Princeton University

Ph205
Mechanics
Problem Set 2

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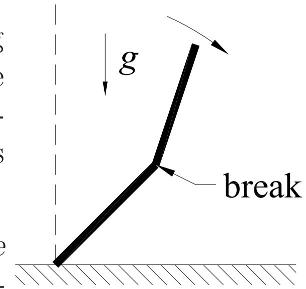
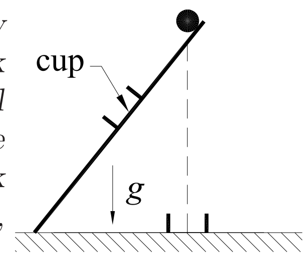
http://kirkmcd.princeton.edu/examples/

1. **Falling Chimney**

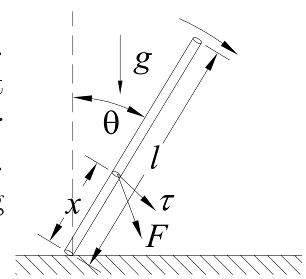
If a chimney is undermined on one side, so that it falls, rotating about its base, it usually snaps before hitting the ground. We can estimate the most likely position of the break by an extension of the principles of statics to a dynamic situation. This is the spirit of D'Alembert.



You might wish to convince yourself that the above picture shows the behavior after the break by performing a home experiment. A ball rests on the one end of a stick held initially at some angle to the horizontal, with the other end of the stick on the floor. Let the system loose. The stick will appear to fall faster that the ball. A cup placed on the stick can catch the ball after the stick hits the floor. Hence, the top end of the stick falls with acceleration greater than 1 g, and if the stick is weak, it will snap in the sense shown in the first figure



Consider a slice through the chimney a distance x from its base. The internal forces across this slice can be combined into a net force $\mathbf{F}$ applied at the center of the slice, and a net torque $\boldsymbol{\tau}$ acting about the center of the slice — a principle of statics. "Clearly" $\boldsymbol{\tau}$ is perpendicular to the vertical plane of the falling chimney.



The chimney might break at x for any of 3 reasons:

1. The tension $F_{\|}$ along the chimney is too great for the mortar between the bricks to sustain. However, $F_{\|}$ is compressive in the case of the falling chimney, and cannot lead to a break.
2. The shear $F_{\perp}$ across the slice is too great.
3. The torque $\boldsymbol{\tau}$ is too great and the chimney bends and snaps.

Show that for an unbroken, falling chimney of mass m at angle θ to the vertical,

$$\tau(x) = \frac{mgx(l-x)^2 \sin\theta}{4l^2}, \qquad \text{and} \qquad F_{\perp} = \frac{mgx(l-x)(l-3x)\sin\theta}{4l^2}, \tag{1}$$

such that the chimney most like breaks at $x = l/3$ if torque matters, but at $x = 2l/3$ if shear matters.

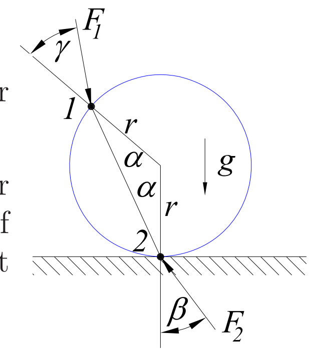
Empirically, most chimneys break near $x = 1/3$, suggesting that they break due to the torque effect.

Hint: Consider torque analyses of the entire chimney, and of the two portions described above.

2. **The Napkin Ring**

This everyday problem requires careful analysis.

A napkin ring in the form of a cylindrical shell of mass m and radius r rests on a horizontal table at point 2. You press on it with force $\mathbf{F}_1$ at some point 1. Discuss the conditions required so that the ring does not move. The possible motion for a solid sphere, rather than a cylindrical ring, will be considered in Prob. 4, Set 4.



Let μ_1 and μ_2 be the coefficients of friction at points 1 and 2. First, what is the direction of $\mathbf{F}_1$, and the value of $\mu_{1,\mathrm{min}}$, for static equilibrium?

If $\mu_1 < \mu_{1,\mathrm{min}}$, then the ring will slip (with respect to your finger pressing on it) at point 1 no matter how small F_1 is. Show that if you start with F_1 small and increase it, the ring will roll without slipping at point 2.

Next, suppose that $\mu_1 < \mu_{1,\mathrm{min}}$ such that the ring never slips at point 1. What about slipping at point 2?

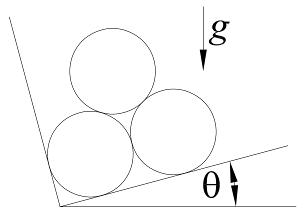
Let α and β be the angles shown in the figure. Show that for static equilibrium,

$$\cot\beta = \cot\alpha + \frac{mg}{F_1 \sin\alpha}\,. \tag{2}$$

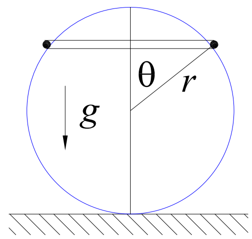
3. Consider an arbitrary motion of a rigid body (recall Chasles' theorem[1]). Show that for any pair of particles i and j within the body, the work done by the forces they exert on another during a small displacement of the body obeys $\delta W_{ij} = \mathbf{f}_{ij} \cdot \delta \mathbf{r}_i + \mathbf{f}_{ji} \cdot \delta \mathbf{r}_j = 0$. Hence, $\delta W_{\rm internal} = \sum_{ij} \Delta W_{ij} = 0$, as claimed on p. 32 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l3.pdf.

[1] M. Chasles, *Note sur les propriétés générales du systéme de deux corps semblables entr'eux*, Bull. Sci. Math. Astr. Phys. Chem. **14**, 321 (1830), http://kirkmcd.princeton.edu/examples/mechanics/chasles_bsmpc_14_321_30.pdf
This theorem was first proved by G. Mozzi, *Discorso matematico sopra il rotamento momentaneo dei corpi* (Naples, 1763), http://kirkmcd.princeton.edu/examples/mechanics/mozzi_discorso.pdf
although the result seems to have been known to Leonardo da Vinci. See also, M. Ceccarelli, *Screw axis defined by Giulio Mozzi in 1763 and early studies on helicoidal motion*, Mech. Machine Theory **35**, 761 (2000), http://kirkmcd.princeton.edu/examples/mechanics/ceccarelli_mmt_35_761_00.pdf

4. Three identical, cylindrical logs rest on the tilted bed of a lumber truck, as shown below. Ignoring friction, what is the minimum angle θ such that all three logs remain touching?



5. A rubber band of mass m, rest length l_0 and spring constant k lies without friction on a billiard ball of radius r, What is the polar angle θ from the vertical axis of the ball to the band?

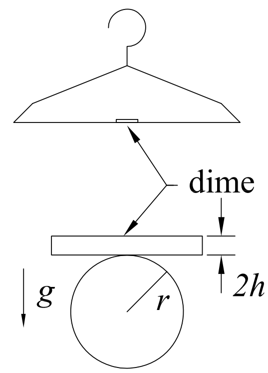


6. An equilibrium is said to be stable if $\delta W \geq 0$ in any small displacement from the equilibrium.

   (a) A dime of thickness $2h$ is balanced on a coat hanger whose wire has radius a. What is the condition for stability?

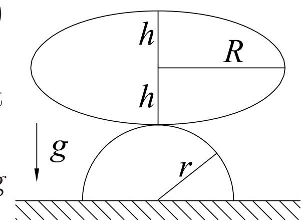
   Try it!

   If done well, you can twirl the coat hanger about a horizonal axis and the dime will stay on.



   (b) An oblate spheroid has height $2h$ along its axis, and radius R. It is balance on a sphere of radius r. Show that the equilibrium is stable if,

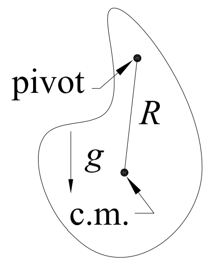
$$\frac{1}{h} > \frac{1}{r} + \frac{1}{\rho}, \tag{3}$$



   where ρ is the radius of curvature of the spheroid at the point of contact. What is the value of ρ?

   If the equilibrium is stable, the spheroid is a rocking stone; if not, a rolling stone.

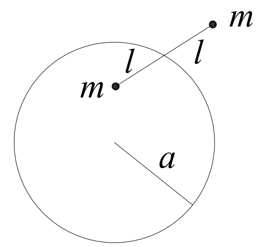
7. A physical pendulum has mass m and moment of inertia I about a pivot point.



The center of mass is at distance R from the pivot point. Where is the center of oscillation (*i.e.*, the location of a point mass m that has the same period as that of the physical pendulum)?

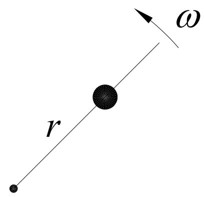
Next, suppose that the physical pendulum is hung from the center of oscillation found above. Show that the old pivot point is the new center of oscillation, and hence the period of oscillation is unchanged.

8. Two "point" masses m are joined by a "massless" rod of length $2l$, the center of which is constrained to move in a circle of radius a. There are no external forces.



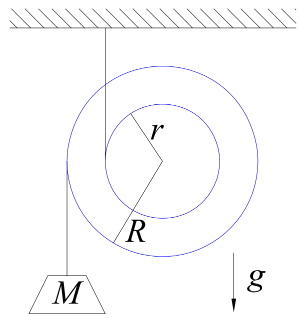
Describe the possible motion using the technique of separation into center-of-mass motion and motion relative to the c.m. Verify your analysis using Lagrange's method. Identify the generalized forces.

9. A mass m slides without friction along a rod that is constrained to rotate in a plane with constant angular velocity ω. There are no external forces.



Use Lagrange's method to find $r(t)$ if $r(0) = r_0$ and $\dot{r}(0) = v_0$.

10. A double cylinder of mass m and moment of inertia I hangs from a string wrapped around the cylinder of radius r. A mass M is suspended from a string wrapped around the cylinder of radius R.



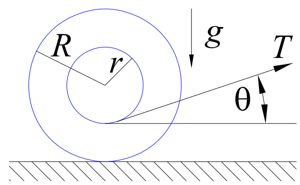
(a) What is the condition of static equilibrium? Discuss the special case that $r = R$.

(b) Use elementary (Newtonian) methods to find the acceleration of the center of mass of the cylinder if the system is not in equilibrium.

(c) Use Lagrange's method to find the acceleration.

The Newtonian method requires consideration of the tensions in the strings, while Lagrange's method avoid this, as the tensions are constraint forces.

11. The double cylinder of the previous problem rests on a horizontal surface with friction. A string wrapped around the cylinder of radius r make angle θ to the horizontal and is pulled with constant tension T.



(a) What is the condition of static equilibrium? What is the minimum coefficient of friction required for this?

(b) Suppose that the system is not in equilibrium, and the coefficient of friction is great enough that the cylinder rolls without slipping. Find the acceleration of the cylinder by Newtonian methods. For what angle θ would the motion be the same even if the friction were zero?

(c) Find the acceleration using Lagrange's method, by constructing a "potential" V such that the work done by the tension T during a small displacement of the system is $dW = -dV$. What is the generalized force associated with this "potential"?

This problem is called **Grandma and the Cat**. *Grandma drops her spool of thread and the cat paws it out of reach. Can she retrieve the spool simply by pulling on the thread? Does she need Lagrange to figure it out?*

Solutions

1. The torque equation for the entire (unbroken) chimney about its base is,

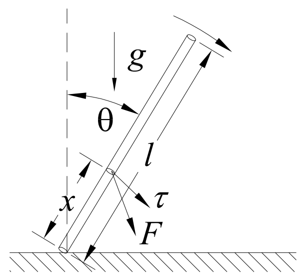
$$\frac{ml^2}{3}\ddot{\theta} = mg\frac{l}{2}\sin\theta, \qquad \ddot{\theta} = \frac{3g\sin\theta}{2l}\,, \tag{4}$$

noting that $\mathbf{F}$ and $\boldsymbol{\tau}$ are zero at the top of the chimney,

We next consider the torque equation for lower portion of the chimney, from 0 to x, again taking the point of reference as the base of the chimney,

$$m\frac{x}{l}\frac{x^2}{3}\ddot{\theta} = m\frac{x}{l}g\frac{x}{2}\sin\theta + F_\perp x - \tau, \qquad \frac{mgx^3\sin\theta}{2l^2} - \frac{mgx^2\sin\theta}{2l} = F_\perp x - \tau. \tag{5}$$

taking $\boldsymbol{\tau}$ to be out of the page.



We also consider the torque equation for the upper portion of the chimney (before it breaks), from x to l. The center of mass of this segment is accelerating, so we use its center of mass as the reference point. Noting that $\mathbf{F}$ and $\boldsymbol{\tau}$ at x on the upper segment are equal and opposite to those on the lower segment, we have,

$$m\frac{l-x}{l}\frac{(l-x)^2}{12}\ddot{\theta} = F_\perp\frac{l-x}{2} + \tau, \qquad \frac{mg(l-x)^3\sin\theta}{8l^2} = F_\perp\frac{l-x}{2} + \tau \tag{6}$$

Adding eqs. (5) and (6),

$$\begin{aligned} F_\perp\frac{l+x}{2} &= \frac{mg\sin\theta}{l}\left(\frac{x^3}{2l} - \frac{x^2}{2} + \frac{l^3 - 3l^2x + 3lx^2 - x^3}{8l}\right) \\ = \frac{mg\sin\theta}{l}\frac{l^3 - 3l^2x - lx^2 + 3x^3}{8l} &= \frac{mg\sin\theta}{l}(l+x)\frac{l^2 - 4lx + 3x^2}{8l} \\ &= \frac{mg\sin\theta}{l}(l+x)(l-x)\frac{l-3x}{8l}\,. \end{aligned} \tag{7}$$

$$F_\perp = \frac{mg\sin\theta(l-x)(l-3x)}{4l^2}\,. \tag{8}$$

Then, using eq. (6),

$$\begin{aligned} \tau = \frac{mg(l-x)^3\sin\theta}{8l^2} - F_\perp\frac{l-x}{2} &= \frac{mg(l-x)\sin\theta}{8l^2}[l^2 - 2lx + x^2 - (l^2 - 4lx + 3x^2)] \\ &= \frac{mgx(l-x)^2\sin\theta}{4l^2}\,. \end{aligned} \tag{9}$$

$F_\perp$ is maximum at $x = 2l/3$, while τ is maximum at $x = l/3$.

The literature on the falling chimney includes:

E.J. Routh, *The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 7^{th} ed. (Macmillan, 1905), Art. 150,
http://kirkmcd.princeton.edu/examples/mechanics/routh_elementary_rigid_dynamics.pdf

R.M. Sutton, *Concerning Falling Chimneys*, Science **84**, 246 (1936),
http://kirkmcd.princeton.edu/examples/mechanics/sutton_science_84_246_36.pdf

J.B. Reynolds, *Falling Chimneys*, Science **87**, 186 (1938),
http://kirkmcd.princeton.edu/examples/mechanics/reynolds_science_87_186_38.pdf

F.P. Bundy, *Stress in Freely Falling Chimneys and Columns*, J. Appl. Phys. **11**, 112 (1940), http://kirkmcd.princeton.edu/examples/mechanics/bundy_jap_11_112_40.pdf

A.T. Jones, *The Falling Chimney*, Am. J. Phys. **14**, 275 (1946),
http://kirkmcd.princeton.edu/examples/mechanics/jones_ajp_14_275_46.pdf

A.A. Bartlett, *More on the falling chimney*, Phys. Teach. **14**, 351 (1975),
http://kirkmcd.princeton.edu/examples/mechanics/bartlett_pt_14_351_75.pdf

E.L. Madsen, *Theory of chimney breaking while falling*, Am. J. Phys. **45**, 182 (1977),
http://kirkmcd.princeton.edu/examples/mechanics/madsen_ajp_45_182_77.pdf

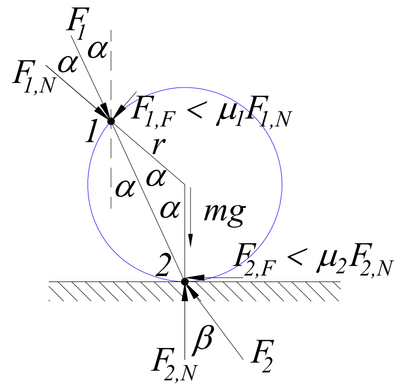
J. Walker, *Strange to relate, smokestacks and pencil points break in the same way*, Sci. Am. **240**(2), 158 (1979), http://kirkmcd.princeton.edu/examples/mechanics/walker_sa_240-2_158_79.pdf

G. Varieschi and K. Kamiya, *Toy models for the falling chimney*, Am. J. Phys. **71**, 1025 (2003), http://kirkmcd.princeton.edu/examples/mechanics/varieschi_ajp_71_1025_03.pdf

G. Varieschi and I.R. Jully, *Toy Blocks and Rotational Physics*, Phys. Teach. **43**, 360 (2005), http://kirkmcd.princeton.edu/examples/mechanics/varieschi_pt_43_360_05.pdf

2. *This problem is from p. 109 of J.H. Jeans, Theoretical Mechanics (Ginn and Co., 1907),* http://kirkmcd.princeton.edu/examples/mechanics/jeans_mechanics_07.pdf

For static equilibrium, the sum of the torques about any point must be zero. About point 2, the torques due to gravity and $\mathbf{F}_2$ are zero, so that due to $\mathbf{F}_1$ must also be zero. That is, $\mathbf{F}_1$ (if nonzero) points to 2, and angle $\gamma = \alpha$.



At point 1, the components of $\mathbf{F}_1$ are, for static equilibrium, related by,

$$\tan\alpha = \frac{F_{1,F}}{F_{1,N}} \le \frac{\mu_1 F_{1,N}}{F_{1,N}}, \tag{10}$$

which only holds if $\mu_1 \ge \mu_{1,\rm min} = \tan\alpha$.

If $\mu_1 < \mu_{1,\rm min}$, the ring slips at point 1 for any value of the normal force $F_{1,N}$, and the ring takes on an initial velocity v_0 to the right, and an initial angular velocity ω_0, which is defined to be positive for backspin ($\boldsymbol{\omega}_0$ out of the page). Aspects of the subsequent motion will be the topic of Prob. 4, Set 4.

If $\mu_1 > \mu_{1,\rm min}$ we consider whether the ring slips at point 2 for large $\mathbf{F}_1$ applied at angle α to the radius vector to point 1, as in the figure.

We have that the normal force at point 2 is,

$$F_{2,N} = F_1\cos\alpha + mg, \tag{11}$$

and for static equilibrium the frictional force at point 2 must be,

$$F_{2,F} = F_1\sin\alpha \le \mu_2 F_{2,N} = \mu_2(F_1\cos\alpha + mg). \tag{12}$$

Static equilibrium is possible for,

$$F_1 \le \frac{\mu_2 mg}{\sin\alpha - \mu_2\cos\alpha}. \tag{13}$$

For what it's worth, if static equilibrium holds, then,

$$\cot\beta = \frac{F_{2,N}}{F_{2,F}} = \cot\alpha + \frac{mg}{F_1\sin\alpha} < \cot\alpha, \qquad \beta > \alpha. \tag{14}$$

3. During a small displacement of a rigid body, the change in position of the location $\mathbf{r}_i$ of a point in that body can be written, according to Chasles' theorem, as,

$$\delta\mathbf{r}_i = \delta\mathbf{r}_0 + \delta\boldsymbol{\theta} \times (\mathbf{r}_i - \mathbf{r}_0), \tag{15}$$

for some point 0 fixed with respect to the body (although not necessarily inside it)

$$\mathbf{f}_{ij} = -\mathbf{f}_{ji}. \tag{16}$$

Hence,

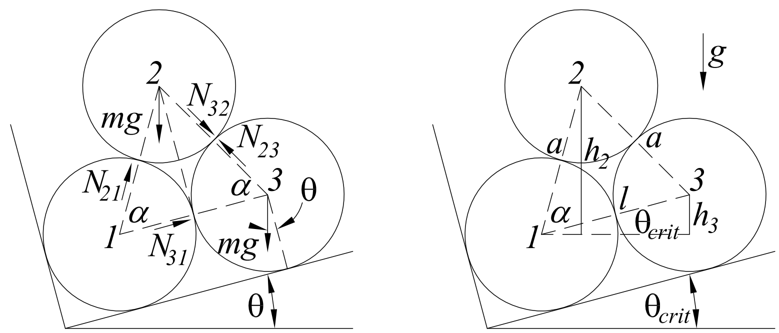
$$\begin{aligned}\delta W_{ij} = \mathbf{f}_{ij} \cdot \delta\mathbf{r}_i + \mathbf{f}_{ji} \cdot \delta\mathbf{r}_j &= (\mathbf{f}_{ij} + \mathbf{f}_{ji}) \cdot \delta\mathbf{r}_0 + \mathbf{f}_{ij} \cdot \delta\boldsymbol{\theta} \times (\mathbf{r}_i - \mathbf{r}_0) + \mathbf{f}_{ji} \cdot \delta\boldsymbol{\theta} \times (\mathbf{r}_j - \mathbf{r}_0) \\ &= \mathbf{f}_{ij} \cdot \delta\boldsymbol{\theta} \times (\mathbf{r}_i - \mathbf{r}_j) = \delta\boldsymbol{\theta} \cdot (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{f}_{ij}. \end{aligned} \tag{17}$$

In classical mechanics, we suppose that the force between points i and j lies along their line of centers, in which case $(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{f}_{ij} = 0$ and $\delta W_{ij} = 0$.

However, if we consider the particles to have electric charges q_i and q_j, and they are in motion, then, in general, $\mathbf{f}_{ij}$ includes terms not along their line of centers, $(\mathbf{r}_i - \mathbf{r}_j)$. Consistency with classical mechanics is obtained by supposing that the electromagnetic field, as well as the moving masses m_i and m_j, carries momentum.

In this course, we will only consider electrically neutral bodies, and will not discuss field momentum further.

4. We can solve this problem by a force analysis at the critical equilibrium, noting that the contact force N_{31} of log 1 on log 3 goes to zero at the critical angle $\theta_{\rm crit}$.



When all the logs touch one another, the forces on log 3 (of mass m) parallel to the bed of the truck are related by (ignoring friction),

$$N_{31} + N_{32}\cos\alpha = mg\sin\theta. \qquad (18)$$

To find N_{32}, which is equal in magnitude to N_{23}, we consider the forces on log 2 parallel and perpendicular to the bed of the truck,

$$N_{21}\cos\alpha - N_{23}\cos\alpha = mg\sin\theta, \qquad (19)$$

$$N_{21}\sin\alpha + N_{23}\sin\alpha = mg\cos\theta, \qquad (20)$$

and hence, $$2N_{23}\cos\alpha\sin\alpha = mg\cos\theta\cos\alpha - mg\sin\theta\sin\alpha. \qquad (21)$$

Combining eqs. (18) and (21) for $\theta = \theta_{\rm crit}$ and $N_{31} = 0$, we have,

$$\frac{mg\cos\theta_{\rm crit}\cot\alpha}{2} - \frac{mg\sin\theta_{\rm crit}}{2} = mg\sin\theta_{\rm crit}, \qquad \cot\theta_{\rm crit} = 3\tan\alpha = 3\sqrt{3}. \qquad (22)$$

However, it is more elegant to apply the principle of virtual work to a small displacement of the logs when the bed of the truck is at the critical angle $\theta_{\rm crit}$. In this displacement (from an equilibrium configuration), $\theta_{\rm crit}$ remains fixed, log 1 remains fixed, logs 1 and 2 remain touching with centers at distance a apart, logs 2 and 3 remain touching, but the distance l between the centers of logs 1 and 3 increases. The angle α between the lines 1-2 and 1-3 decreases from its equilibrium value of $\tan\alpha = \sqrt{3}$.

Then, the virtual work done by gravity during the small displacement from an equilibrium configuration is zero. Ignoring friction, this implies,

$$\delta W = mg(\delta h_2 + \delta h_3) = 0, \qquad (23)$$

where $h_{2,3}$ is the height of the center of log 2,3 above the center of log 1. Now,

$$h_2 = a\sin(\alpha + \theta_{\rm crit}), \quad \delta h_2 = a\cos(\alpha+\theta_{\rm crit})\,\delta\alpha, \quad h_3 = l\sin\theta_{\rm crit}, \quad \delta h_3 = \delta l\sin\theta_{\rm crit}, \qquad (24)$$

$$\frac{l}{2} = a\cos\alpha, \qquad \delta l = -2a\sin\alpha\,\delta\alpha, \qquad \delta h_3 = -2a\sin\alpha\sin\theta_{\rm crit}\,\delta\alpha, \qquad (25)$$

From eq. (23), $-\delta h_3 = \delta h_2$, and hence,

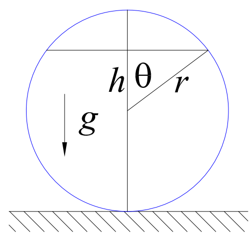
$$2\sin\alpha\sin\theta_{\rm crit} = \cos(\alpha+\theta_{\rm crit}) = \cos\alpha\cos\theta_{\rm crit} - \sin\alpha\sin\theta_{\rm crit}, \qquad (26)$$

$$\cot\theta_{\rm crit} = 3\tan\alpha = 3\sqrt{3}, \qquad \tan\theta_{\rm crit} = \frac{\sqrt{3}}{9}, \qquad \theta_{\rm crit} = 10.9^\circ. \qquad (27)$$

5. As the band falls and stretches, gravitational potential energy is transforms into spring energy. In the absence of friction these changes are equal and opposite. A calculation of this related the initial, unstretched band to its final configuration is messy, but the calculation for a small step is easy.

   For a change with respect to the final, static equilibrium, we invoke the principle of virtual work,

$$\delta W = 0 = \delta(mgh) + \delta\left(\frac{k(l-l_0)^2}{2}\right) = mg\,\delta h + k(l-l_0)\,\delta l. \tag{28}$$



   From the geometry of a band with minor radius small compared to that of the ball,

$$h = r\cos\theta, \qquad l = 2\pi r\sin\theta, \tag{29}$$

$$\delta h = -r\sin\theta\,\delta\theta, \qquad \delta l = 2\pi r\cos\theta\,\delta\theta, \tag{30}$$

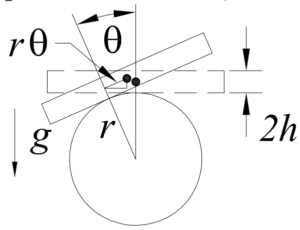
   and from eqs. (28) and (29),

$$mgr\sin\theta\,\delta\theta = 2\pi k(l-l_0)r\cos\theta\,\delta\theta, \tag{31}$$

$$l = 2\pi r\sin\theta = l_0 + \frac{mg}{2\pi k}\tan\theta. \tag{32}$$

6. The equilibrium of the dime or spheroid is stable if its center of mass rises as the object rolls (without slipping) away from equilibrium.

   (a) As the dime rolls through angle θ on the wire of the coat hanger, the point of contact of the dime with the wire moves by distance $r\theta$. This implies that the center of mass of the dime is at distance $r\theta$ from the radial line from the center of the wire through the point of contact, as shown in the figure below.

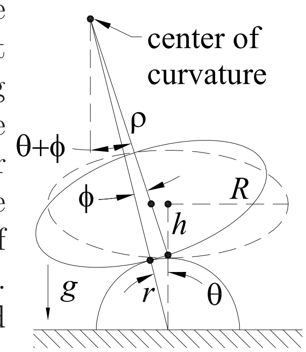


Hence, the center of mass of the dime moves from $H_0 = r + h$ above the center of the wire to height H, where,

$$H = (r+h)\cos\theta + r\theta\sin\theta \approx (r+h)\left(1 - \frac{\theta^2}{2}\right) + r\theta^2 = r + h + (r-h)\frac{\theta^2}{2}. \quad (33)$$

The equilibrium is stable if $H > H_0$, *i.e.*, if $r > h$ (which is satisfied by many hangers).

   (b) Let ρ be the radius of curvature of the spheroid at the equilibrium point of contact. If the point of contact rotates by angle θ about the center of the supporting sphere, then the height of the center of curvature above the center of the sphere is $(r+\rho)\cos\phi$, and the center of mass of the spheroid is below this by vertical distance $(\rho - h)\cos(\theta + \phi$, where ϕ is the angle about the center of curvature between the initial and final points of contact. Also, by the law of sines for the triangle containing θ and ϕ, $\phi \approx r\theta/\rho$.



Hence, the height above the center of the sphere of the center of mass of the rotated spheroid is,

$$H = (r+\rho)\cos\theta - (\rho - h)\cos(\theta+\phi) \approx (r+\rho)\left(1 - \frac{\theta^2}{2}\right) + (h-\rho)\left(1 - \frac{\theta^2}{2}\left(1 + \frac{r}{\rho}\right)^2\right)$$

$$= r + h - \frac{r\theta^2}{2} - \frac{\rho\theta^2}{2} + \frac{(\rho-h)\theta^2}{2}\left(1+\frac{r}{\rho}\right)^2. \quad (34)$$

For stability, we need $H > r + h$, and hence,

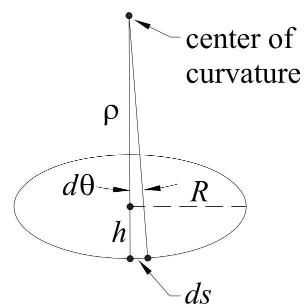
$$-(r+\rho) + \frac{(\rho-h)}{\rho^2}(r+\rho)^2 > 0, \qquad \frac{(\rho-h)}{\rho^2}(r+\rho) > 1, \quad (35)$$

$$r\rho > hr + h\rho, \qquad \frac{1}{h} > \frac{1}{r} + \frac{1}{\rho}. \quad (36)$$

If the spheroid were a sphere of radius R, then $\rho = h = R$ and there is no stability for finite r.

One student felt the preceding solution was not mathematical enough, and submitted the version on the following, handwritten pages.

The radius ρ of curvature at a point on the spheroid is related by $\rho = ds/d\theta$ where ds is the arc length along the spheroid from that point subtended by small angle $d\theta$ about the center of curvature.
Taking the origin at the center of the spheroid, its equation at equilibrium in the (vertical) x-y plane is,



$$\frac{x^2}{R^2} + \frac{y^2}{h^2} = 1, \qquad y = \pm h\sqrt{1 - \frac{x^2}{R^2}}\,. \tag{37}$$

At the point $(x, y) = (0, -h)$, we have that, $ds \approx dx$ and,

$$d\theta = \frac{y(dx) - y(0)}{dx} \approx \frac{-h}{dx}\left(1 - \frac{dx^2}{R^2} - 1\right) = \frac{h\,dx}{R^2}\,, \tag{38}$$

and (in the limit of zero dx),

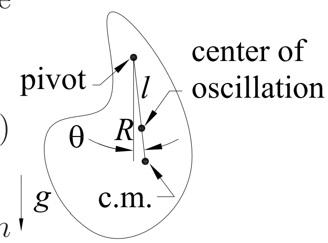
$$\rho = \frac{ds}{d\theta} = \frac{R^2}{h}\,. \tag{39}$$

7. The center of oscillation is the point such that is all the mass of the physical pendulum were concentrated there, the angular frequency of small oscillations would be unchanged.

The torque equation for the physical pendulum, of mass m and moment of inertia I_p with respect to the pivot point is,

$$I_p\ddot{\theta} = -mgR\sin\theta \approx -mgR\theta, \tag{40}$$

where θ is the angle of the center of mass from the vertical. The angular frequency of small oscillation is,



$$\omega_p = \sqrt{\frac{mgR}{I_p}}, \tag{41}$$

whereas this would be $\sqrt{g/l}$ for a simple pendulum of mass m and length l. Hence, the center of oscillation is at distance,

$$l = \frac{I_p}{mR}, \tag{42}$$

from the pivot, along the line to the center of mass.

By the parallel axis theorem, the moment of inertia of the physical pendulum about its center of oscillation is,

$$I_{\rm co} = I_{cm} + m(R-l)^2 = I_p - mR^2 + m(R-l)^2 = I_p + m(l^2 - 2lR). \tag{43}$$

If the physical pendulum were hung from the center of oscillation, at distance $R - l$ from the center of mass, the angular frequency of small oscillations would be,

$$\omega_{\rm co} = \sqrt{\frac{mg(R-l)}{I_{\rm co}}}, \tag{44}$$

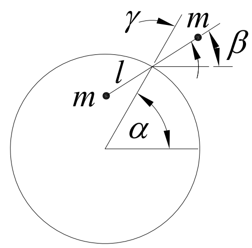
and the new center of oscillation would be at distance,

$$l' = \frac{I_{\rm co}}{m(R-l)} = \frac{lR + l^2 - 2lR}{R-l} = -l, \tag{45}$$

from the original center of oscillation. The new center of oscillation is at distance $(R-l) - l' = R$ from the center of mass, which is just the position of the original pivot point.

8. In this idealized problem there are no torques about either the center of the rod of length $2l$ or the center of the fixed circle of radius a on which the pivot of the rod slides. Hence, the rod rotates with constant angular velocity ω about the pivot point, which is the center of mass of the moving rods. Also, the pivot point moves in a circle with (a different) constant angular velocity Ω.

   The angular momentum of the center-of-mass motion is $L_{rmcm} = 2ma^2\Omega$, while the angular momentum of the motion relative to the c.m. is $L_{\rm rel} = 2ml^2\omega$. The total angular momentum is the sum of these two.



   To use Lagrange's method, we note that this system has two degrees of freedom, which we take to be the angle α of the radius of the circle to the pivot point, and angle β of the rod relative to the reference direction for angle θ. The kinetic energy is the sum of that of the center of mass motion plus that of the motion relative to the center of mass,

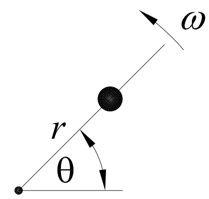
$$T = ma^2\dot\alpha^2 + ml^2\dot\beta^2, \tag{46}$$

   the potential energy V is 0, and the Lagrangian is simply $\mathcal{L} = T$. Then, since $\partial\mathcal{L}/\partial\alpha = 0 = \partial\mathcal{L}/\partial\beta$, there are two constant canonical momenta,

$$\frac{\partial\mathcal{L}}{\partial\dot\alpha} = 2ma^2\dot\alpha = L_{\rm cm}, \qquad \text{and} \qquad \frac{\partial\mathcal{L}}{\partial\dot\beta} = 2ma^2\dot\beta = L_{\rm rel}. \tag{47}$$

   We might have taken the second angle to be γ $(= \alpha - \beta)$, between the rod and the radius vector to the pivot. Then, the Lagrangian would be $\mathcal{L} = ma^2\dot\alpha^2 + ml^2(\dot\alpha - \dot\gamma)^2$, which leads to conserved canonical momentum that are linear combinations of the angular momenta of eq. (47).

9. This system has a single degree of freedom, r, the distance of mass m from the center of rotation at constant angular velocity $\omega = d\theta/dt$.



The kinetic energy is,

$$T = \frac{m\dot{r}^2}{2} + \frac{mr^2\omega^2}{2}, \tag{48}$$

and the potential energy V is zero. Lagrange's equation of motion is, for $\mathcal{L} = T - V = T$,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\ddot{r} = \frac{\partial \mathcal{L}}{\partial r} = m\omega^2 r. \tag{49}$$

This has solutions of the form,

$$r = A\,e^{\omega t} + B\,e^{-\omega t}, \tag{50}$$

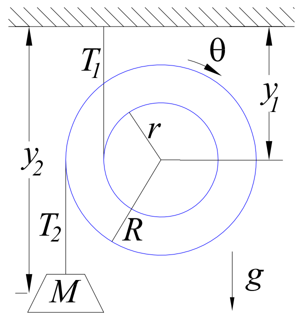
so for $r(0) = r_0$ and $\dot{r}(0) = v_0$, we have,

$$A + B = r_0, \qquad A - B = \frac{v_0}{\omega}, \qquad A = \frac{r_0}{2} + \frac{v_0}{2\omega}, \qquad B = \frac{r_0}{2} - \frac{v_0}{2\omega}, \tag{51}$$

and finally,

$$r = r_0 \cosh \omega t + \frac{v_0}{\omega} \sinh \omega t. \tag{52}$$

10. .



(a) If the cylinder drops by Δy_1, then it rotates by angle $\Delta\theta = \Delta y_1/r$ with respect to a fixed direction. Meanwhile, the coordinate of mass M changes by,

$$\Delta y_2 = \Delta y_1 - R\Delta\theta = -\Delta y_1 \frac{R-r}{r}. \tag{53}$$

The Newtonian equations of motion of masses M and m are,

$$M\ddot{y}_2 = -M\ddot{y}_1\frac{R-r}{r} = Mg - T_2, \qquad T_2 = Mg + M\ddot{y}_1\frac{R-r}{r}, \tag{54}$$

$$m\ddot{y}_1 = mg - T_1 + T_2, \qquad T_1 = mg + Mg + M\ddot{y}_1\frac{R-r}{r} - m\ddot{y}_1, \tag{55}$$

$$I\ddot{\theta} = \frac{I\ddot{y}_1}{r} = T_1 r - T_2 R$$

$$= r(m+M)g + M\ddot{y}_1(R-r) - mr\ddot{y}_1 - RMg - RM\ddot{y}_1\frac{R-r}{r}, \tag{56}$$

and hence,

$$\ddot{y}_1\left(\frac{I}{r} - M(R-r) + mr + RM\frac{R-r}{r}\right) = (mr - M(R-r))g \tag{57}$$

$$\ddot{y}_1\left(mr + M\frac{(R-r)^2}{r} + \frac{I}{r}\right) = (mr - M(R-r))g \tag{58}$$

(b) For static equilibrium $\ddot{y}_1 = 0$, and the masses must be related by,

$$m = M\frac{R-r}{r}. \tag{59}$$

If $r = R$, equilibrium is possible only for a massless cylinder (and massless string).

(c) For Lagrange's method with for the single degree of freedom y_1, we note that the kinetic energy is,

$$T = \frac{m\dot{y}_1^2}{2} + \frac{I\dot{\theta}^2}{2} + \frac{M\dot{y}_2^2}{2} = \frac{\dot{y}_1^2}{2}\left(m + \frac{I}{r^2} + M\frac{(R-r)^2}{r^2}\right), \tag{60}$$

and the potential energy can be written as,

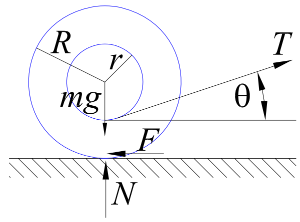
$$V = -mgy_1 - Mgy_2 + \text{constant} = \left(-m + M\frac{R-r}{r}\right)gy_1. \tag{61}$$

Lagrange's equation of motion is then, for $\mathcal{L} = T - V$,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{y}_1} = \frac{d}{dt}\frac{\partial T}{\partial\dot{y}_1} = \ddot{y}_1\left(m + \frac{I}{r^2} + M\frac{(R-r)^2}{r^2}\right) = \frac{\partial\mathcal{L}}{\partial y_1} = -\frac{\partial V}{\partial y_1} = \left(m - M\frac{R-r}{r}\right)g, \tag{62}$$

which is eq. (58) divided by r.

11. The forces on the spool are sketched in the figure below.



(a) The force and torque equations of motion are, for rolling without slipping where the angular acceleration of the cylinder is $\alpha = -a/R$ with a positive to the right and $\boldsymbol{\alpha}$ positive when out of the paper,

$$T\cos\theta - F = ma, \tag{63}$$

$$T\sin\theta + N - mg = 0, \tag{64}$$

$$rT - RF = I\alpha = -\frac{Ia}{R}, \tag{65}$$

using the center of mass of the cylinder as the reference point for the torques. For static equilibrium, $a = 0$, and hence,

$$\frac{F}{T} = \cos\theta = \frac{RF}{r}, \qquad \cos\theta = \frac{r}{R}, \tag{66}$$

$$T\cos\theta = F \le \mu N = \mu(mg - T\sin\theta), \qquad \mu \ge \frac{T\cos\theta}{mg - T\sin\theta}, \tag{67}$$

and $T\sin\theta \le mg$ so the cylinder stays on the floor.

(b) For rolling without slipping with nonzero acceleration (with $T\sin\theta \le mg$), we can combine eqs. (63) and (65) to find,

$$a = \frac{R}{I}(R(T\cos\theta - ma) - rT), \qquad a = \frac{RT(R\cos\theta - r)}{I + mR^2} = \frac{T}{m}\frac{\cos\theta - r/R}{1 + I/mR^2}. \tag{68}$$

If there were no friction, we would also have $ma = T\cos\theta$, and hence,

$$ma = T\frac{\cos\theta - r/R}{1 + I/mR^2} = T\cos\theta, \qquad \cos\theta = \frac{r/R}{-I/mR^2} = -\frac{mrR}{I}. \tag{69}$$

(c) To use Lagrange's method with a constant tension T in the string as it winds at fixed angle θ, we can associate a "potential" V with it, such that the work done by the tension when the center of the cylinder moves by dx (to the right) is,

$$dW = -dV = -\frac{dV}{dx}\,dx = F_{\text{generalized}}\,dx. \tag{70}$$

When the cylinder moves by dx, the outer cylinder of radius R rolls without slipping, and length dx of the string becomes wound onto the inner cylinder, so the cylinder rotates by angle $d\phi = -dx/R$. During this rotation, the torque of the string does work,

$$dW_\tau = \tau\, d\phi = -\frac{rT\, dx}{R}\,. \tag{71}$$

Meanwhile, the point of contact of the string with the inner cylinder has moved distance $d\mathbf{x} = dx\,\hat{\mathbf{x}}$ so the tension $\mathbf{T}$ has done work of translation,

$$dW_T = \mathbf{T}\cdot d\mathbf{x} = T\cos\theta\, dx. \tag{72}$$

The total work done is,

$$dW = T\left(\cos\theta - \frac{r}{R}\right) dx = -\frac{dV}{dx}\, dx, \tag{73}$$

so the "potential" can be written as,

$$V = -Tx\left(\cos\theta - \frac{r}{R}\right). \tag{74}$$

The kinetic energy of the motion is,

$$\text{KE} = \frac{m\dot{x}^2}{2} + \frac{I\dot{\phi}^2}{2} = \left(m + \frac{I}{R^2}\right)\frac{\dot{x}^2}{2}\,, \tag{75}$$

so that Lagrange's equation of motion is, with $\mathcal{L} = \text{KE} - V$ and $\ddot{x} = a$,

$$\left(m + \frac{I}{R^2}\right) a = T\left(\cos\theta - \frac{r}{R}\right) = F_{\text{generalized}}, \tag{76}$$

in agreement with eq. (69).