\boldsymbol{\theta}} = -\hat{\mathbf{x}}$, and,

$$\mathbf{F}(\theta = 90^\circ) = -mr\dot\theta^2\,\hat{\mathbf{r}} + mr\ddot\theta\,\hat{\boldsymbol{\theta}} + mg\,\hat{\mathbf{y}} = \frac{8mg}{3}\,\hat{\mathbf{y}} + mg\,\hat{\mathbf{y}} = \frac{11mg}{3}\,\hat{\mathbf{y}}. \tag{38}$$

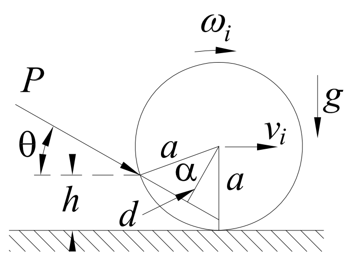
[3] The result for $\ddot\theta$ could also be obtained from the torque equation about the pivot point.

$$\tau = -rmg\cos\theta = I\ddot\theta = \frac{3mr^2\ddot\theta}{2}, \qquad \ddot\theta = -\frac{2g\cos\theta}{3r}. \tag{35}$$

4. An early study of the mechanics of billiards was G. Coriolis, *Théorie Mathématique des Effets du Jeu de Billard* (Paris, 1835),
http://kirkmcd.princeton.edu/examples/mechanics/coriolis_billard_35.pdf

(a) **Returning Ball.**

The cue imparts impulse P to the cue ball of mass m and radius a, at angle θ to the horizontal.



IF friction can be ignored during the impulse (a doubtful assumption), the resulting initial velocity v_i of the (center of mass of the) ball just after the impulse is related by,

$$P\cos\theta = mv_i, \qquad v_i = \frac{P\cos\theta}{m}\,. \tag{39}$$

and the angular velocity ω_i is related by,

$$Pd = Pa\cos\alpha = -I\omega_i = \frac{2}{5}ma^2\omega_i, \tag{40}$$

$$\omega_i = -\frac{5P\cos\alpha}{2ma} = -\frac{5v_i}{2a}\frac{\cos\alpha}{\cos\theta}\,, \tag{41}$$

where ω_i is positive for rotation as shown in the figure, *i.e.*, for $\boldsymbol{\omega}_i$ into the paper (parallel to $\mathbf{g}\times\mathbf{v}_i$). The minus sign in eq. (41) holds for $h < a(1+\sin\theta)$, as in the figure.

After the impulse is over, the ball rolls subject to sliding friction of magnitude μmg at the point of contact with the table. The direction of this friction is opposite to the velocity $\mathbf{v}_C$ of the point on the ball in instantaneous contact with the table,

$$\mathbf{v}_C = \mathbf{v} + \boldsymbol{\omega}\times\mathbf{a}, \qquad v_c = v - a\omega, \tag{42}$$

where $\mathbf{v}$ is the velocity of the center of mass of the ball, and $\mathbf{a}$ is the vector from the center of mass of the ball to the point of contact with the table.

Just after the impulse, eqs. (39)-(42) imply that $\mathbf{v}_{C,}$ is in the same direction as $\mathbf{v}_i$, which we define to be the $+x$ direction. Initially, the sliding frictional force μmg is in the $-x$ direction, which decreases both v and $|\omega|$, but increases ω as this is initially negative.

The equations of motion after the impulse are,

$$m\dot{v} = -\mu mg, \qquad v(t) = v_i - \mu mgt, \tag{43}$$

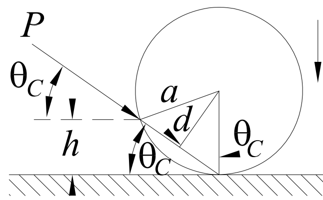
$$I\dot{\omega} = \frac{2}{5}ma^2\dot{\omega} = \mu mga, \qquad \omega(t) = \omega_i + \frac{5}{2}\frac{\mu mgt}{a}\,. \tag{44}$$

Rolling without slipping commences at time t_r when $v(t_r) = a\omega(t_r)$,

$$v_i - \mu mgt_r = a\omega_1 + \frac{5}{2}\mu mgt_r, \qquad t_r = \frac{2}{7}\frac{v_i - a\omega_i}{\mu mg}, \tag{45}$$

$$v(t_r) = v_i - \frac{2}{7}(v_i - a\omega_i) = \frac{5}{7}v_i + \frac{2}{7}a\omega_i = \frac{5}{7}v_i + \frac{2}{7}\frac{5}{2}v_i\frac{\cos\alpha}{\cos\theta} = \frac{5}{7}v_i\left(1 - \frac{\cos\alpha}{\cos\theta}\right). \tag{46}$$

For a returning ball, $v(t_r)$ must be less than zero, which requires $\alpha < \theta$.
The critical case is when $\alpha = \theta_C$, which corresponds to the impulse being directed towards the point of contact of the ball with the table.



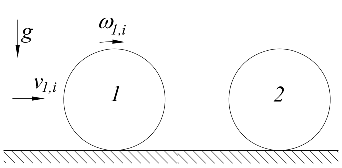
Here, the chord along the direction of the impulse has length $2a\sin\theta$ and,

$$h = 2a\sin^2\theta_C, \qquad \sin\theta_C = \sqrt{\frac{h}{2a}}, \qquad \tan\theta_C = \frac{1}{\sqrt{2a/h - 1}}. \tag{47}$$

In our model, the ball returns if $\theta > \theta_C$, *i.e.*, $\tan\theta > 1/\sqrt{2a/h - 1}$.
Taking friction into account, the critical angle is larger than that given by eq. (47).

(b) **Follow Shots and Draw Shots.**
If the elastic collision between balls 1 and 2 is along their line of centers, and friction can be ignored during the collision, then the angular velocity of ball 1 in unchanged during the collision, while the velocity of ball 1 is transferred to ball 2, whose angular velocity is zero. That is, just after the collision the velocities and angular velocities of the two balls are,

$$v_{1,a} = 0, \qquad \omega_{1,a} = \omega_{1,i}, \qquad v_{2,a} = v_{1,i}, \qquad \omega_{2,a} = 0. \tag{48}$$



Because ball 1 is still spinning after the collision at time $t = 0$, it experiences sliding friction μmg with the table, which propels the ball forward is $\omega_{1,i} > 0$, as in the figure, and ball 1 follows ball 2. In this case, the motion of ball 1 prior to rolling without slipping is,

$$v_1(t) = \mu gt, \qquad \omega_1(t) = \omega_{1,i} - \frac{2}{5}\frac{\mu gt}{a}, \tag{49}$$

Rolling without slipping commences for ball 1 at time t_1 such that,

$$v_1(t_1) = \mu gt_1 = a\omega_1(t_1) = a\omega_{1,i} - \frac{2}{5}\mu gt_1, \quad t_1 = \frac{5}{7}\frac{a\omega_{1,i}}{\mu g}, \quad v_1(t_1) = \frac{5}{7}a\omega_{1,i}. \tag{50}$$

Meanwhile, ball 2 also experiences sliding friction, opposite to its direction of motion,

$$v_2(t) = v_{1,i} - \mu g t, \qquad \omega_2(t) = \frac{2}{5}\frac{\mu g t}{a}, \tag{51}$$

Rolling without slipping commences for ball 2 at time t_2 such that,

$$v_2(t_1) = v_{1,i} - \mu g t_2 = a\omega_2(t_2) = \frac{2}{5}\mu g t_2, \qquad t_2 = \frac{5}{7}\frac{v_{1,i}}{\mu g}, \qquad v_2(t_2) = \frac{2}{7}v_{1,i}. \tag{52}$$

If $a\omega_{1,i} > 2v_{1,i}/5$, then ball 1 eventually collides again with ball 1 (and we could continue the analysis....).

On the other hand, is $\omega_{1,i} < 0$, then ball 1 is propelled backwards by sliding friction after the collision, and we speak of a draw shot. The analysis of eq. (50) holds, but with the signs reversed.

(c) **English.** See Art. 239, p. 183 of E.J. Routh, *The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 6th ed. (Macmillan, 1905),
http://kirkmcd.princeton.edu/examples/mechanics/routh_advanced_rigid_dynamics.pdf

If the cue is not horizontal, and not pointing at the center of the ball, the latter acquires "english", and does not move in a straight line. We wish to deduce the path of the center of mass of the ball.

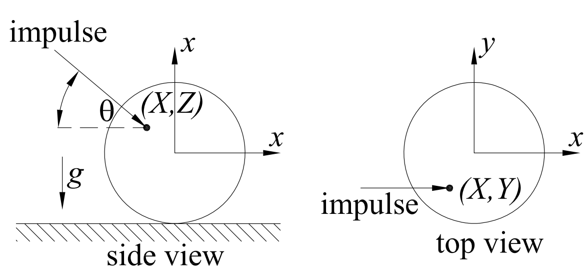
If the ball is not to move in straight line, there must be some friction perpendicular to $\mathbf{v}_{\rm cm}$ of the ball. This requires there to be some rotation about an axis parallel to $\mathbf{v}_{\rm cm}$, which rotation is called "english".

The concept of "english" also includes rotation of the ball about the vertical axis, but this does not influence the motion of the ball between collisions. If the area of contact of the ball with the table is small, there is negligible torque about the vertical (z) axis, so ω_z never changes, and does not affect $\mathbf{v}_{\rm cm}$.

A uniform sphere, of moment of inertia $I = ma^2/5$ about its center, has angular momentum $\mathbf{L} = I\boldsymbol{\omega}$ for any direction of angular velocity $\boldsymbol{\omega}$.

We first discuss the motion of the ball, first deduce its initial motion after being struck by the cue, and then deduce the subsequent motion (changes in which are caused by friction of the table against the ball.

We use a coordinate system with z vertical, and the cue in the x-x plane, with the origin at the center of the ball (initially at rest). Let $\mathbf{R} = (X, Y, Z)$ be the coordinates of the point of contact of the cue with the ball, $X^2 + Y^2 + Z^2 = a^2$.



If P is the magnitude of the impulse, and the cue makes angle θ to the horizontal, then $\mathbf{P} = (P\cos\theta, 0, -P\sin\theta)$.

We again suppose that friction can be ignored during the impulse. Then, the velocity and angular momentum of the ball just after the impulse are given by,

$$v_{i,x} = \frac{P_x}{m} = \frac{P\cos\theta}{m}, \qquad (v_{i,y} = 0 = v_{i,z}), \qquad \mathbf{L}_i = I\boldsymbol{\omega}_i = \mathbf{R}\times\mathbf{P}, \tag{53}$$

$$\omega_{i,x} = \frac{5}{2ma^2}(YP_z - ZP_y) = -\frac{5}{2ma^2}PY\sin\theta, \tag{54}$$

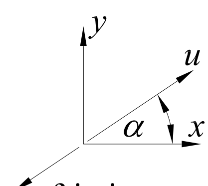
$$\omega_{i,y} = \frac{5}{2ma^2}(ZP_x - XP_z) = -\frac{5}{2ma^2}P(Z\cos\theta + X\sin\theta), \tag{55}$$

$$\omega_{i,z} = \frac{5}{2ma^2}(XP_y - YP_x) = -\frac{5}{2ma^2}PY\cos\theta. \tag{56}$$

In the approximation of negligible area of contact of the ball with the table, friction causes no torque about the z-axis, such that $\omega_x = \omega_{z,i}$ is constant. However, both ω_x and ω_y vary with time.

To discuss the subsequent motion of the ball, it is helpful to define $\mathbf{u}$ to be the velocity of the point on the ball in instantaneous contact with the table ($\mathbf{u} = 0$ for rolling without slipping). We also define α as the angle of $\mathbf{u}$ to the x-axis,

$$u_x = u\cos\alpha, \qquad u_y = u\sin\alpha. \tag{57}$$



Then, with $\mathbf{r} = (0, 0, -a)$ as the vector from the center of mass of the ball to the point of contact with the table,

$$\mathbf{u} = \mathbf{v} + \boldsymbol{\omega}\times\mathbf{r}, \qquad \dot{u}_x = \dot{v}_x - a\dot{\omega}_y, \qquad \dot{u}_y = \dot{v}_y + a\dot{\omega}_x. \tag{58}$$

The initial value of $\mathbf{u}$, just after the collision, is related by,

$$u_{x,i} = v_{x,i} - a\omega_{y,i} = \frac{P\cos\theta}{m} + \frac{5}{2ma}P(Z\cos\theta + X\sin\theta), \tag{59}$$

$$u_{y,i} = u\sin\alpha = v_{y,i} + a\omega_{x,i} = -\frac{5}{2ma}PY\sin\theta. \tag{60}$$

The force of sliding friction is then $\mathbf{F} = m\dot{\mathbf{v}} = -\mu mg\,\hat{\mathbf{u}} = -\mu mg(\cos\alpha\,\hat{\mathbf{x}} + \sin\alpha\,\hat{\mathbf{y}})$, so the equations of motion can be written as $m\dot{\mathbf{v}} = \mathbf{F}$ and $I\dot{\boldsymbol{\omega}} = \mathbf{r}\times\mathbf{F}$.

From the force equation, we have,

$$\dot{v}_x = -\mu g\cos\alpha, \qquad \dot{v}_y = -\mu g\sin\alpha, \tag{61}$$

From the torque equation,

$$I\dot{\omega}_x = r_y F_z - r_z F_y = aF_y = -\mu amg \sin\alpha, \qquad \dot{\omega}_x = -\frac{5}{2a}\mu g \sin\alpha, \tag{62}$$

$$I\dot{\omega}_y = r_z F_x - r_x F_z = -aF_x = \mu amg \cos\alpha, \qquad \dot{\omega}_y = \frac{5}{2a}\mu g \cos\alpha, \tag{63}$$

$$I\dot{\omega}_z = r_x F_y - r_y F_x = 0. \tag{64}$$

Then, from eqs. (59)-(63),

$$\dot{u}_x = \dot{v}_x - a\dot{\omega}_y = -\mu g\cos\alpha - \frac{5}{2}\mu g\cos\alpha = -\frac{7}{2}\mu g\cos\alpha, \tag{65}$$

$$\dot{u}_y = \dot{v}_y + a\dot{\omega}_x = -\mu g\sin\alpha - \frac{5}{2}\mu g\sin\alpha = -\frac{7}{2}\mu g\sin\alpha \tag{66}$$

Also, from eq. (57),

$$\dot{u}_x = \dot{u}\cos\alpha - u\sin\alpha\,\dot{\alpha}, \qquad \dot{u}_y = \dot{u}\sin\alpha + u\cos\alpha\,\dot{\alpha}. \tag{67}$$

Together, eqs. (65)-(67) imply that,

$$\dot{u} = -\frac{7}{2}\mu g, \qquad u = u_i - \frac{7}{2}\mu g t, \qquad \dot{\alpha} = 0, \qquad \alpha = \text{constant}. \tag{68}$$

Rolling without slipping commences when $u = 0$, at time,

$$t_r = \frac{2u_i}{7\mu g}, \tag{69}$$

after the impulse. After time t_r, the path of the ball on the table is a straight line with,

$$v_x(t > t_r) = v_{i,x} - \mu g\cos\alpha t_r = \frac{P\cos\theta}{m} - \frac{2}{7}u_i\cos\alpha \tag{70}$$

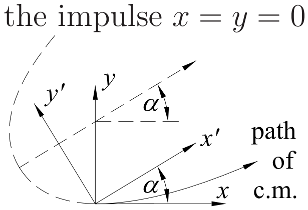
$$v_y(t > t_r) = v_{i,y} - \mu g\sin\alpha t_r = -\frac{2}{7}u_i\sin\alpha, \tag{71}$$

where v_i is given by eq. (53).
Hence, in a coordinate system with x' along $\mathbf{u}$,

$$\dot{v}_{y'} = 0, \qquad v_{y'} = v_{y',i}, \qquad y' = v_{y',i}t, \qquad t = \frac{y'}{v_{y',i}}, \tag{72}$$

$$\dot{v}_{x'} = -\mu g, \qquad v_{x'} = v_{x',i} - \mu g t, \qquad x' = v_{x',i}t - \frac{\mu g t^2}{2} = \frac{v_{x',i}}{v_{y',i}}y' - -\frac{\mu g y'^2}{2v_{y',i}^2}, \tag{73}$$

recalling that just after the impulse $x = y = 0 = x' = y'$.



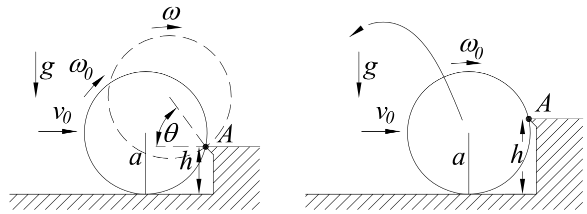
The path of the ball is a parabola whose axis makes angle α to the x-axis.
Recalling eqs. (59)-(60), we also have that,

$$\tan\alpha = \frac{u_{y,i}}{u_{x,i}} = \frac{-\frac{5}{2}\frac{Y}{a}\sin\theta}{\left(1 + \frac{5}{2}\frac{Z}{a}\right)\cos\theta + \frac{5}{2}\frac{X}{a}\sin\theta}. \tag{74}$$

If either $\sin\theta = 0$ or $Y = 0$, then $\alpha = 0$ and the path reduces to a straight line.

5. This problem is from Art. 174, p. 142 of E.J. Routh, *The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 7th ed. (Macmillan, 1905),
http://kirkmcd.princeton.edu/examples/mechanics/routh_elementary_rigid_dynamics.pdf

A ball of radius a collides with a bumper of height h. Just before the collision the center of the ball has velocity $\mathbf{v}_0$ perpendicular to the bumper, and angular velocity $\boldsymbol{\omega}_0$ directed along $\mathbf{v}_0 \times \mathbf{g}$, as in the figure. Suppose there is no slipping at the bumper during, or after, the collision.



(a) Angular momentum is conserved during the collision about the point of contact A of the ball with the bumper (whether or not slipping occurs at point A, but $L_f = I_A\,\omega$ only for no slipping),

$$L_i = I\omega_0 + mv_0(a-h) = L_f = I_A\,\omega = (I+ma^2)\,\omega_f = \frac{7}{5}ma^2\omega_f, \qquad (75)$$

where $I = 2ma^2/5$, $I_A = 7ma^2/5$, and ω is the angular velocity of the ball just after the collision. Hence,[4]

$$\omega_f = \frac{5}{7ma^2}\left(\frac{2ma^2\,\omega_0}{5} + mv_0(a-h)\right) = \frac{2}{7}\omega_0 + \frac{5v_0(a-h)}{7a^2}. \qquad (76)$$

If the ball is to jump up onto the cushion/step, its potential energy must increase by at least mgh during the collision, where h is the height of the step. For this, the kinetic energy just after the collision must satisfy,

$$\mathrm{KE}_f = \frac{I_A\,\omega_f^2}{2} \geq mgh, \qquad \omega_f^2 \geq \frac{2mgh}{I_A} = \frac{10gh}{7a^2}, \qquad (77)$$

$$4a^4\omega_0^2 + 20\frac{\omega_0 v_0 a^2(a-h)}{a^2} + 25v_0^2(a-h)^2 \geq 70a^2gh. \qquad (78)$$

If $\omega_0 = 0$, the condition is,

$$v_0 \geq \frac{a}{a-h}\sqrt{\frac{14gh}{5}}. \qquad (79)$$

Of course, we must have $v_0 > 0$ for the ball to collide with the bumper, which requires $h < a$ (for $\omega_0, 0$.

If $\omega_0 = v_0/a$, the condition (77) is,

$$\omega_f = \frac{v_0}{7a}\left(2 + \frac{5(a-h)}{a}\right) = \frac{v_0(7a-5h)}{7a^2} \geq \sqrt{\frac{10gh}{7a^2}} \qquad v_0 \geq \frac{a}{7a-5h}\sqrt{70gh}. \ (80)$$

---

[4]If ω_0 were negative, ω could be negative, with the implication that after the collision with the bumper, the ball would rotate into the table. We avoid discussion of this ambiguous case by supposing that $\omega_0 > 0$.

The requirement that $v_0 > 0$ implies that $h < 7a/5$ for collisions when $\omega_0 = v_0/a$.

Does the ball stay in contact with point A after the collision?

The mechanical energy of the ball is conserved after the collision, so if the ball remains in contact with the bumper,

$$E = \frac{I_A\,\omega_f^2}{2} + mg(a-h) = \frac{I_A\,\omega^2}{2} + mga\sin\theta, \tag{81}$$

where θ is the angle of the center of the ball from the horizontal at point A (see the figure on the previous page), and $\omega = \dot\theta$.

The radial force at A is nonzero, and the ball is in contact with A, only if $ma\omega^2 = ma\dot\theta^2 < mg\sin\theta$. In particular, just after the collision (when $\omega = \omega_f$), we must have $\sin\theta_f > 0$, *i.e.*, $h < a$, for the ball to remain in contact with point A.

For the case of $\omega_0 = v_0/a$ and $a < h < 7a/5$, $\sin\theta_f < 0$ and the ball rises after the collision, with $v_x < 0$, but immediately loses contact with point A and subsequently falls back onto the table to the left of the bumper.

In general, since $\omega = \dot\theta$ decreases as the ball rises after the collision, if the ball loses contact with point A after the collision, it does so immediately.

(b) We consider $h < a$, where $\sin\theta_f = (a-h)/a$.

For $\omega_0 = 0$ and $v_0 = v_{0,\min}$ of eq. (79) the critical condition that the ball remain in contact with point A after the collision is,

$$\begin{aligned} ma\omega_f^2 &= ma\left(\frac{5v_{0,\min}(a-h_c)}{7a^2}\right)^2 = ma\left(\frac{5}{7a}\sqrt{\frac{14gh_c}{5}}\right)^2 = \frac{10mgh_c}{7a} \\ &= mga\sin\theta_c = mg(a-h_c), \qquad h_c = \frac{7a}{17}. \end{aligned} \tag{82}$$

The ball flies off the bumper if,

$$\frac{h}{a} > \frac{h_c}{a} = \frac{7}{17}. \tag{83}$$

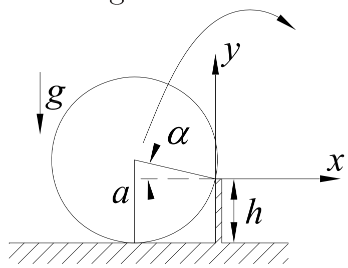
Similarly, for $\omega_0 = v_0/a$ and $v_0 = v_{0,\min}$ of eq. (80) the critical condition that the ball remain in contact with point A after the collision is,

$$\begin{aligned} ma\omega_f^2 &= ma\left(\frac{v_{0,\min}(7a-5h_c)}{7a^2}\right)^2 = ma\left(\frac{\sqrt{70gh_c}}{7a}\right)^2 = \frac{10mgh}{7a} \\ &= mga\sin\theta_c = mg(a-h_c), \qquad h_c = \frac{7a}{17}. \end{aligned} \tag{84}$$

Again, the ball flies off the bumper if,

$$\frac{h}{a} > \frac{h_c}{a} = \frac{7}{17}. \tag{85}$$

(c) We now suppose the step is just a narrow slat of height $h < a$, and again there is no slipping at the bumper during the collision. The velocity v of the center of the ball after the collision is large enough that the ball immediately flies into the air. That is, $v^2 > ag \sin\alpha$, recalling that the ball flies if $ma\,\omega^2 = mv^2/a > mg \sin\alpha$.



We use a coordinate system centered on the point of contact of the ball with the slat, and α is the initial angle of the center of mass of the ball with respect to the $-x$ axis.

The position of the center of the ball, while in flight after the collision at time $t = 0$ is,

$$x = -a\cos\alpha + v\sin\alpha\, t, \qquad y = a\sin\alpha + v\cos\alpha t - \frac{1}{2}gt^2, \tag{86}$$

will hit the slat at some positive time t if $X(t) = 0 = Y(t)$, at which time,

$$\begin{aligned} a^2 = x^2 + y^2 &= (-a\cos\alpha + v\sin\alpha\, t)^2 + (a\sin\alpha + v\cos\alpha t - gt^2/2)^2 \\ &= a^2 + v^2t^2 + \frac{g^2t^4}{4} - ga\sin\alpha\, t^2 - gv\cos\alpha\, t^3. \end{aligned} \tag{87}$$

The second collision with the slat occurs at the smaller, positive, real root (if it exists) of the equation,[5]

$$\frac{1}{4}g^2t^2 - gvt\cos\alpha + v^2 - ag\sin\alpha = 0, \tag{88}$$

$$\begin{aligned} t = \frac{2}{g^2}\left(gv\cos\alpha - \sqrt{g^2v^2\cos^2\alpha - g^2(v^2 - ag\sin\alpha)}\right) \\ = \frac{2}{g}\left(v\cos\alpha - \sqrt{ag\sin\alpha - v^2\sin^2\alpha}\right). \end{aligned} \tag{89}$$

The second collision occurs for v in the range,

$$\frac{ag}{\sin\alpha} > v^2 > ag\sin\alpha. \tag{90}$$

[5]If the second collision occurs (at the origin) for some time t, then at a larger time another point on the circumference of the ball would pass through the origin if somehow that slat had been removed before the second collision. Hence, there are two positive, real roots to eq. (88), or none.

6. **Relativistic Rocket.** The relativistic rocket may have first been discussed by J. Ackeret, *Zur Theorie der Raketen*, Helv. Phys. Acta **19**, 103 (1946), http://kirkmcd.princeton.edu/examples/GR/ackeret_hpa_19_103_46.pdf

   In the rest frame of the rocket, let $m^\star$ be the rest mass of the rocket (plus fuel) at some time $t^\star$. After a short time, the rocket has mass $m_1^\star$, and velocity $dv^\star$ as a result of spewing out exhaust of mass $dm^\star < 0$ at velocity $u > 0$ relative to the rocket.

   In general, $dm^\star \neq m_1^\star - m^\star$ in general, but this holds to a first approximation when $dv^\star$ is small (compared to c, the speed of light).

   Conservation of momentum, in this approximation, is that,

$$m_1^\star\, dv \approx m^\star dv^\star \approx -dm^\star\, u, \qquad dv^\star \approx -u\frac{dm^\star}{m^\star}\,. \tag{91}$$

   In the lab frame, the velocity of the rocket changes from v to $v + dv$ during the above process. The relativistic velocity transformation of $dv^\star$ to the frame where the rocket has velocity v tells us that,

$$v + dv = \frac{v + dv^\star}{1 + v\, dv^\star/c^2}\,, \qquad v + dv^\star \approx v + dv + \frac{v^2\, dv^\star}{c^2}\,, \tag{92}$$

$$dv^\star \approx \frac{dv}{1 - v^2/c^2} \equiv \gamma^2 dv = c\, d\left(\tanh^{-1}\frac{v}{c}\right). \tag{93}$$

   combining eqs. (91) and (93), we integrate (noting the $-$ sign in eq. (91) to find,

$$\tanh^{-1}\frac{v}{c} = \frac{u}{c}\ln\frac{m_0^\star}{m^\star}\,, \tag{94}$$

$$\frac{v}{c} = \tanh\left(\frac{u}{c}\ln\frac{m_0^\star}{m^\star}\right) = \frac{\left(\frac{m_0^\star}{m^\star}\right)^{u/c} - \left(\frac{m_0^\star}{m^\star}\right)^{-u/c}}{\left(\frac{m_0^\star}{m^\star}\right)^{u/c} + \left(\frac{m_0^\star}{m^\star}\right)^{-u/c}} = \frac{\left(\frac{m_0^\star}{m^\star}\right)^{2u/c} - 1}{\left(\frac{m_0^\star}{m^\star}\right)^{2u/c} + 1}\,, \tag{95}$$

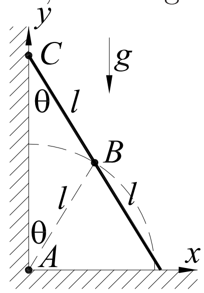
   where $m^\star = m_0^\star$ when $v = 0$.

   The result (95) can also be written as,

$$\frac{m_0^\star}{m^\star} = \left(\frac{1 + v/c}{1 - v/c}\right)^{c/2v}\,. \tag{96}$$

   This form follows that of Tsiolkovsky, https://en.wikipedia.org/wiki/Tsiolkovsky_rocket_equation, who analyzed nonrelativistic rockets in 1903.

7. We use Lagrange's method for the ladder problem, with angle θ as the single generalized coordinate. The ladder has mass m, and length $2l$.



The ladder loses contact with the wall when the normal force at the moving point C vanishes. It is awkward to include this force as a constraint force, so we first deduce the equation of motion of the ladder when in contact with the wall via Lagrange's method, and then consider the normal force.

The center of mass of the ladder is at point B moves in a circle of radius l about the fixed point A. The moment of inertia of the ladder about its center of mass is,

$$I_{\rm cm} = \frac{m\,l^2}{3}\,. \tag{97}$$

The kinetic energy T of the sliding ladder consists of the kinetic energy of the motion of the center of mass plus the kinetic energy of rotation about the center of mass,

$$T = \frac{m\,v_{\rm cm}^2}{2} + \frac{I_{\rm cm}\,\dot\theta^2}{2} = \frac{m\,(l\dot\theta)^2}{2} + \frac{m\,l^2\,\dot\theta^2}{6} = \frac{2m\,l^2\,\dot\theta^2}{3}\,. \tag{98}$$

The gravitational potential energy V of the ladder relative to the floor is,

$$V = m\,g\,l\,\cos\theta. \tag{99}$$

The equation of motion of the ladder follows from Lagrange's equation,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} = \frac{\partial\mathcal{L}}{\partial\theta}\,, \tag{100}$$

where the Lagrangian is $\mathcal{L} = T - V$. From eqs. (98)-(100) we find that,

$$\ddot\theta = \frac{3g}{4l}\sin\theta. \tag{101}$$

The ladder loses contact with the vertical wall when (horizontal) contact force F_C vanishes. This contact force causes the horizontal acceleration of the center of mass,

$$F_C = m\,a_x. \tag{102}$$

The x coordinate of the center of mass (so long as the ladder remains in contact with the wall) is,

$$x_{\rm cm} = l\sin\theta, \qquad a_x = \ddot x_{\rm cm} = l\cos\theta\,\ddot\theta - l\sin\theta\,\dot\theta^2 = \frac{3g}{4}\cos\theta\sin\theta - l\sin\theta\,\dot\theta^2. \tag{103}$$

The angular velocity $\dot{\theta}$ of the ladder follows from conservation of energy,

$$E_0 = m\,g\,l\cos\theta_0 = E = T + V = \frac{2m\,l^2\,\dot{\theta}^2}{3} + m\,g\,l\cos\theta, \tag{104}$$

so that,[6]

$$\dot{\theta}^2 = \frac{3g}{2l}(\cos\theta_0 - \cos\theta). \tag{105}$$

The ladder loses contact with the vertical wall when $F_C = m\,a_x$ vanishes, which occurs when

$$a_x = 0 = \frac{3g}{4}\cos\theta\sin\theta - \frac{3g}{2}\sin\theta(\cos\theta_0 - \cos\theta), \tag{106}$$

$$\cos\theta = \frac{2}{3}\cos\theta_0. \tag{107}$$

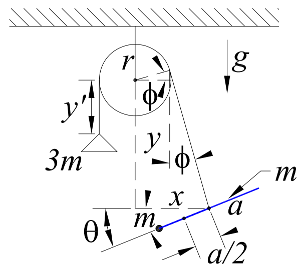
As the above analysis involved some use of Newtonian methods, the reader might wish to consider use of Newtonian methods only. This could involve a torque analysis, which is straightforward for torques computed about either points A or B. But, if torques are computed about an accelerating point of than the center of mass, such as C, "fictitious forces" must be included in the analysis. See http://kirkmcd.princeton.edu/examples/ladder.pdf.

For general comments on torque analyses with accelerated reference points, see http://kirkmcd.princeton.edu/examples/torque.pdf.

[6]The equation of motion (101) can also be deduced by taking the time derivative of eq. (105), or conversely the energy equation (104) could be obtained by integrating the equation of motion (101).

8. This problem is Ex. 12, p. 170 of E.J. Routh, *The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 7^{th} ed. (Macmillan, 1905),
http://kirkmcd.princeton.edu/examples/mechanics/routh_elementary_rigid_dynamics.pdf

After the mass at the left end of the rod has fallen off, we describe the system by five coordinates, x and y of the center of the rod relative to the center of the pulley, the angle θ of the rod to the horizontal, the angle ϕ to the vertical of the string from the pulley to the rod, and the vertical coordinate y' of the mass $3m$. We suppose that the latter mass has no horizontal motion.



If the initial coordinates are $x_0 = r$, y_0, $\theta_0 = 0 = \phi_0$ and y'_0 (which also hold just after one mass m falls off), then the length of the string is $y_0 + y'_0 + 2\pi r$, which we take to be independent of time. In this case, y' is not an independent coordinate.

At a time after one mass m has fallen off, the length of the string from the pulley to the center of the rod is,

$$l = \sqrt{(x - r\cos\phi)^2 + (y + r\sin\phi)^2} = \sqrt{x^2 + y^2 + r^2 - 2rx\cos\phi + 2ry\sin\phi}. \qquad (108)$$

Then,

$$\begin{aligned} y' &= y_0 + y'_0 + 2\pi r - y - r(2\pi - \phi) - l \\ &= y_0 + y'_0 + r\phi - \sqrt{x^2 + y^2 + r^2 - 2rx\cos\phi + 2ry\sin\phi}. \end{aligned} \qquad (109)$$

The general motion is complex, and deducing the equations of motion for all four independent coordinates x, y, θ and ϕ is tedious. However, if we confine our attention to the motion just after one mass falls off, we see that $\ddot{x} = 0 = \ddot{\phi}$, and only $\ddot{y}$ and $\ddot{\theta}$ are nonzero then. This simplification permits a Newtonian analysis.

The equation of motion for the mass $3m$ is,

$$3m\ddot{y}' = -3m\ddot{y} = 3mg - T, \qquad (110)$$

where T is the tension in the string, and $\ddot{y}' = -\ddot{y}$ holds just after one mass falls off.

For the rod of mass m with another mass m at one end, we note that its center of mass is at distance $a/2$ from the center of the rod. In general,

$$y_{\rm cm} = y + \frac{a}{2}\sin\theta, \qquad \dot{y}_{\rm cm} = \dot{y} + \frac{a}{2}\cos\theta\,\dot{\theta}, \qquad \ddot{y}_{\rm cm} = \ddot{y} + \frac{a}{2}\cos\theta\,\ddot{\theta} - \frac{a}{2}\sin\theta\,\dot{\theta}^2, \qquad (111)$$

and just after one mass falls off, $\ddot{y}_{\rm cm} = \ddot{y} + (a/2)\,\ddot{\theta}$. Hence the equation of motion for the center of mass of the rod system, just after one mass falls off, is,

$$2m\ddot{y}_{\rm cm} = 2m\ddot{y} + ma\,\ddot{\theta} = 2mg - T. \qquad (112)$$

Combining eqs. (110) and (112),

$$\ddot{y} = -\frac{g + a\,\ddot{\theta}}{5}\,. \tag{113}$$

To determine $\ddot{\theta}$, we consider the torque equation about the center of mass of the rod system, whose moment of inertia (about the cm) is,

$$I_{\rm cm} = I_{\rm rod,cm} + m\left(\frac{a}{2}\right)^2 + m\left(\frac{a}{2}\right)^2 = \frac{5ma^2}{6}\,. \tag{114}$$

recalling that $I_{\rm rod,cm} = ma^2/3$. Then, just after one mass falls off,

$$\begin{aligned} I_{\rm cm}\,\ddot{\theta} = \frac{5ma^2}{6}\,\ddot{\theta} &= mg\frac{a}{2} - mg\frac{a}{2} + T\frac{a}{2} = (3mg + 3m\ddot{y})\frac{a}{2} \\ &= \left[3mg - \frac{3}{5}m(g + a\,\ddot{\theta})\right]\frac{a}{2} = (12mg - 3ma\,\ddot{\theta})\frac{a}{10}\,, \qquad (115) \\ \ddot{\theta} &= \frac{18g}{17a}\,. \qquad (116) \end{aligned}$$

Finally, the tension in the string just after one mass falls off is,

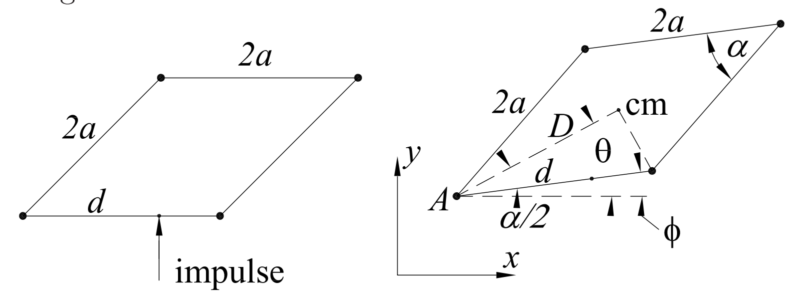
$$T = 3mg + 3m\ddot{y} = \frac{12mg - 3ma\,\ddot{\theta}}{5} = \frac{mg}{5}\left(12 - \frac{54}{17}\right) = \frac{30mg}{17}\,. \tag{117}$$

For a Lagrangian analysis to find the behavior just after one mass falls off, we could suppose that $x = x_0$ and $\phi = 0$ always, although this wouldn't give correct results for any finite time after the mass fell off.

We might include the tension in the analysis via a Lagrange multiplier related to the constraint $y' + y - y_0 - y_0' = 0$ (supposing $x = x_0$ and $\phi = 0$ always), and taking y, y' and θ as independent variables. This is the approach used in
http://kirkmcd.princeton.edu/examples/Ph205/ph205sol4.pdf.

9. This problem is Ex. 18, p. 180 of E.J. Routh, *The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 7th ed. (Macmillan, 1905),
http://kirkmcd.princeton.edu/examples/mechanics/routh_elementary_rigid_dynamics.pdf

The general motion of the deforming, rotating, translating rhombus can be described by 4 coordinates q_j, such as the coordinates $x_{\rm cm}$, $y_{\rm cm}$, the interior, acute angle α, and the angle θ of the line from point A to the center of mass to the original direction of the side struck by the impulse. Of these coordinates, perhaps only the choice of angle θ is not straightforward. For example, we might instead use angle ϕ between the struck side and its original direction.



The kinetic energy of translation of the center of mass, with no change in the angles, is just $2m(\dot x^2 + \dot y^2)$, where m is the mass of each side of the rhombus.

The kinetic energy of rotation about a fixed center of mass, with no deformation (α constant), is $I_{\rm cm}\dot\theta^2/2 = 4(ma^2/3 + ma^2)\,\dot\theta^2/2 = 8ma^2\dot\theta^2/3$, since the center of each side is at distance a from the center of mass. This shows that θ rather than ϕ is a desirable coordinate.

The kinetic energy of deformation (α variable while x, y and θ constant) is $4ma^2(\dot\alpha/2)^2/2 + 4I_{\rm side}(\dot\alpha/2)^2/2 = 4(4ma^2/3)(2\dot\alpha^2) = ma^2\,\dot\alpha^2/2 + 4(ma^2/3)\,\dot\alpha^2/8 = 2ma^2\,\dot\alpha^2/3$.

Lagrange's equations for the impulsive motion can be written as,

$$\Delta\frac{\partial T}{\partial \dot q_j} = I_j, \tag{118}$$

where I_j is the generalized impulse associate with coordinate j,

$$I_j = \mathbf{P}\cdot\frac{\partial \mathbf{r}_{\rm P}}{\partial q_j}\,, \tag{119}$$

where $\mathbf{P} = \int \mathbf{F}_{\rm impulse}\,dt$ is the 3-force impulse, and $\mathbf{r_P}$ is its point of application.

If $I_j = 0$ for some coordinate j, then there is no impulsive motion associated with that coordinate.

In the present example, noting that $D = 2a\cos(\alpha/2)$,

$$\begin{aligned}\mathbf{r_P} &= (x_A + d\cos(\theta-\alpha/2), y_A + d\sin(\theta-\alpha/2)) \\ &= (x_{\rm cm} - D\cos\theta + d\cos(\theta-\alpha/2), y_{\rm cm} - D\sin\theta + d\sin(\theta-\alpha/2)) \\ &= (x_{\rm cm} - 2a\cos(\alpha/2)\cos\theta + d\cos(\theta-\alpha/2), y_{\rm cm} - 2a\cos(\alpha/2)\sin\theta + d\sin(\theta-\alpha/2)).\end{aligned} \tag{120}$$

If we want the rhombus to move without deformation after the impulse, then we require that $I_\alpha = 0$. If we use coordinate θ, then,

$$0 = I_\alpha = \mathbf{P} \cdot \frac{\partial \mathbf{r}_{\rm P}}{\partial \alpha} = P \frac{\partial r_{R,y}}{\partial \alpha} = P\left[a \sin\frac{\alpha}{2}\sin\theta - \frac{d}{2}\cos\left(\theta - \frac{\alpha}{2}\right)\right], \tag{121}$$

for the initial values of α and θ, namely $\theta = \alpha/2$. Hence,

$$d = 2a\sin^2\frac{\alpha}{2} = a(1 - \cos\alpha). \tag{122}$$

If instead we desire the diagonal of the rhombus not to rotate after the impulse, then $I_\theta = 0$,

$$0 = I_\theta = \mathbf{P} \cdot \frac{\partial \mathbf{r}_{\rm P}}{\partial \theta} = P \frac{\partial r_{R,y}}{\partial \theta} = P\left[-2a \cos\frac{\alpha}{2}\cos\theta + d\cos\left(\theta - \frac{\alpha}{2}\right)\right], \tag{123}$$

for the initial values of α and θ, namely $\theta = \alpha/2$. Hence,

$$d = 2a\cos^2\frac{\alpha}{2} = a(1 + \cos\alpha), \tag{124}$$

such that the impulse points directly at the center of mass of the initial rhombus. Indeed, we might have concluded this with no calculation, since for no rotation of the rhombus as a whole, the impulse must exert no torque about its center of mass.

10. The physics of an idealized Super-Ball was first discussed by G.L. Strobel, *Matrices and Superballs*, Am. J. Phys. **50**, 856 (1988),
http://kirkmcd.princeton.edu/examples/mechanics/strobel_ajp_36_834_68.pdf
and by R.L. Garwin, *Kinematics of an Ultraelastic Rough Ball*, Am. J. Phys. **37**, 88 (1969), http://kirkmcd.princeton.edu/examples/mechanics/garwin_ajp_37_88_69.pdf.

Angular momentum is conserved about the point of contact of the Super-Ball with floor during the collision,

$$L_z = -mrv_x + kmr^2\,\omega = L'_z = -mrv'_x + kmr^2\,\omega', \qquad v'_x - v_x = kr(\omega' - \omega), \quad (125)$$

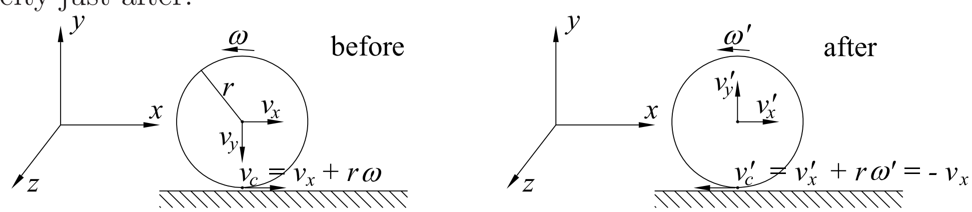
where the Super-Ball has mass m, radius r, $k = 2/5$ for a uniform sphere, and $v_x\,(v'_x)$ and $\omega\,(\omega')$ are its x-velocity and the z-component of its angular velocity just before(after) the collision (with $\omega_x = \omega_y = 0 = \omega'_x = \omega'_y$).

We first suppose that **energy of the y-motion is conserved** in the collision with the floor, and, separately, the energy of the x-motion plus rotational motion is conserved.

Then, the vertical velocity after the collision with the floor is,

$$v'_y = -v_y > 0, \quad (126)$$

where $v_y < 0$ is the vertical velocity just before the collision and v'_y is the vertical velocity just after.



Before (after) the collision with the floor, the center of the Super-Ball has horizontal velocity $v_x\,(v'_x)$.

Conservation of energy of the x-motion and the rotational motion implies that,

$$\frac{mv_x^2}{2} + \frac{kmr^2\,\omega^2}{2} = \frac{mv'^2_x}{2} + \frac{kmr^2\,\omega'^2}{2}, \qquad v'^2_x - v_x^2 = kr^2(\omega^2 - \omega'^2), \quad (127)$$

$$(v'_x - v_x)(v'_x + v_x) = kr^2(\omega - \omega')(\omega + \omega'). \quad (128)$$

Dividing eq. (128) by (125), we have that,

$$v'_x + v_x = -r(\omega + \omega'), \qquad v_c = v_x + r\,\omega = -(v'_x + r\,\omega') = -v'_c, \quad (129)$$

where v_c is the (horizontal) velocity of the point on the ball just before the collision that will become the point of contact with the floor. Thus, we find that the velocity of the point of contact of the ball with the floor reverses during the collision (which is not consistent with no slipping there).

We can now eliminate v'_x from eqs. (125) and (129) to find,

$$r\,\omega' = -\frac{1-k}{1+k}r\,\omega - \frac{2}{1+k}v_x = -\frac{3}{7}r\,\omega - \frac{10}{7}v_x. \quad (130)$$

Finally, eq. (130) and either of eqs. (125) or (129) gives,

$$v'_x = -\frac{2k}{1+k} r\,\omega + \frac{1-k}{1+k} v_x = -\frac{4}{7} r\,\omega + \frac{3}{7} v_x. \tag{131}$$

We now consider some examples:

(a) A Super-Ball of mass m and radius r is dropped vertically with spin from height h onto a hard floor.

If the spin had a vertical component, the idealization of no slipping at the point of contact implies that this component would disappear during the first bound on the floor.

So, it suffices to consider that the initial angular velocity $\boldsymbol{\omega}$ of the Super-Ball has only a horizontal component, ω Then, the initial horizontal velocity is $v_x = 0$, and the vertical velocity is $y_y = -\sqrt{2gh}$ when the ball strikes the floor.

From eqs. (126), (130) and (131), the motion just after the first collision is,

$$v'_x = -\frac{4}{7} r\omega, \qquad v'_y = \sqrt{2gh}, \qquad r\,\omega' = -\frac{3}{7} r\,\omega. \tag{132}$$

After the collision, the ball rises again to its original height h, and falls back onto the floor after time,

$$t = \frac{2v'_y}{g} = \frac{2}{g}\sqrt{2gh}, \tag{133}$$

during which time the ball moves horizontally by distance,

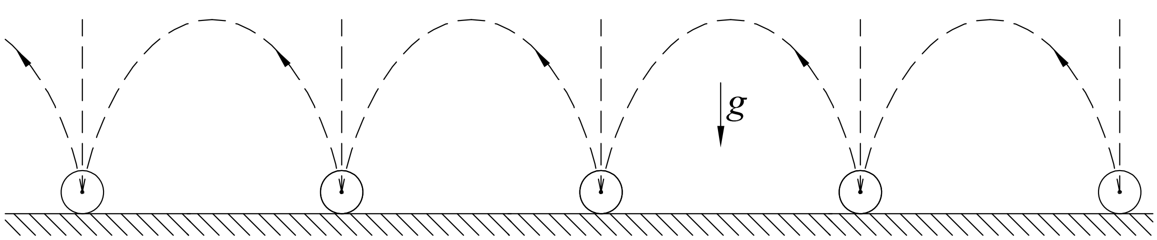
$$d = v'_x t = -\frac{8r\,\omega}{7}\sqrt{\frac{2h}{g}}\,. \tag{134}$$

Then, just after the second collision of the ball with the floor,

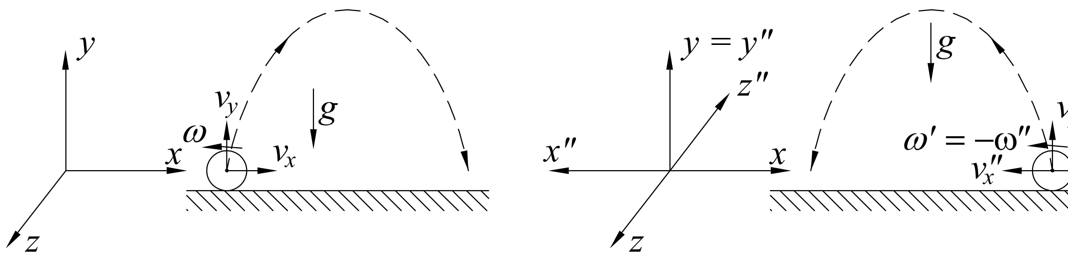
$$v''_x = -\frac{4}{7} r\,\omega' + \frac{3}{7} v'_x = \frac{3}{7}\frac{4}{7} r\,\omega + \frac{4}{7}\frac{3}{7} r\,\omega = 0, \qquad v''_y = -(-v'_y) = \sqrt{2gh}, \tag{135}$$

$$r\,\omega'' = -\frac{3}{7} r\,\omega' - \frac{10}{7} v'_x = \frac{3}{7}\frac{3}{7}, \omega' + \frac{4}{7}\frac{10}{7} r\,\omega = r\,\omega. \tag{136}$$

The ball rises again to height h, with zero horizontal velocity, and its initial angular velocity, but to the left of its original position by the distance of eq. (134). After this, the ball falls again, and again bounces to the left, in a perpetual sequence if energy is conserved.



(b) For the ball to bounce back and forth on the same trajectory, as in the figure below, both v_x and ω must reverse during a collision with the floor.



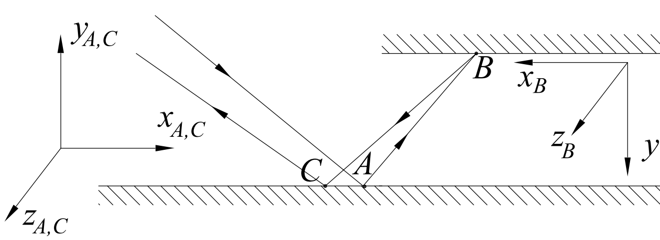
That is, after the collision in the right figure above, the situation must look the same as in the left figure when the right figure is viewed from the other side, which changes x to x'' and z to z'', and consequently $v'_x \to -v''_x$ and $\omega' \to -\omega''$, while we desire that $v''_x = v_x$ and $\omega'' = \omega$.

According to eqs. (130) and (131), this requires,

$$r\,\omega' = -\frac{3}{7} r\,\omega - \frac{10}{7} v_x = -r\,\omega, \qquad r\,\omega = \frac{10}{4} v_x, \tag{137}$$

$$v'_x = -\frac{4}{7} r\,\omega + \frac{3}{7} v_x = -v_x, \qquad v_x = \frac{4}{10} r\,\omega. \tag{138}$$

(c) We trace the results of 3 collisions of the ball, first with the floor at point A, then with the underside of the table at point B, and again with floor at point C.



In the present model $|v_y|$ is essentially constant if the velocity is large enough that effects of gravity are negligible.

The results of the collision at point A, from eqs. (130) and (131), are, for initial spin $\omega = 0$ and initial horizontal velocity v_x,

$$r\,\omega'_A = -\frac{10}{7} v_x, \qquad v'_{A,x} = \frac{3}{7} v_x. \tag{139}$$

To use these results as input to eqs. (130) and (131) for the collision at point B under the table, we must transform them to a right-handed coordinate system (x_B, y_B, z_B) associated with that point, with the y_B axis downward, as in the figure, and the x_B axis opposite to axis x_A. That is,

$$r\,\omega_B = r\,\omega'_A = -\frac{10}{7} v_x, \qquad v_{B,x} = -v'_{A,x} = -\frac{3}{7} v_x. \tag{140}$$

Then, the results of the collision at point B are,

$$r\,\omega'_B = -\frac{3}{7} r\omega_B - \frac{10}{7} v_{B,x} = \frac{3}{7}\frac{10}{7} v_x + \frac{10}{7}\frac{3}{7} v_x = \frac{60}{49} v_x, \tag{141}$$

$$v'_{B,x} = -\frac{4}{7} r\omega_B + \frac{3}{7} v_{B,x} = \frac{4}{7}\frac{10}{7} v_x - \frac{3}{7}\frac{3}{7} v_x = \frac{31}{49} v_x. \tag{142}$$

The transform of these results to the input for collision C with the floor is,

$$r\,\omega_C = r\,\omega'_B = \frac{60}{49}v_x, \qquad v_{C,x} = -v'_{B,x} = -\frac{31}{49}v_x, \tag{143}$$

and the results of collision C are,

$$r\,\omega'_C = -\frac{3}{7}r\,\omega_C - \frac{10}{7}v_{C,x} = -\frac{3}{7}\frac{60}{49}v_x + \frac{10}{7}\frac{31}{49}v_x = \frac{130}{343}v_x, \tag{144}$$

$$v'_{C,x} = -\frac{4}{7}r\,\omega_C + \frac{3}{7}v_{C,x} = -\frac{4}{7}\frac{60}{49}v_x - \frac{3}{7}\frac{31}{49}v_x = -\frac{333}{343}v_x \approx -v_x, \tag{145}$$

$$v'_{C,y} = -v_y. \tag{146}$$

The prediction is that the ball has almost exactly the reverse of its initial velocity after the 3 bounces A, B and C.

We next consider the alternative hypothesis of **no slipping at the point of contact** and conservation only of the total mechanical energy.

In this model, the velocity of the point on the ball in contact with the floor is zero during, and just after, the collision,

$$0 = v'_c = v'_x + r\,\omega', \qquad v'_x = -r\,\omega', \tag{147}$$

rather than eq. (129).

Just after the collision, the motion is rigid-body rotation about the point of contact, plus translation perpendicular to the floor. Conservation of angular momentum about the point of contact, eq. (125), can now be written as,

$$L_z = -mrv_x + kmr^2\,\omega = L'_z = -mrv'_x + kmr^2\,\omega' = (1+k)mr^2\,\omega', \tag{148}$$

$$r\omega' = \frac{k}{1+k}r\,\omega - \frac{v_x}{1+k} = \frac{2}{7}r\,\omega - \frac{5}{7}v_x, \tag{149}$$

$$v'_x = -r\omega' = -\frac{2}{7}r\,\omega + \frac{5}{7}v_x. \tag{150}$$

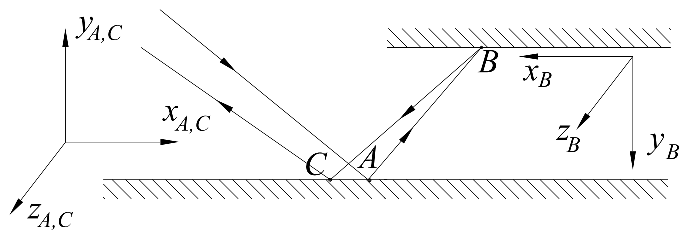
Finally, we determine v'_y by conservation of total mechanical energy,

$$\begin{aligned}\frac{mv_x^2}{2} + \frac{mv_x^2}{2} + \frac{kmr^2\,\omega^2}{2} &= \frac{mv'^2_x}{2} + \frac{mv'^2_y}{2} + \frac{kmr^2\,\omega'^2}{2} \\ &= \frac{(1+k)m}{2}\left(\frac{k}{1+k}r\,\omega - \frac{v_x}{1+k}\right)^2 + \frac{mv'^2_y}{2}\end{aligned} \tag{151}$$

$$v'^2_y = v_y^2 + \frac{(kr\,\omega - v_x)^2}{1+k} = v_y^2 + \frac{(2r\,\omega - 5v_x)^2}{35}. \tag{152}$$

We now reconsider the third example, of a ball thrown without spin onto the floor and up under a table. We will find that the ball does not return, which disfavors the model of no slippage at the point of contact.

We again trace the results of 3 collisions of the ball, first with the floor at point A, then with the underside of the table at point B, and again with floor at point C.



Again, $|v_y|$ is essentially constant if the velocity is large enough that effects of gravity are negligible.

The results of the collision at point A, eqs. (149), (150) (152), are, for initial spin $\omega = 0$ initial horizontal velocity $v_x > 0$ and initial vertical velocity $y_y < 0$,

$$r\,\omega'_A = -\frac{5}{7}v_x, \qquad v'_{A,x} = -r\,\omega'_A = \frac{5}{7}v_x, \qquad v'^2_y = v_y^2 + \frac{5v_x^2}{7}\,. \tag{153}$$

To use these results as input for the collision at point B under the table, we must transform them to a right-handed coordinate system (x_B, y_B, z_B) associated with that point, with the y_B axis downward, as in the figure, and the x_B axis opposite to axis x_A. That is,

$$r\,\omega_B = r\,\omega'_A = -\frac{5}{7}v_x, \quad v_{B,x} = -v'_{A,x} = -\frac{5}{7}v_x, \quad v_{B,y}^2 = v'^2_{A,y} = v_y^2 + \frac{5v_x^2}{7}\,. \tag{154}$$

Then, the results of the collision at point B are,

$$r\,\omega'_B = \frac{2}{7}r\omega_B - \frac{5}{7}v_{B,x} = -\frac{2}{7}\frac{5}{7}v_x + \frac{5}{7}\frac{5}{7}v_x = \frac{15}{49}v_x, \tag{155}$$

$$v'_{B,x} = -r\,\omega'_B = -\frac{15}{49}v_x, \tag{156}$$

$$v'^2_{B,y} = v_{B,y}^2 + \frac{5v_{B,x}^2}{7} = v_y^2 + \frac{5v_x^2}{7} - \frac{5}{7}\frac{25}{49}v_x^2 = v_y^2 + \frac{120}{343}v_x^2. \tag{157}$$

The negative value of $v'_{B,x}$ indicates that the ball will not return in this model. But, we proceed to collision C.

The transform of these results to the input for collision C with the floor is,

$$r\,\omega_C = r\,\omega'_B = \frac{15}{49}v_x, \quad v_{C,x} = -v'_{B,x} = \frac{15}{49}v_x, \quad v_{C,y}^2 = v'^2_{B,y} = v_y^2 + \frac{120}{343}v_x^2. \tag{158}$$

and the results of collision C are,

$$r\,\omega'_C = \frac{2}{7}r\,\omega_C - \frac{5}{7}v_{C,x} = \frac{2}{7}\frac{15}{49}v_x - \frac{5}{7}\frac{15}{49}v_x = -\frac{45}{343}v_x, \tag{159}$$

$$v'_{C,x} = -r\,\omega'_C = \frac{45}{343}v_x, \tag{160}$$

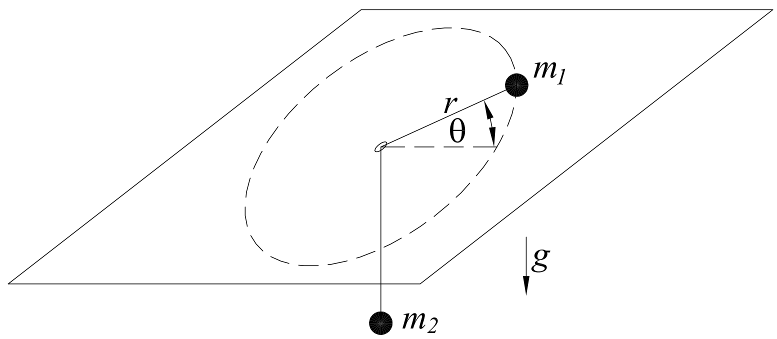
$$v'^2_{C,y} = v_{C,y}^2 + \frac{5v_{C,x}^2}{7} = v_y^2 + \frac{120}{343}v_x^2 + \frac{5}{7}\frac{225}{2401}v_x^2 = v_y^2 + \frac{7005}{16807}v_x^2. \tag{161}$$

In the model with no slippage at the point of contact, the ball does not return after collision C, in disagreement with experiment.

That, is, the model of no slippage is disfavored![7]

[7]It could be that while there is no slippage at the point of contact at the time of peak forces during the collision, some slippage occurs during the beginning and end of the collision, such that, in effect, the condition (147) does not hold.

11. (a) We use Lagrange's methods, with coordinates r and θ of mass m_1, which is connected to mass m_2 by a string of length l.



The kinetic energy is,

$$T = \frac{m_1}{2}(\dot{r}^2 + r^2\,\dot{\theta}^2) + \frac{m_2}{2}\dot{r}^2. \tag{162}$$

and the potential energy can be written as,

$$V = m_2 g(r - l). \tag{163}$$

Then, Lagrange's equations for $\mathcal{L} = T - V$ are,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = (m_1 + m_2)\ddot{r} = \frac{\partial \mathcal{L}}{\partial r} = m_1 r\,\dot{\theta}^2 - m_2 g, \tag{164}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{d}{dt}(m_1 r^2 \dot{\theta}) = \frac{\partial \mathcal{L}}{\partial \theta} = 0, \qquad m_1 r^2\,\dot{\theta} = L_0 = \text{constant}. \tag{165}$$

$$(m_1 + m_2)\ddot{r} = \frac{L_0^2}{m_1 r^3} - m_2 g = -\frac{d}{dr}\left(\frac{L_0^2}{2m_1 r^2} + m_2 r g\right) \equiv -\frac{dV_{\text{eff}}}{dr}\,. \tag{166}$$

The equilibrium radius is, for $\ddot{r} = 0$,

$$r_0 = \sqrt[3]{\frac{L_0^2}{m_1 m_2 g}}\,, \qquad L_0^2 = m_1 m_2 g r_0^3, \tag{167}$$

and the equilibrium angular velocity $\dot{\theta}_0$ is related by $L_0 = m_1 r_0^2 \dot{\theta}_0$, such that,

$$v_0 = r_0\dot{\theta}_0 = \frac{L_0}{m_1 r_0} = \sqrt{\frac{m_2 g r_0}{m_1}}\,, \qquad T_0 = \frac{m_1 v_0^2}{2} = \frac{m_2 g r_0}{2}\,. \tag{168}$$

To determine the character of small oscillations about the equilibrium radius r_0, we expand the effective potential V_{eff} about this,

$$V_{\text{eff}}(r) \approx V_{\text{eff}}(r_0) + \frac{1}{2}\frac{d^2V_{\text{eff}}(r_0)}{dr^2}(r - r_0)^2. \tag{169}$$

which is a springlike potential with effective spring constant,

$$k_{\text{eff}}(r) = \frac{d^2V_{\text{eff}}(r_0)}{dr^2} = \frac{3L_0^2}{m_1 r_0^4} \tag{170}$$

The angular frequency of small oscillations about r_0 is,

$$\omega = \sqrt{\frac{k_{\rm eff}}{m_1 + m_2}} = \sqrt{\frac{3L_0^2}{m_1 r_0^4}} = \sqrt{\frac{3m_2 g}{(m_1+m_2)r_0}}\,. \tag{171}$$

(b) If m_1 initially moves in a circle of radius r_0 the energy of the system is, recalling eq. (168),

$$E_0 = T_0 + V_0 = \frac{m_1}{2} r_0^2 \dot\theta_0^2 + m_2 g(r_0 - l) = \frac{3m_2 g r_0}{2} - m_2 g l. \tag{172}$$

If this system later comes to rest due to friction of the sprinkled dust, the energy would be only $E_{\rm final} = -m_2 g l$. That is, the energy dissipated by the dust is,

$$\Delta E = E_0 - E_{\rm final} = \frac{3m_2 g r_0}{2}\,. \tag{173}$$

The force of friction is $F = \mu m_1 g$, so the work done by friction as mass m_1 slides distance d is $W = \Delta E = Fd$, so the distance traveled by m_1 until stops is,

$$d = \frac{\Delta E}{F} = \frac{3m_2 g r_0}{2\mu m_1 g} = \frac{3m_2 r_0}{2\mu m_1}\,. \tag{174}$$

The rate of dissipation of energy by friction is,

$$\frac{dE}{dt} = -Fv, \tag{175}$$

where $v \approx r\dot\theta$ is the velocity of mass m_1. Assuming that the motion is always approximately circular, eqs. (168) and (172) hold for all r, and,

$$\frac{dE}{dt} = \frac{3m_2 g}{2}\frac{dr}{dt} = -\mu m_1 g \sqrt{\frac{m_2 g r}{m_1}}\,, \qquad \frac{dr}{\sqrt{r}} = -\frac{2\mu}{3}\sqrt{\frac{m_1 g}{m_2}}\,dt, \tag{176}$$

$$2(\sqrt{r} - \sqrt{r_0}(= -\frac{2\mu}{3}\sqrt{\frac{m_1 g}{m_2}}(t - t_0), \qquad \Delta t = \frac{3}{\mu}\sqrt{\frac{m_2 r_0}{m_1 g}}\,, \tag{177}$$

where Δt is the time interval over which the system comes to rest due to friction.