Princeton University

Ph205
Mechanics
Problem Set 11

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(1988)

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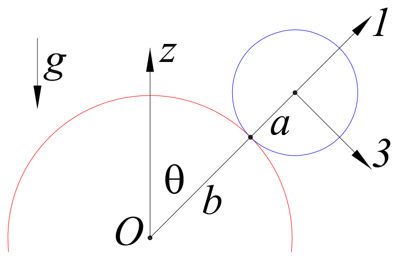
http://kirkmcd.princeton.edu/examples/

1. **Spinning Basketballs.**

The Harlem Globetrotters can balance a basketball stably on a finger by spinning the ball. That stability is possible if the basketball acts like a gyroscope and precesses, rather than falling off the finger.



Consider a sphere, of mass m and radius a with moment of inertia I about its center, that rolls without slipping on a fixed sphere of radius b. Derive, and decompose into components, the (vector) equations of motion.

Show that the total angular velocity $\boldsymbol{\omega}$ obeys $\boldsymbol{\omega} \cdot d\hat{\mathbf{1}}/dt = 0 = \hat{\mathbf{1}} \cdot d\boldsymbol{\omega}/dt$, where $\hat{\mathbf{1}}$ points outward along the line of centers of the two spheres and makes angle θ to the vertical, $\hat{\mathbf{z}}$, and hence,

$$\boldsymbol{\omega} = \omega_1 \hat{\mathbf{1}} + \frac{a+b}{a} \hat{\mathbf{1}} \times \frac{d\hat{\mathbf{1}}}{dt}, \tag{1}$$

where $\omega_1 = \boldsymbol{\omega} \cdot \hat{\mathbf{1}} =$ constant, and that,

$$\left(I + ma^2\right) \frac{a+b}{a} \hat{\mathbf{1}} \times \frac{d^2\hat{\mathbf{1}}}{dt^2} + I\,\omega_1 \frac{d\hat{\mathbf{1}}}{dt} + mga\,\hat{\mathbf{2}} = 0. \tag{2}$$

Note that $\hat{\mathbf{1}}$ rotates about $\hat{\mathbf{z}}$ at rate $\dot{\phi}$ and about $\hat{\mathbf{2}} = \hat{\mathbf{1}} \times \hat{\mathbf{z}}$ at rate $\dot{\theta}$ (be careful with signs).

After obtaining the 3 component equations of motion, first consider steady motion, $\dot{\theta} = 0$, $\dot{\phi} = \Omega =$ constant, to show that ω_1 must satisfy,

$$\omega_1 > \frac{2}{I}\sqrt{mg(a+b)\,(I + ma^2)\cos\theta_0}, \tag{3}$$

for steady motion.

The spinning sphere will fall off the fixed sphere if the force of contact between them vanishes. Show that this happens (during steady motion) if,

$$\Omega^2 < \frac{g\cos\theta_0}{(a+b)\sin^2\theta_0}. \tag{4}$$

Use the relation between Ω and ω_1 to show this indicates that too much spin is bad, as well as too little.

Consider nutations about steady precession,

$$\theta = \theta_0 + \epsilon \sin \alpha t, \qquad \dot{\phi} = \Omega + \delta \sin \alpha t, \tag{5}$$

for small constants ϵ and δ to show that $\alpha^2 > 0$ for large enough ω_1, in which case the nutations are stable.

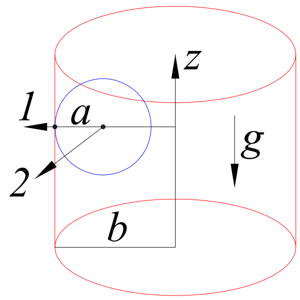
For a basketball of radius $a = 12$ cm, which is a hollow sphere with $I = 2mq^2/3$, balanced vertically on a finger of radius of curvature $b \approx 1$ cm, the spin required for gyroscopic stability is greater than 6 revolutions per second, which seems higher than in videos of "balanced", spinning basketballs. That is, their stability is due to active stabilization by horizontal motion of the support finger rather than gyroscopic effects.

One of many YouTube videos on how to spin a basketball,
https://www.youtube.com/watch?v=lLxUq6nhkb4
in which the spin seems to be only 1-2 revolutions per second.

2. **The Golfer's Nemesis.**

Can a golf ball roll into the cup, roll around on its vertical wall and pop back out?[1]

Consider a sphere of radius a that rolls without slipping inside a vertical cylinder of radius $b > a$.



If $\Omega = \dot{\phi}$ = angular velocity of the point of contact about the vertical, $\hat{\mathbf{1}}$ points from the center of the sphere to the point of contact, $\hat{\mathbf{z}}$ is vertical, and $\hat{\mathbf{2}} = \hat{\mathbf{z}} \times \hat{\mathbf{1}}$, show that the component equations of motion are,

$$\hat{\mathbf{z}}: \qquad \dot{\Omega} = 0, \tag{6}$$

$$\hat{\mathbf{1}}: \qquad a\,\dot{\omega}_1 = \Omega z, \tag{7}$$

$$\hat{\mathbf{2}}: \qquad \left(I + ma^2\right)\ddot{z} = -ma^2 g - I a\,\omega_1\,\Omega. \tag{8}$$

Show that z of the center of mass executes simple harmonic motion, and if at $t = 0$, $z = 0$, $\dot{z} = \dot{z}_0$, and $\omega_1 = \omega_{10}$, then,

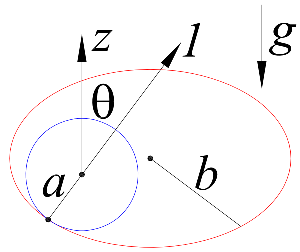
$$z = \frac{ma^2 g + I a\,\Omega\,\omega_{10}}{I\,\Omega^2}(\cos\alpha t - 1) + \frac{\dot{z}_0}{\alpha}\sin\alpha t, \qquad \text{where} \qquad \alpha = \Omega\sqrt{\frac{I}{I + ma^2}}\,. \tag{9}$$

With what velocity and angular velocity must the ball arrive at the rim of the cup to fall in and execute the above oscillatory motion, and possibly pop back out?

[1] This behavior is distinct from the possibility that the ball bounces off the flagpole in the hole, or the plastic insert therein, as occurs from time to time.

3. **Off the Rim.**

A frequent occurrence in basketball or golf is that the ball rolls around in the rim of the hoop/cup for a while, then sometimes goes in, sometimes not...



Consider a sphere of radius a that rolls without slipping on a horizontal hoop of radius $b > a$. An equilibrium of steady rolling exist with zero "spin" component, $\omega_0 = \boldsymbol{\omega} \cdot \hat{\mathbf{1}} = 0$, where $\boldsymbol{\omega}$ is the total angular velocity of the sphere and $\hat{\mathbf{1}}$ is directed from the point of contact with the hoop to the center of the sphere. Show that in the case the angular velocity of the point of contact about the vertical is,

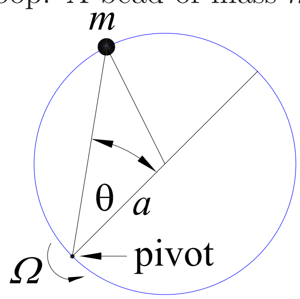
$$\Omega = \sqrt{\frac{3g \tan\theta_0}{5(b - a\sin\theta_0)}}, \tag{10}$$

for a spherical shell, where θ is the angle of $\hat{\mathbf{1}}$ to the vertical.

For a basketball of radius 12 cm and a hoop of radius 24 cm, $\Omega_0 \approx 0.6$ revolution per second at $\theta_0 = 45°$.

Show that this equilibrium is unstable (for $b/a = 2$). That is, for Ω greater/less than ω_0, the sphere rises/falls, and only in the latter case does it pass through the hoop as desired.

4. A circular hoop of radius a rotates constant angular velocity Ω in a horizontal plane about a fixed point on the hoop. A bead of mass m slides freely on the hoop.



(a) Use θ as shown in the figure as the coordinate with Lagrange's method to deduce the equation of motion.

(b) Show that the Hamiltonian is,

$$H = \frac{p_\theta^2}{8ma^2} - p_\theta \, \Omega \cos\theta - \frac{ma^2}{2} \, \Omega^2 \sin^2 2\theta, \tag{11}$$

and that Hamilton's equations lead to the equation of motion found in part (a).

(c) Deduce the equation of motion via an analysis in the rotating frame of the hoop.

5. **The Piano.**

A piano wire is struck by a sharp blow from a hammer, and a fairly pure note is produced.[2] This is perhaps surprising in view of the analysis on p. 229 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l21.pdf of the effect of an impulse. Helmholtz[3] has suggested that a better approximation to the effect of the hammer is that it exerts a force,

$$F(x,t) = \begin{cases} F\,\delta(x-b)\sin\frac{2\pi t}{T} & (0 < t < T/2), \\ 0 & \text{(otherwise)}. \end{cases} \tag{12}$$

That is, the force goes through one half period of s sinusoidal oscillation.

The force is applied at distance b from one end of a wire of length l and mass density ρ per unit length, which is fixed at both end and subject to a tension that makes the transverse wave velocity equal to c.

Consider a Fourier analysis of the vibrations, $s(x,t) = \sum_n \phi_n(t)\sin(n\pi x/l)$, and use Green's method[4] to solve the differential equations for the ϕ_n to show that,

$$s(x,t) = \frac{2FT}{\pi^2 c\,\rho}\sum_n \frac{1}{n(1-(\frac{ncT}{2l})^2)}\sin\frac{n\pi b}{l}\cos\frac{n\pi cT}{4l}\sin\frac{n\pi x}{l}\sin\frac{n\pi c(t-T/4)}{l}\,. \tag{13}$$

If we take $b = l/2$, the midpoint, and $T = 2l/c$, the fundamental period, then,

$$s(x,t) = \frac{Fl}{\pi^2 T}\sum_n \frac{\sin n\pi}{n(1-n^2)}\sin n\pi x\,\sin\frac{n\pi c(t-T/4)}{l}\,, \tag{14}$$

so all harmonics vanish except $n = 1$, since $\lim_{n\to 1}\frac{\sin n\pi}{1-n^2} = \frac{\pi\cos n\pi}{-2n} = \frac{\pi}{2}$.

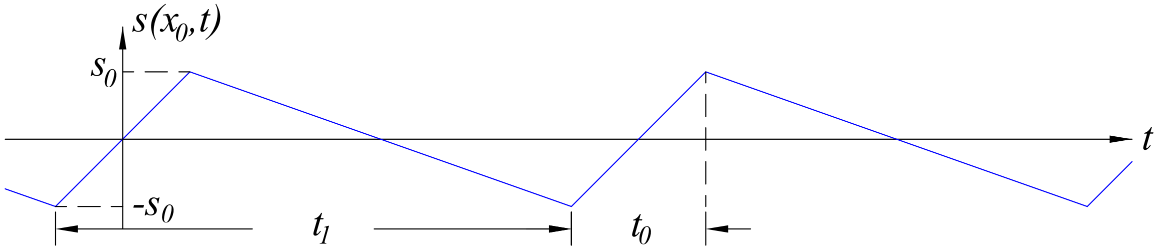
[2] *Awaiting the sensation of a short sharp shock, from a cheap and chippy chopper on a big black block!* — *The Mikado*, Act 1, Gilbert & Sullivan (1885).

[3] H.L.F. Helmholtz, *On the Sensations of Tone*, 2nd English ed. (Longmans, Green, 1885), pp. 380-394, http://kirkmcd.princeton.edu/examples/mechanics/helmholtz_85.pdf. See also, pp. 74-80 (and pp. 545-546), particularly the footnotes, which recount interest in England in Helmholtz' theories of the piano in the years 1883-1885.

[4] p. 145 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l13.pdf

6. **The Violin.**

From experiments, Helmholtz deduced that the action of the bow of a violin is to force the string of length l into a transverse vibration at x_0, the point of application of the bow ($0 < x_0 < l$), with the approximate form,



which is periodic with the period of the fundamental, free oscillation, $t_1 = 2l/c$, where c is the velocity of transverse waves on the stretched string. The rising motion occupies time $t_0 < t_1$ related by $x_0/l = t_0/t_1$.

Make a Fourier analysis in time of the motion of the point of contact to show that,

$$s(x_0, t) = \frac{2s_0 t_1^2}{\pi^2 t_0(t_1 - t_0)} \sum_n \frac{1}{n^2} \sin\frac{n\pi x_0}{l} \sin\frac{2n\pi t}{t_1}. \tag{15}$$

In general, we expect the motion of the entire string to the analyzable as,

$$s(x, t) = \sum_n \sin\frac{n\pi x}{l}\left(A_n \cos\frac{2n\pi t}{t_1} + B_n \sin\frac{2n\pi t}{t_1}\right), \tag{16}$$

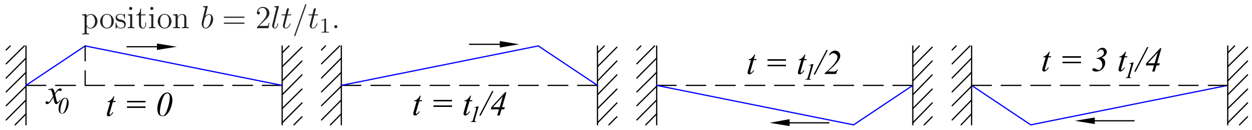
so it follows from eq. (15) that $A_n = 0$ and,

$$s(x, t) = \frac{2s_0 t_1^2}{\pi^2 t_0(t_1 - t_0)} \sum_n \frac{1}{n^2} \sin\frac{n\pi x}{l} \sin\frac{2n\pi t}{t_1}. \tag{17}$$

On p. 228 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l21.pdf, we saw that a string plucked at $x = b$ at time $t = 0$ has the Fourier analysis,

$$s(x, t) = \frac{2s_0 l^2}{\pi^2 b(l - b)} \sum_n \frac{1}{n^2} \sin\frac{n\pi x}{l} \sin\frac{n\pi b}{l}. \tag{18}$$

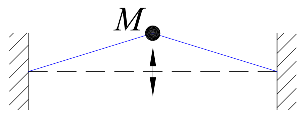
Hence, at any time t the violin string looks like the initial form of a string plucked at position $b = 2lt/t_1$.



The crest of the wave motion moves along the string with velocity $c = 2l/t_1$. The "vibration" is better described as a traveling wave than a standing wave.

7. A string of length l is fixed at both ends and stretched with tension T.

   (a) A mass M is attached to the midpoint of the string.



   Ignoring the mass of the string, show that transverse oscillations of mass M have angular frequency $\Omega_0 = 2\sqrt{T/lM}$.

   (b) Suppose mass M attached at distance $b < l$ from one end of the string (of mass m).

   Consider the intervals $[0, b]$ and $[b, l]$ to show that the normal (angular) frequencies Ω obey the transcendental equation,

$$\Omega \sin \frac{\Omega b}{c} \sin \Omega \frac{l-b}{c} = \frac{t}{Mc} \sin \frac{\Omega l}{c}, \tag{19}$$

   where $c = \sqrt{Tl/m}$ is the velocity of waves on the string.

   (c) Consider again the case where mass M is attached at $b = l/2$, but don't neglect the mass m of the string.

   Show that there are two classes of solutions, for which the angular frequency Ω obeys:

   i. Mss M does not move, and $\Omega = 2n\pi c/l$.

   ii. Mass M moves, and,

$$\frac{\Omega l}{2c} \tan \frac{\Omega l}{2c} = \frac{m}{M}. \tag{20}$$

   (d) If $M \ll m$, show that the lowest frequency is,

$$\Omega \approx \frac{\pi l}{c}\left(1 - \frac{M}{m}\right), \tag{21}$$

   which implies that $\Omega = \pi c/l$ when $M = 0$.

   (e) If $m \ll M$, keep enough higher-order terms to show that the lowest frequency is,

$$\Omega \approx \Omega_0 \left(1 - \frac{m}{6M}\right) \approx 2\sqrt{\frac{T}{l(M + m/3)}}, \tag{22}$$

   so that (for this mode) the mass of the string appears as a correction $m/3$ to mass M.

8. A uniform bar of mass m has rest length l_0. One end is fixed and the other end is attached to a mass M.

   Set up the boundary conditions, and solve the wave equation for longitudinal (spring-like) oscillations, ignoring gravity.

   Show that,

$$\cot(\Omega l_0) = \frac{M}{m}\Omega l_0, \qquad \text{where} \qquad \Omega = \frac{\omega}{l_0}\sqrt{\frac{m}{k}}\,, \tag{23}$$

   k is the spring constant of the bar, and ω is the angular frequency of the oscillations.

   By suitable approximation, show that the angular frequency of the lowest mode is,

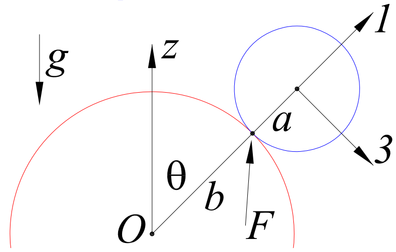
$$\omega \approx \sqrt{\frac{k}{M + m/3}}\,, \tag{24}$$

   as found in Prob. 5, Set 1, http://kirkmcd.princeton.edu/examples/ph205set1.pdf.

Solutions

1. **Spinning Basketballs.**

This problem is the Example on p. 354, §415 of E.A. Milne, *Vectorial Mechanics* (Metheun; Interscience Publishers, 1948),
http://kirkmcd.princeton.edu/examples/mechanics/milne_mechanics.pdf



We consider a sphere, of mass m and radius a with moment of inertia I about its center, that rolls without slipping on a fixed sphere of radius b. We use a set of principal axes (but not body axes) about the center of the sphere of radius a, where $\hat{\mathbf{1}}$ points outward along the line of centers of the two spheres and makes angle θ to the vertical, $\hat{\mathbf{z}}$. Also, $\hat{\mathbf{2}} = \hat{\mathbf{1}} \times \hat{\mathbf{z}} / \sin\theta$ (which is always horizontal), and $\hat{\mathbf{3}} = \hat{\mathbf{1}} \times \hat{\mathbf{2}}$ (which lies in the vertical plane of $\hat{\mathbf{1}}$ and $\hat{\mathbf{z}}$).

The center of the sphere of radius a is at position $\mathbf{r} = (a + b)\,\hat{\mathbf{1}}$ with respect to the center of the fixed sphere of radius b, which we take as the origin of coordinates in the lab frame. Then, the velocity of the center of the sphere of radius a is,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (a+b)\frac{d\hat{\mathbf{1}}}{dt}\,. \tag{25}$$

The (nonholonomic) constraint of rolling without slipping is that the point of contact on the spinning sphere of radius a with the sphere of radius b is instantaneously at rest in the lab frame,

$$\mathbf{v}_{\rm contact} = 0 = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{a} = (a+b)\frac{d\hat{\mathbf{1}}}{dt} - a\boldsymbol{\omega} \times \hat{\mathbf{1}}, \tag{26}$$

where $\boldsymbol{\omega}$ is the total angular velocity of the sphere radius a in the lab frame, and $\mathbf{a} = -a\,\hat{\mathbf{1}}$ is the vector from the center of the sphere of radius a to the point of contact.[5]

[5]At this point in the analysis we could also note that $\mathbf{v} = -(a+b)\dot\phi \sin\theta\,\hat{\mathbf{2}} + \dot\theta\,\hat{\mathbf{3}}$ where $\dot\phi$ is the angular velocity of the center of the spinning sphere about the z-axis. Then (25) implies eq. (47) below. We could also use eq. (26) to find,

$$\hat{\mathbf{1}} \times (\boldsymbol{\omega} \times \mathbf{a}) = -a\,\boldsymbol{\omega} - \omega_1\,\mathbf{a} = -\hat{\mathbf{1}} \times \mathbf{v} = (a+b)\,\dot\theta\,\hat{\mathbf{2}} + (a+b)\,\dot\phi \sin\theta\,\hat{\mathbf{3}}, \quad \boldsymbol{\omega} = \omega_1\,\hat{\mathbf{1}} - \frac{a+b}{a}\,\dot\theta\,\hat{\mathbf{2}} - \frac{a+b}{a}\,\dot\phi\,\hat{\mathbf{3}}, \tag{27}$$

where $\omega_1 = \boldsymbol{\omega} \cdot \hat{\mathbf{1}}$, in agreement with eq. (45) below.

The force and torque equations of motion of (center of) the sphere of radius a are,

$$m\frac{d\mathbf{v}}{dt} = m(a+b)\frac{d^2\hat{\mathbf{1}}}{dt^2} = \mathbf{F} - mg\,\hat{\mathbf{z}}, \qquad \mathbf{F} = m(a+b)\frac{d^2\hat{\mathbf{1}}}{dt^2} + mg\,\hat{\mathbf{z}}, \tag{28}$$

$$\frac{d\mathbf{L}}{dt} = I\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\tau} = \mathbf{a}\times\mathbf{F} = -ma(a+b)\hat{\mathbf{1}}\times\frac{d^2\hat{\mathbf{1}}}{dt^2} - mga\,\hat{\mathbf{1}}\times\hat{\mathbf{z}}. \tag{29}$$

From eq. (26) we have that,

$$\boldsymbol{\omega}\cdot\frac{d\hat{\mathbf{1}}}{dt} = 0, \tag{30}$$

while from eq. (29) we have that,

$$\hat{\mathbf{1}}\cdot\frac{d\boldsymbol{\omega}}{dt} = 0. \tag{31}$$

Hence,

$$\frac{d}{dt}(\boldsymbol{\omega}\cdot\hat{\mathbf{1}}) = \frac{d\omega_1}{dt} = 0, \tag{32}$$

and $\omega_1 = \boldsymbol{\omega}\cdot\hat{\mathbf{1}}$ is constant.

Also, we can multiply eq. (26) by $\hat{\mathbf{1}}$ to find that,

$$\boldsymbol{\omega} = \omega_1\,\hat{\mathbf{1}} + \frac{a+b}{a}\hat{\mathbf{1}}\times\frac{d\hat{\mathbf{1}}}{dt}, \qquad \frac{d\boldsymbol{\omega}}{dt} = \omega_1\frac{d\hat{\mathbf{1}}}{dt} + \frac{a+b}{a}\hat{\mathbf{1}}\times\frac{d^2\hat{\mathbf{1}}}{dt^2}, \tag{33}$$

and then rewrite the equation of motion (29) as,

$$\left(I + ma^2\right)\frac{a+b}{a}\hat{\mathbf{1}}\times\frac{d^2\hat{\mathbf{1}}}{dt^2} + I\,\omega_1\frac{d\hat{\mathbf{1}}}{dt} + mga\,\hat{\mathbf{1}}\times\hat{\mathbf{z}} = 0. \tag{34}$$

For steady motion, with $\theta = \theta_0 =$ constant, the spinning sphere, and the triad of principal axes, precess about the vertical at constant angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}_{123} = \Omega\,\hat{\mathbf{z}}$, and hence,

$$\frac{d\hat{\mathbf{1}}}{dt} = \boldsymbol{\omega}_{123}\times\hat{\mathbf{1}} = \boldsymbol{\Omega}\times\hat{\mathbf{1}} = \Omega\,\hat{\mathbf{z}}\times\hat{\mathbf{1}}, \tag{35}$$

$$\frac{d^2\hat{\mathbf{1}}}{dt} = \boldsymbol{\Omega}\times\frac{d\hat{\mathbf{1}}}{dt} = \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\hat{\mathbf{1}}) = (\boldsymbol{\Omega}\cdot\hat{\mathbf{1}})\boldsymbol{\Omega} - \Omega^2\,\hat{\mathbf{1}} = \Omega^2(\cos\theta_0\,\hat{\mathbf{z}} - \hat{\mathbf{1}}). \tag{36}$$

$$\hat{\mathbf{1}}\times\frac{d^2\hat{\mathbf{1}}}{dt} = \Omega^2\cos\theta_0\,\hat{\mathbf{1}}\times\hat{\mathbf{z}}. \tag{37}$$

Then, all terms in the equation of motion (34) are proportional to $\hat{\mathbf{1}}\times\hat{\mathbf{z}}$, and we have,

$$\left(I + ma^2\right)\frac{a+b}{a}\Omega^2\cos\theta_0 - I\,\omega_1\,\Omega + mga = 0, \tag{38}$$

$$\Omega = \frac{I\omega_1 \pm \sqrt{I^2\,\omega_1^2 - 4\,(I+ma^2)\,(a+b)mg\cos\theta_0}}{2\,(I+ma^2)\,\frac{a+b}{a}\cos\theta_0} = \frac{\omega_1 \pm \sqrt{\omega_1^2 - \omega_{1,\text{“min”}}^2}}{I\omega_{1,\text{“min”}}^2/2mga}, \tag{39}$$

where for steady precession at rate Ω to exist, we must have,

$$\omega_1 \geq \omega_{1,\text{“min”}} = \frac{2}{I}\sqrt{mg(a+b)\,(I+ma^2)\cos\theta_0}, \qquad (40)$$

which (not surprisingly) limits steady motion to angle $\theta_0 < 90°$.

For a basketball of radius $a = 12$ cm, which is a hollow sphere with $I = 2ma^2/3$ ($k = 2/3$), balanced vertically ($\theta_0 = 0$) on a finger of radius of curvature $b \approx 1$ cm $\ll a$, the minimum ω_1 required for gyroscopic stability is about 6 revolutions per second.[6] This seems higher that the rotation rates of spinning basketballs in online videos,[7] so it seems likely that their apparent stability is due to active stabilization by horizontal motion of the supporting finger, rather than gyroscopic stabilization.

The spinning sphere remains in contact with the fixed sphere only if the outward force of contact, $\mathbf{F}\cdot\hat{\mathbf{1}}$, is positive. From eqs. (28) and (36), we have for steady motion,

$$\mathbf{F} = m(a+b)\frac{d^2\hat{\mathbf{1}}}{dt^2} + mg\,\hat{\mathbf{z}} = m(a+b)\Omega^2(\cos\theta_0\,\hat{\mathbf{z}} - \hat{\mathbf{1}}), +mg\,\hat{\mathbf{z}}, \qquad (41)$$

$$\mathbf{F}\cdot\hat{\mathbf{1}} = mg\cos\theta_0 + m(a+b)\Omega^2(\cos^2\theta_0 - 1) = mg\cos\theta_0 - m(a+b)\Omega^2\sin^2\theta_0. \qquad (42)$$

Hence, the spinning sphere flies off the fixed sphere if,[8]

$$\Omega^2 > \frac{g\cos\theta_0}{(a+b)\sin^2\theta_0}. \qquad (43)$$

In particular, if ω_1 is $\omega_{1,\text{“min”}}$ of eq. (40), the spinning sphere flies of when,

$$\frac{ma^2\sin^2\theta_0}{(I+ma^2)\cos^2\theta_0} > 0, \qquad (44)$$

so only at $\theta_0 = 0$ can there be steady motion with $\omega_1 = \omega_{1,\text{“min”}}$.

That is, the true minimum of ω_1 for steady motion in contact with the fixed sphere is the root of the quartic equation obtained by combining eqs. (39) and (43). A numerical study[9] indicates that spinning sphere always flies off for Ω with the positive root in eq. (39), while for the negative root, steady motion in contact with the fixed sphere is possible for any $\theta_0 < 90°$ for large enough ω_1 (much larger than $\omega_{1,\text{“min”}}$ of eq. (40) as θ_0 approaches $90°$).

To discuss nutation about steady motion, we note that the angular velocity $\boldsymbol{\omega}_{123}$ of the principal axes consists of the term $-\dot{\theta}\,\hat{\mathbf{2}}$, together with their rotation $\dot{\phi}\,\hat{\mathbf{z}} = \dot{\phi}(\cos\theta\,\hat{\mathbf{1}} -$

[6] The Ω corresponding to this minimum ω_1 is $3\omega_1/4$, which describes the rotation of the mathematical triad $\hat{\mathbf{1}}$-$\hat{\mathbf{2}}$-$\hat{\mathbf{3}}$. However, ω_1 describes the rotation of the physical sphere, as visible to observers of spinning basketballs.

[7] Many videos include remarks that higher spin makes the ball more stable.

[8] For $\theta_0 = 0$, the spinning sphere will never fly off.

[9] http://kirkmcd.princeton.edu/examples/basketball.xlsx

$\sin\theta\,\hat{\mathbf{3}})$.[10] Also, the total angular velocity $\boldsymbol{\omega}$ of the sphere of radius a consists of the "spin" angular velocity ω_s of the sphere about axis $\hat{\mathbf{1}}$ relative to the principal axes, together with $(a+b)/a$ times the angular velocity $\boldsymbol{\omega}_{123}$ of the principal axes relative to the lab frame (which subtle relation is inferred from eqs. (33) and (46)). Hence,

$$\boldsymbol{\omega}_{123} = \dot\phi\cos\theta\,\hat{\mathbf{1}} - \dot\theta\,\hat{\mathbf{2}} - \dot\phi\sin\theta\,\hat{\mathbf{3}}, \qquad \boldsymbol{\omega} = \omega_s\,\hat{\mathbf{1}} + \frac{a+b}{a}\left(\dot\phi\cos\theta\,\hat{\mathbf{1}} - \dot\theta\,\hat{\mathbf{2}} - \dot\phi\sin\theta\,\hat{\mathbf{3}}\right). \tag{45}$$

The time rate of change of the principal axes is related by,

$$\frac{d\hat{\mathbf{i}}}{dt} = \boldsymbol{\omega}_{123}\times\hat{\mathbf{i}}, \tag{46}$$

$$\frac{d\hat{\mathbf{1}}}{dt} = (\dot\phi\cos\theta\,\hat{\mathbf{1}} - \dot\theta\,\hat{\mathbf{2}} - \dot\phi\sin\theta\,\hat{\mathbf{3}})\times\hat{\mathbf{1}} = -\dot\phi\sin\theta\,\hat{\mathbf{2}} + \dot\theta\,\hat{\mathbf{3}}, \tag{47}$$

$$\frac{d\hat{\mathbf{2}}}{dt} = (\dot\phi\cos\theta\,\hat{\mathbf{1}} - \dot\theta\,\hat{\mathbf{2}} - \dot\phi\sin\theta\,\hat{\mathbf{3}})\times\hat{\mathbf{2}} = -\dot\phi\sin\theta\,\hat{\mathbf{1}} + \dot\phi\cos\theta\,\hat{\mathbf{3}}, \tag{48}$$

$$\frac{d\hat{\mathbf{3}}}{dt} = (\dot\phi\cos\theta\,\hat{\mathbf{1}} - \dot\theta\,\hat{\mathbf{2}} - \dot\phi\sin\theta\,\hat{\mathbf{3}})\times\hat{\mathbf{3}} = -\dot\theta\,\hat{\mathbf{1}} - \dot\phi\cos\theta\,\hat{\mathbf{2}}. \tag{49}$$

$$\begin{aligned}\frac{d^2\hat{\mathbf{1}}}{dt^2} &= (-\ddot\phi\sin\theta - \dot\phi\,\dot\theta\cos\theta)\,\hat{\mathbf{2}} + \ddot\theta\,\hat{\mathbf{3}} + \dot\phi^2\sin^2\theta\,\hat{\mathbf{1}} - \dot\phi^2\sin\theta\cos\theta\,\hat{\mathbf{3}} - \dot\theta^2\,\hat{\mathbf{1}} - \dot\theta\,\dot\phi\cos\theta\,\hat{\mathbf{2}} \\ &= (\dot\phi^2\sin^2\theta - \dot\theta^2)\,\hat{\mathbf{1}} - (\ddot\phi\sin\theta + 2\dot\theta\,\dot\phi\cos\theta)\,\hat{\mathbf{2}} + (\ddot\theta - \dot\phi^2\sin\theta\cos\theta)\,\hat{\mathbf{3}},\end{aligned} \tag{50}$$

$$\hat{\mathbf{1}}\times\frac{d^2\hat{\mathbf{1}}}{dt^2} = (\dot\phi^2\sin\theta\cos\theta - \ddot\theta)\,\hat{\mathbf{2}} - (\ddot\phi\sin\theta + 2\dot\theta\,\dot\phi\cos\theta)\,\hat{\mathbf{3}}. \tag{51}$$

Using eqs. (47) and (51), and recalling that $\hat{\mathbf{1}}\times\hat{\mathbf{z}} = \sin\theta\,\hat{\mathbf{2}}$, we see that the equation of motion (34) has nonzero $\hat{\mathbf{2}}$- and $\hat{\mathbf{3}}$- components,

$$\left(I + ma^2\right)\frac{a+b}{a}(\dot\phi^2\sin\theta\cos\theta - \ddot\theta) - I\,\omega_1\dot\phi\sin\theta + mga\sin\theta = 0, \tag{52}$$

$$\left(I + ma^2\right)\frac{a+b}{a}(\ddot\phi\sin\theta + 2\dot\theta\,\dot\phi\cos\theta) - I\,\omega_1\dot\theta = 0. \tag{53}$$

For steady motion, $\theta = \theta_0 =$ constant, $\dot\theta = 0$, $\dot\phi = \Omega =$ constant, eq. (53) is trivial, while eq. (52) leads to eq. (38).

We also digress to consider a use of Lagrange's method, with coordinates θ, $\phi =$ angle of $\hat{\mathbf{2}}$ to the x-axis, and $\psi =$ angle of rotation of the sphere about the $\hat{\mathbf{1}}$ axis.

The center of the sphere of radius a is at distance $a+b$ from the origin = center of fixed sphere of radius b, and hence the velocity of its center can be written as $\mathbf{v} = (a+b)(\dot\theta\,\hat{\mathbf{3}} - \dot\phi\sin\theta\,\hat{\mathbf{2}})$. The kinetic energy of the center-of-mass motion is,[11]

$$T_{\rm cm} = \frac{mv^2}{2} = \frac{ma^2}{2}\frac{(a+b)^2}{a^2}\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right). \tag{54}$$

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[10]We could continue using the triad $\hat{\mathbf{1}}$, $\hat{\mathbf{z}}$, $\hat{\mathbf{1}}\times\hat{\mathbf{z}}$ as in eq. (34), but since $\hat{\mathbf{1}}$ and $\hat{\mathbf{z}}$ are not orthogonal, the algebra is somewhat more intricate.

[11]As a check, we note that the rolling constraint (26) can be written as $\mathbf{v} = \mathbf{a}\times\boldsymbol{\omega} = -a\,\hat{\mathbf{1}}\times\boldsymbol{\omega}$, and hence the kinetic energy of the motion of the center of mass is $T_{\rm cm} = mv^2/2 = ma^2(\omega^2 - \omega_1^2)/2$. Using eq. (45) we again obtain eq. (54).

The kinetic energy of rotation is, recalling eq. (45) and noting that $\omega_s = \dot\psi$,

$$T_{\rm rot} = \frac{I\,\omega^2}{2} = \frac{I}{2}\left(\dot\psi + \frac{a+b}{a}\dot\phi\cos\theta\right)^2 + \frac{I}{2}\frac{(a+b)^2}{a^2}\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right), \tag{55}$$

and the potential energy is $V = mg(a+b)\cos\theta$. *The Lagrangian is,*

$$\mathcal{L} = T_{\rm cm} + T_{\rm rot} - V =$$
$$\frac{I}{2}\left(\dot\psi + \frac{a+b}{a}\dot\phi\cos\theta\right)^2 + \frac{(I+ma^2)}{2}\frac{(a+b)^2}{a^2}\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) - mg(a+b)\cos\theta. \tag{56}$$

The Lagrangian does not depend on ψ, *so* $\partial\mathcal{L}/\partial\dot\psi = I\left[\dot\psi - ((a+b)/a)\dot\phi\cos\theta\right] = I\,\omega_1$ *is a conserved generalized momentum, and* ω_1 *is constant, as found above.*

The equation of motion for coordinate θ *is,*

$$\frac{\partial\mathcal{L}}{\partial\theta} = -I\,\omega_1\frac{a+b}{a}\,\dot\phi\sin\theta + \left(I+ma^2\right)\frac{(a+b)^2}{a^2}\dot\phi^2\sin\theta\cos\theta + mg(a+b)\sin\theta$$
$$= \frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = \left(I+ma^2\right)\frac{(a+b)^2}{a^2}\,\ddot\theta, \tag{57}$$
$$\left(I+ma^2\right)\frac{a+b}{a}\left(\dot\phi^2\sin\theta\cos\theta - \ddot\theta\right) - I\,\omega_1\,\dot\phi\sin\theta + mga\sin\theta = 0 \tag{58}$$

in agreement with eq. (52).

The equation of motion for coordinate ϕ *is, recalling that* ω_1 *is constant,*

$$\frac{\partial\mathcal{L}}{\partial\phi} = 0 = \frac{d}{dt}\frac{\partial L}{\partial\dot\phi} = \frac{d}{dt}\left[\frac{a+b}{a}I\,\omega_1\cos\theta + \left(I+ma^2\right)\frac{(a+b)^2}{a^2}\sin^2\theta\,\dot\phi\right]$$
$$= -\frac{a+b}{a}I\omega_1\,\dot\theta\sin\theta + \left(I+ma^2\right)\frac{(a+b)^2}{a^2}\left(\ddot\phi\sin^2\theta + 2\dot\theta\,\dot\phi\sin\theta\cos\theta\right), \tag{59}$$
$$\left(I+ma^2\right)\frac{a+b}{a}\left(\ddot\phi\sin\theta + 2\dot\theta\,\dot\phi\cos\theta\right) - I\omega_1\,\dot\theta = 0, \tag{60}$$

as found above in eq. (53).

The difficult step in the Lagrangian method is arriving at eq. (45) for the total angular velocity $\boldsymbol{\omega}$, *for which the vectorial method, and awareness of the rolling constraint, is helpful.*

We now consider nutations of the form,

$$\theta = \theta_0 + \epsilon\sin\alpha t, \qquad \dot\phi = \Omega + \delta\sin\alpha t, \tag{61}$$
$$\sin\theta \approx \sin\theta_0 + \epsilon\cos\theta_0\sin\alpha t, \qquad \cos\theta \approx \cos\theta_0 - \epsilon\sin\theta_0\sin\alpha t, \tag{62}$$

for small constants ϵ and δ. Then, to first order in ϵ and δ, eq. (53) becomes,

$$\left(I+ma^2\right)\frac{a+b}{a}(\alpha\,\delta\sin\theta_0\cos\alpha t + 2\alpha\,\epsilon\,\Omega\cos\theta_0\cos\alpha t) - \alpha\epsilon I\,\omega_1\cos\alpha t = 0, \tag{63}$$
$$\delta = \epsilon\frac{I\omega_1}{(I+ma^2)\,\frac{a+b}{a}\sin\theta_0} - \frac{2\epsilon\,\Omega\cos\theta_0}{\sin\theta_0}, \tag{64}$$

and eq. (52) becomes, recalling eq. (38) for the 0$^{\rm th}$-order terms,

$$\left(I + ma^2\right) \frac{a+b}{a}[(\Omega^2 + 2\Omega\,\delta\sin\alpha t)(\sin\theta_0 + \epsilon\cos\theta_0\sin\alpha t)(\cos\theta_0 - \epsilon\sin\theta_0\sin\alpha t) + \epsilon\alpha^2\sin\alpha t]$$
$$-I\,\omega_1\,(\Omega + \delta\sin\alpha t)(\sin\theta_0 + \epsilon\cos\theta_0\sin\alpha t) + mga(\sin\theta_0 + \epsilon\cos\theta_0\sin\alpha t) = 0, \qquad (65)$$

$$\left(I + ma^2\right) \frac{a+b}{a}(\epsilon\Omega^2\cos 2\theta_0 + 2\delta\,\Omega\sin\theta_0\cos\theta_0 + \epsilon\alpha^2)$$
$$-I\,\omega_1\,(\epsilon\Omega\cos\theta_0 + \delta\sin\theta_0) + \epsilon\, mga\cos\theta_0 = 0, \qquad (66)$$

$$\alpha^2 = -\Omega^2\cos 2\theta_0 - 2\Omega\cos\theta_0 \frac{I\,\omega_1}{(I + ma^2)\,\frac{a+b}{a}} + 4\Omega^2\cos^2\theta_0$$
$$+\frac{I\,\omega_1}{(I + ma^2)\,\frac{a+b}{a}}\left(\Omega\cos\theta_0 + \frac{I\,\omega_1}{(I + ma^2)\,\frac{a+b}{a}} - 2\Omega\cos\theta_0\right) - \frac{mga\cos\theta_0}{(I + ma^2)\,\frac{a+b}{a}} \qquad (67)$$
$$= \Omega^2(1 + 2\cos^2\theta_0) + \frac{I^2\,\omega_1^2}{(I + ma^2)^2\left(\frac{a+b}{a}\right)^2} - \frac{3I\,\omega_1\,\Omega\cos\theta_0}{(I + ma^2)\,\frac{a+b}{a}} - \frac{mga\cos\theta_0}{(I + ma^2)\,\frac{a+b}{a}}\,. \qquad (68)$$

For sufficiently large ω_1, $\alpha^2 > 0$, and the nutations exist as ongoing, small oscillations. However, the condition for this is not simple.

We can extract a somewhat simpler condition if we restrict our attention to the "minimum" ω_1 for steady motion, as found in eq. (40) above. For this case, the associated Ω is given by eq. (39), and is called $\Omega_{\rm min}$ here,

$$\Omega_{\rm min} = \frac{I_1\,\omega_{1,\text{“min”}}}{2\,(I + ma^2)\,\frac{a+b}{a}\cos\theta_0}\,. \qquad (69)$$

Using this in eq. (68), we have (after some algebra),

$$\alpha^2 = \frac{I^2\,\omega_{1,\text{“min”}}^2}{4\,(I + ma^2)^2\left(\frac{a+b}{a}\right)^2\cos^2\theta_0} - \frac{mga\cos\theta_0}{(I + ma^2)\,\frac{a+b}{a}}\,. \qquad (70)$$

Since $\cos\theta_0 \le 1$, this condition for stable nutations is slightly weaker than the condition (40) for the existence of steady motion. That is, whenever steady motion is possible, nutations about this motion are stable.

It was noted in sec. 41, p. 101 of H. Lamb, *Higher Mechanics* (Cambridge U. Press, 1920), http://kirkmcd.princeton.edu/examples/mechanics/lamb_higher_mechanics.pdf *that if we restrict our attention to motion in which angle θ is very small (as for balanced, spinning basketballs), we can give an analysis using only x-y-z coordinates.*

The center of the spinning sphere is at $\mathbf{r} = (x, y, z)$ where $r = a + b$. The rolling constraint (26) can then be written as,

$$\mathbf{v} = \dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z}) = \mathbf{a}\times\boldsymbol{\omega} = \frac{a}{a+b}(z\,\omega_y - y\,\omega_z, x\,\omega_z - z\,\omega_x, y\,\omega_x - x\,\omega_y), \qquad (71)$$

noting that $\mathbf{a} = -a\mathbf{r}/(a+b)$ and $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$.

The general equations of motion are, taking the torque about the center of the sphere,

$$m\ddot{\mathbf{r}} = \mathbf{F} - mg\,\hat{\mathbf{z}}, \qquad I\,\dot{\boldsymbol{\omega}} = \mathbf{a}\times\mathbf{F} = -\frac{a}{a+b}\mathbf{r}\times\mathbf{F}. \tag{72}$$

For motion with small θ, we have that $z \approx a+b$, $F_z \approx mg$, and $\omega_z \approx$ constant. The constraint relation (71) reduces to,

$$\dot{x} = a\,\omega_y - \frac{a\,\omega_z}{a+b}y, \qquad \dot{y} = -a\,\omega_x + \frac{a\,\omega_z}{a+b}x, \tag{73}$$

and the equations of motion (72) reduce to $m\ddot{x} = F_x$, $m\ddot{y} = F_y$ and,

$$I\,\dot{\omega}_x = aF_y - \frac{mga}{a+b}y = ma\ddot{y} - \frac{mga}{a+b}y, \quad I\,\dot{\omega}_y = -aF_x + \frac{mga}{a+b}x = -ma\ddot{x} + \frac{mga}{a+b}x. \tag{74}$$

Using eq. (74) in the time derivative of eq. (73), we find,

$$\ddot{x} = a\,\dot{\omega}_y - \frac{a\,\omega_z}{a+b}\dot{y} = -\frac{ma^2}{I}\ddot{x} + \frac{mga^2}{I(a+b)}x - \frac{a\,\omega_z}{a+b}\dot{y}, \tag{75}$$

$$\left(I + ma^2\right)\frac{a+b}{a}\ddot{x} + I\,\omega_z\dot{y} - magx = 0, \tag{76}$$

$$\ddot{y} = -a\,\dot{\omega}_x + \frac{a\,\omega_z}{a+b}\dot{x} = -\frac{ma}{I}\ddot{y} - \frac{mga^2}{I(a+b)}y + \frac{a\,\omega_z}{a+b}\dot{x}, \tag{77}$$

$$\left(I + ma^2\right)\frac{a+b}{a}\ddot{y} - I\,\omega_z\dot{x} - magy = 0. \tag{78}$$

Lamb noted that it is clever to introduce the complex variable $\zeta = x + iy$ where $i = \sqrt{-1}$ here. Then, eqs. (76) and (78) combine into the form,

$$\left(I + ma^2\right)\frac{a+b}{a}\ddot{\zeta} - iI\,\omega_z\,\dot{\zeta} - mag\,\zeta = 0. \tag{79}$$

We seek oscillatory behavior with $\zeta \propto e^{i\alpha t}$, which implies that,

$$\left(I + ma^2\right)\frac{a+b}{a}\alpha^2 - I\,\omega_z\,\alpha + mag = 0, \tag{80}$$

$$\alpha = \frac{I\omega_z \pm \sqrt{I^2\,\omega_z^2 - 4\left(I + ma^2\right)(a+b)mg}}{2\left(I + ma^2\right)\frac{a+b}{a}}. \tag{81}$$

This oscillatory behavior (nutation) exists for,

$$\left(I + ma^2\right)\frac{a+b}{a}\alpha^2 - I\,\omega_z\,\alpha + mag = 0, \tag{82}$$

$$\omega_z > \frac{2}{I}\sqrt{\left(I + ma^2\right)(a+b)mg}, \tag{83}$$

which is the same condition found in eq. (40) for $\theta_0 = 0$, where $\omega_1 = \omega_z$.

Lamb noted that if the real values of eq. (81) are $\alpha_\pm$, then the trajectory of the center of the spinning sphere has the form,

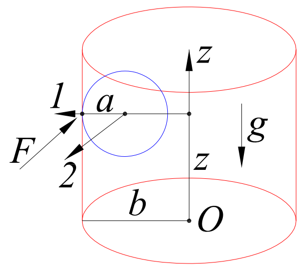
$$x = A_+ \cos(\alpha_+ t + \beta_+) + A_- \cos(\alpha_- t + \beta_-), \quad y = A_+ \sin(\alpha_+ t + \beta_+) + A_- \sin(\alpha_- t + \beta_-), \tag{84}$$

which describe an epicyclic curve.

He also considered a velocity-dependent friction somehow acting only on the center of the sphere, for which the mathematics is analytically tractable and implies that one of the oscillations, with angular frequency α_+ or α_-, is exponentially damped, while the other grows exponentially until the spinning sphere flies off the fixed one.

2. **The Golfer's Nemesis.**

This problem is discussed in §*421, p. 357 of* E.A. Milne, *Vectorial Mechanics* (Metheun; Interscience Publishers, 1948),
http://kirkmcd.princeton.edu/examples/mechanics/milne_mechanics.pdf



We consider a sphere, of mass m and radius a with moment of inertia I about its center, that rolls without slipping on a fixed, vertical cylinder of radius $b > a$. We use a set of principal axes (but not body axes) about the center of the sphere of radius a, where $\hat{\mathbf{1}}$ points outward along the horizontal line from the center of the spheres to the point of contact with the cylinder. Axis $\hat{\mathbf{3}}$ is vertical (parallel to $\hat{\mathbf{z}}$), and axis $\hat{\mathbf{2}} = \hat{\mathbf{z}} \times \hat{\mathbf{1}}$ is also horizontal).

The center of the sphere of radius a is at position $\mathbf{r} = (b-a)\,\hat{\mathbf{1}} + z\,\hat{\mathbf{z}}$ with respect to the origin at the bottom center of the cylinder. Then, the velocity of the center of the sphere of radius a is,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (b-a)\frac{d\hat{\mathbf{1}}}{dt} + \dot{z}\,\hat{\mathbf{z}}\,. \tag{85}$$

The (nonholonomic) constraint of rolling without slipping is that the point of contact of the sphere of radius with the cylinder is instantaneously at rest in the lab frame,

$$\mathbf{v}_{\rm contact} = 0 = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{a} = (b-a)\frac{d\hat{\mathbf{1}}}{dt} + \dot{z}\,\hat{\mathbf{z}} + a\boldsymbol{\omega} \times \hat{\mathbf{1}}, \tag{86}$$

where $\boldsymbol{\omega}$ is the total angular velocity of the sphere radius a in the lab frame, and $\mathbf{a} = a\,\hat{\mathbf{1}}$ is the vector from the center of the sphere of radius a to the point of contact.

The force and torque equations of motion for (the center of) the sphere are,

$$m\frac{d\mathbf{v}}{dt} = m(b-a)\frac{d^2\hat{\mathbf{1}}}{dt^2} + m\ddot{z}\,\hat{\mathbf{z}} = \mathbf{F} - mg\,\hat{\mathbf{z}}, \qquad \mathbf{F} = m(b-a)\frac{d^2\hat{\mathbf{1}}}{dt^2} + m(g+\ddot{z})\,\hat{\mathbf{z}}, \tag{87}$$

$$\frac{d\mathbf{L}}{dt} = I\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\tau} = \mathbf{a} \times \mathbf{F} = ma(b-a)\,\hat{\mathbf{1}} \times \frac{d^2\hat{\mathbf{1}}}{dt^2} - m(g+\ddot{z})a\,\hat{\mathbf{2}}, \tag{88}$$

where I is the moment of inertia of the sphere about its center.

We define $\boldsymbol{\Omega} = \Omega\,\hat{\mathbf{z}}$ as the angular velocity of the center of the sphere (and also of the point of contact, as well as of the triad $\hat{\mathbf{1}}$-$\hat{\mathbf{2}}$-$\hat{\mathbf{3}}$) about the vertical axis, such that,

$$\frac{d\hat{\mathbf{1}}}{dt} = \boldsymbol{\Omega} \times \mathbf{1} = \Omega\,\hat{\mathbf{2}}, \quad \frac{d^2\hat{\mathbf{1}}}{dt} = \dot{\Omega}\,\hat{\mathbf{2}} + \Omega\,\boldsymbol{\Omega} \times \hat{\mathbf{2}} = -\Omega^2\hat{\mathbf{1}} + \dot{\Omega}\hat{\mathbf{2}}, \quad \hat{\mathbf{1}} \times \frac{d^2\hat{\mathbf{1}}}{dt} = \dot{\Omega}\,\hat{\mathbf{z}}. \tag{89}$$

The velocity (85) of the center of the sphere can now be written as,

$$\mathbf{v} = -\Omega(b-a)\,\hat{\mathbf{2}} + \dot{z}\,\hat{\mathbf{z}}\,, \tag{90}$$

so the $\hat{\mathbf{2}}$-component of the total angular velocity $\boldsymbol{\omega}$ of the sphere about its center (and also that about the point of contact) is $v_z/a = \dot{z}/a$, and the $\hat{\mathbf{z}}$-component is $v_2/a = -(b-a)/a$. Thus,

$$\boldsymbol{\omega} = \omega_1\,\hat{\mathbf{1}} + \frac{\dot{z}}{a}\,\hat{\mathbf{2}} - \Omega\frac{b-a}{a}\,\hat{\mathbf{z}}, \qquad \frac{d\boldsymbol{\omega}}{dt} = \dot{\omega}_1\,\hat{\mathbf{1}} + \Omega\,\omega_1\,\hat{\mathbf{2}} + \frac{\ddot{z}}{a}\,\hat{\mathbf{2}} - \frac{\Omega\dot{z}}{a}\,\hat{\mathbf{1}} - \dot{\Omega}\frac{b-a}{a}\,\hat{\mathbf{z}}, \tag{91}$$

With these, the equation of motion (88) becomes,

$$I\left[\left(\dot{\omega}_1 - \frac{\Omega\dot{z}}{a}\right)\,\hat{\mathbf{1}} + \left(\Omega\,\omega_1 + \frac{\ddot{z}}{a}\right)\,\hat{\mathbf{2}} - \dot{\Omega}\frac{b-a}{a}\,\hat{\mathbf{z}}\right] = ma(b-a)\,\dot{\Omega}\,\hat{\mathbf{z}} - m(g+\ddot{z})a\,\hat{\mathbf{2}}, \tag{92}$$

The components of the equation of motion imply,

$$\hat{\mathbf{z}}: \qquad \dot{\Omega} = 0, \qquad \Omega = \text{constant}, \tag{93}$$

$$\hat{\mathbf{1}}: \qquad \dot{\omega}_1 = \frac{\Omega\dot{z}}{a}, \qquad \omega_1 = \frac{\Omega z}{a} + \omega_{10}, \tag{94}$$

$$\hat{\mathbf{2}}: \qquad \left(I + ma^2\right)\ddot{z} + I\,\Omega^2 z = -ma^2 g - I\,\Omega\,\omega_{10}. \tag{95}$$

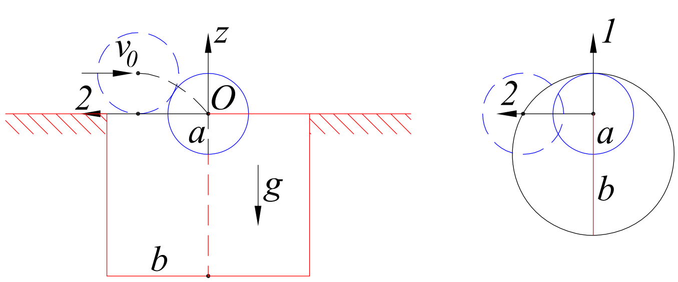
The center of the sphere executes simple harmonic motion in z,[12] and if at time $t = 0$, $z = 0$, $\dot{z} = \dot{z}_0$, $\omega_1 = \omega_{10}$, then,

$$z = \frac{ma^2 g + Ia\,\Omega\,\omega_{10}}{I\,\Omega^2}(\cos\alpha t - 1) + \frac{\dot{z}_0}{\alpha}\sin\alpha t, \qquad \text{where} \qquad \alpha = \Omega\sqrt{\frac{I}{I+ma^2}}\,. \tag{96}$$

We now consider under what conditions a golf ball could roll into a cup/vertical cylinder such that at time $t = 0$ the motion is described by eq. (96).

According to eqs. (90) and (91), the velocity $\mathbf{v}_0$ and the angular velocity $\boldsymbol{\omega}_0$ at this time must be,

$$\mathbf{v}_0 = -\Omega(b-a)\,\hat{\mathbf{2}} + \dot{z}_0\,\hat{\mathbf{z}}\,, \qquad \boldsymbol{\omega}_0 = \omega_{10}\,\hat{\mathbf{1}} + \frac{\dot{z}_0}{a}\,\hat{\mathbf{2}} - \Omega\frac{b-a}{a}\,\hat{\mathbf{z}}. \tag{97}$$



[12]This motion can be regarded as a nutation about steady motion with angular velocity Ω in a horizontal circle at $z = -(ma^2 g + Ia\,\Omega\,\omega_{10})/I\,\Omega^2$.

The figure above shows side and top views of the ball as it enters the cup, after rolling into it from the left while on the horizontal surface. At time $t = 0$, the ball has fallen through height a, so $\dot{z}_0 = -\sqrt{2ag}$. If the ball arrived at the top of the cup with horizontal velocity v_0 (in the $-\hat{\mathbf{2}}$ direction), then this is also the horizontal velocity when the center of the ball has fallen to $z = 0$, and so $\Omega = v_0/(b-a)$. The angular velocity of the ball did not change while it fell into the cup, so the angular velocity at the time of arrival was,

$$\boldsymbol{\omega}_{\rm arrival} = \boldsymbol{\omega}_0 = \omega_{10}\,\hat{\mathbf{1}} - \sqrt{\frac{2g}{a}}\,\hat{\mathbf{2}} - \frac{v_0}{a}\,\hat{\mathbf{z}}, \qquad \mathbf{v}_0 = -\Omega(b-a)\,\hat{\mathbf{2}}. \tag{98}$$

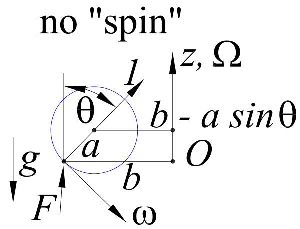
If the ball had been simply rolling without slipping prior to arrival at the cup, then $\omega_{10} = v_0/a$ and the $\hat{\mathbf{2}}$- and $\hat{\mathbf{z}}$-components of $\boldsymbol{\omega}_{\rm arrival}$ would be zero. Hence, only under special conditions of rolling with slipping at the moment of arrival at the cup could the ball roll into it and pop back out after following motion of the form)96.

For a golf ball of uniform mass density, $I = 2ma^2/5$, and $\alpha = \sqrt{2/7}\,\Omega = \Omega/1.87$. If the golf ball does pop out of the hole, it does so in somewhat less than one period of the vertical oscillation, *i.e.*, in less the 1.87 revolutions of the ball around the vertical axis of the cup.

3. **Off the Rim.**

We consider a sphere of radius a that rolls without slipping on a horizontal hoop of radius $b > a$.

Before treating the general motion, we consider the special case of steady motion with no "spin" about the line, $\hat{\mathbf{1}}$, between the point of contact of the sphere with the hoop and the center of the sphere.



In this case, the angle θ_0 between the vertical, $\hat{\mathbf{z}}$, and $\hat{\mathbf{1}}$ is constant, and the center of the sphere moves in a horizontal circle of radius $b - a\sin\theta_0$ with constant angular velocity Ω.

The rolling constraint is,

$$\mathbf{v}_{\rm contact} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{a} = 0, \tag{99}$$

where $\mathbf{v}$ is the velocity of the center of the sphere, $\boldsymbol{\omega}$ is its total angular velocity, and $\mathbf{a} = -a\,\mathbf{1}$ points from the center of the sphere to the point of contact.

In the case of no "spin" about $\hat{\mathbf{1}}$, the angular velocity $\boldsymbol{\omega}$ is perpendicular to $\hat{\mathbf{1}}$ and in the vertical plane that contains the center of the hoop and the point of contact, as shown in the figure above. Then, the rolling constraint (99) implies,

$$\Omega(b - a\sin\theta_0) = \omega a. \tag{100}$$

The torque equation of (steady) motion about the point of contact is,

$$\boldsymbol{\tau}_{\rm contact} = -\mathbf{a} \times m\mathbf{g} = -mag\,\hat{\mathbf{1}} \times \hat{\mathbf{z}} = mag\sin\theta_0\,\hat{\mathbf{2}}$$

$$= \frac{d\mathbf{L}_{\rm contact}}{dt} = I_{\rm contact}\frac{d\boldsymbol{\omega}}{dt} = \left(I + ma^2\right)\Omega\,\hat{\mathbf{z}} \times \boldsymbol{\omega} = \left(I + ma^2\right)\Omega\,\omega\,\cos\theta_0\,\hat{\mathbf{2}}, \tag{101}$$

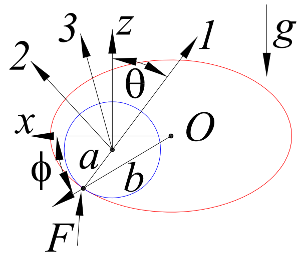
$$\Omega = \frac{mag\sin\theta_0}{(I + ma^2)\,\omega\,\cos\theta_0}, \qquad \Omega^2 = \frac{ma^2 g\tan\theta_0}{(I + ma^2)\,(b - a\sin\theta_0)}, \tag{102}$$

using eq. (100), and defining $\hat{\mathbf{2}} = \hat{\mathbf{z}} \times \hat{\mathbf{1}}/\sin\theta_0$, which is into the page in the figure above.

For a spherical shell of radius $a = 12$ cm, $I = 2ma^2/3$, with $b = 2a$ and $\theta_0 = 45^\circ$, the frequency of revolution of the sphere about the center of the hoop is $2\pi/\Omega = 2\pi\sqrt{5a(4 - \sqrt{2})/6g} \approx 0.6$ Hz.

Also, there is a formal equilibrium with $\theta_0 = 0 = \Omega$, at which the sphere is perched on a point on the rim. For "spin" $\omega_1 = 0$ this equilibrium is unstable, but we need to consider whether it might be stable for large enough ω_1.

Turning to the general case when θ and $\dot\phi = \Omega$ vary with time, we introduce the principal axes (not body axes) $\hat{\mathbf{1}}$,$\hat{\mathbf{2}}$,$\hat{\mathbf{3}}$, with origin at the center of the sphere. $\hat{\mathbf{1}}$ points from the point of contact with the hoop to the center of the sphere, $\hat{\mathbf{2}} = \hat{\mathbf{z}} \times \hat{\mathbf{1}}/\sin\theta$ is horizontal, and $\hat{\mathbf{3}} = \hat{\mathbf{1}} \times \hat{\mathbf{2}}$ is in the vertical plane containing the centers of the hoop and the sphere. Also, $\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{1}} + \sin\theta\,\hat{\mathbf{3}}$.



The velocity of the center of the sphere is,

$$\mathbf{v} = -\dot\phi(b - a\sin\theta)\,\hat{\mathbf{2}} - a\,\dot\theta\,\hat{\mathbf{3}}. \tag{103}$$

where $\Omega = \dot\phi$ is the angular velocity of the sphere about the center of the hoop.

From the rolling constraint (99) we have, recalling that $\mathbf{a} = -a\,\hat{\mathbf{1}}$,

$$\mathbf{1} \times (\boldsymbol{\omega} \times \mathbf{a}) = -a\,\boldsymbol{\omega} - \omega_1\,\mathbf{a} = -\hat{\mathbf{1}} \times \mathbf{v} = -a\,\dot\theta\,\hat{\mathbf{2}} + \dot\phi(b - a\sin\theta)\,\hat{\mathbf{3}}, \tag{104}$$

$$\boldsymbol{\omega} = \omega_1\,\hat{\mathbf{1}} + \dot\theta\,\hat{\mathbf{2}} - \dot\phi\frac{b - a\sin\theta}{a}\,\hat{\mathbf{3}}. \tag{105}$$

The angular velocity of the triad $\hat{\mathbf{1}}$,$\hat{\mathbf{2}}$,$\hat{\mathbf{3}}$ is,

$$\boldsymbol{\omega}_{123} = \dot\theta\,\hat{\mathbf{2}} + \dot\phi\,\hat{\mathbf{z}} = \dot\phi\cos\theta\,\hat{\mathbf{1}} + \dot\theta\,\hat{\mathbf{2}} + \dot\phi\sin\theta\,\hat{\mathbf{3}}. \tag{106}$$

The time rate of change of the principal axes is related by,

$$\frac{d\hat{\mathbf{i}}}{dt} = \boldsymbol{\omega}_{123} \times \hat{\mathbf{i}}, \tag{107}$$

$$\frac{d\hat{\mathbf{1}}}{dt} = (\dot\phi\cos\theta\,\hat{\mathbf{1}} + \dot\theta\,\hat{\mathbf{2}} + \dot\phi\sin\theta\,\hat{\mathbf{3}}) \times \hat{\mathbf{1}} = \dot\phi\sin\theta\,\hat{\mathbf{2}} - \dot\theta\,\hat{\mathbf{3}}, \tag{108}$$

$$\frac{d\hat{\mathbf{2}}}{dt} = (\dot\phi\cos\theta\,\hat{\mathbf{1}} + \dot\theta\,\hat{\mathbf{2}} + \dot\phi\sin\theta\,\hat{\mathbf{3}}) \times \hat{\mathbf{2}} = \dot\phi\sin\theta\,\hat{\mathbf{1}} + \dot\phi\cos\theta\,\hat{\mathbf{3}}, \tag{109}$$

$$\frac{d\hat{\mathbf{3}}}{dt} = (\dot\phi\cos\theta\,\hat{\mathbf{1}} + \dot\theta\,\hat{\mathbf{2}} + \dot\phi\sin\theta\,\hat{\mathbf{3}}) \times \hat{\mathbf{3}} = \dot\theta\,\hat{\mathbf{1}} - \dot\phi\cos\theta\,\hat{\mathbf{2}}. \tag{110}$$

The force and torque equations of motion of (the center of) the sphere of radius a are,

$$\begin{aligned}\mathbf{F} - mg\,\hat{\mathbf{z}} = m\frac{d\mathbf{v}}{dt} &= -m\,\ddot\phi(b - a\sin\theta)\,\hat{\mathbf{2}} + ma\,\dot\phi\,\dot\theta\cos\theta\,\hat{\mathbf{2}} - ma\,\ddot\theta\,\hat{\mathbf{3}} \\ &\quad -m\dot\phi(b - a\sin\theta)(\dot\phi\sin\theta\,\hat{\mathbf{1}} + \dot\phi\cos\theta\,\hat{\mathbf{3}}) - ma\dot\theta(\dot\theta\,\hat{\mathbf{1}} - \dot\phi\cos\theta\,\hat{\mathbf{2}}) \\ &= -m(\dot\phi^2(b - a\sin\theta)\sin\theta + a\,\dot\theta^2)\,\hat{\mathbf{1}} + m(2a\,\dot\phi\,\dot\theta\cos\theta - \ddot\phi(b - a\sin\theta))\,\hat{\mathbf{2}} \\ &\quad -m(a\,\ddot\theta + \dot\phi^2(b - a\sin\theta)\cos\theta)\,\hat{\mathbf{3}},\end{aligned} \tag{111}$$

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} = I\frac{d\boldsymbol{\omega}}{dt} &= I\dot{\omega}_1\,\hat{\mathbf{1}} + I\ddot{\theta}\,\hat{\mathbf{2}} + I\left(\dot{\phi}\,\dot{\theta}\cos\theta - \ddot{\phi}\frac{b - a\sin\theta}{a}\right)\hat{\mathbf{3}} + I\omega_1(\dot{\phi}\sin\theta\,\hat{\mathbf{2}} - \dot{\theta}\,\hat{\mathbf{3}}) \\
&\quad + I\dot{\theta}(\dot{\phi}\sin\theta\,\hat{\mathbf{1}} + \dot{\phi}\cos\theta\,\hat{\mathbf{3}}) - I\dot{\phi}\frac{b - a\sin\theta}{a}(\dot{\theta}\,\hat{\mathbf{1}} - \dot{\phi}\cos\theta\,\hat{\mathbf{2}}) \\
&= I\left(\dot{\omega}_1 - \dot{\phi}\,\dot{\theta}\frac{b - 2a\sin\theta}{a}\right)\hat{\mathbf{1}} + I\left(\ddot{\theta} + \omega_1\,\dot{\phi}\cos\theta + \dot{\phi}^2\frac{b - a\sin\theta}{a}\cos\theta\right)\hat{\mathbf{2}} \\
&\quad + I\left(2\dot{\phi}\,\dot{\theta}\cos\theta - \omega_1\,\dot{\theta} - \ddot{\phi}\frac{b - a\sin\theta}{a}\right)\hat{\mathbf{3}} \\
&= \boldsymbol{\tau} = \mathbf{a}\times\mathbf{F} \qquad (112) \\
&= -ma(2a\,\dot{\phi}\,\dot{\theta}\cos\theta - \ddot{\phi}(b - a\sin\theta))\,\hat{\mathbf{3}} - ma(a\,\ddot{\theta} + \dot{\phi}^2(b - a\sin\theta)\cos\theta)\,\hat{\mathbf{2}} + mag\sin\theta\,\hat{\mathbf{2}},
\end{aligned}$$

where I is the moment of inertia of the sphere about its center. The components of the equation of motion (112) are:

$$\hat{\mathbf{1}}: \qquad \dot{\omega}_1 = \dot{\phi}\,\dot{\theta}\frac{b - 2a\sin\theta}{a}, \tag{113}$$

$$\hat{\mathbf{2}}: \qquad \left(I + ma^2\right)\left(\ddot{\theta} + \dot{\phi}^2\frac{b - a\sin\theta}{a}\cos\theta\right) + I\,\omega_1\,\dot{\phi}\cos\theta = mag\sin\theta, \tag{114}$$

$$\hat{\mathbf{3}}: \qquad \left(I + ma^2\right)\left(2\dot{\phi}\,\dot{\theta}\cos\theta - \ddot{\phi}\frac{b - a\sin\theta}{a}\right) = I\,\omega_1\,\dot{\theta}. \tag{115}$$

For steady motion with $\theta = \theta_0 =$ constant and $\dot{\phi} = \Omega =$ constant, we have that $\omega_1 =$ constant from eq. (113), eq. (115) is trivial, and eq. (114) leads to,

$$\Omega^2 + \frac{I\Omega\,\omega_1}{I + ma^2} = \frac{ma^2 g\tan\theta_0}{(I + ma^2)\,(b - a\sin\theta_0)} \tag{116}$$

which reduces to eq. (102) for the special case of no “spin”, *i.e.*, $\omega_1 = 0$.

The coupled equations of motion (113)-(115) are intricate, and we limit further discussion to two special cases: either the “spin” ω_1 is negligible, or the equilibrium is with $\theta_0 = 0 = \Omega$.

For ω_1 negligible, we consider possible nutations of the form,

$$\theta = \theta_0 + \epsilon\sin\alpha t, \qquad \dot{\phi} = \Omega + \delta\sin\alpha t, \tag{117}$$

$$\sin\theta \approx \sin\theta_0 + \epsilon\cos\theta_0\sin\alpha t, \qquad \cos\theta \approx \cos\theta_0 - \epsilon\sin\theta_0\sin\alpha t, \tag{118}$$

for small constants ϵ and δ. Then, to first order in ϵ and δ, eq. (115) becomes,

$$2\Omega\alpha\,\epsilon\cos\alpha t\cos\theta_0 \approx \alpha\,\delta\cos\alpha t\frac{b - a\sin\theta_0}{a}, \qquad \delta \approx 2\,\epsilon\,\Omega\frac{a}{b - a\sin\theta_0}\cos\theta_0. \tag{119}$$

and eq. (114) becomes,

$$mag(\sin\theta_0 + \epsilon\cos\theta_0\sin\alpha t) \approx -\alpha^2\,\epsilon\left(I + ma^2\right)\sin\alpha t \tag{120}$$
$$+\left(I + ma^2\right)(\Omega^2 + 2\Omega\,\delta\sin\alpha t)\left(\frac{b}{a} - \sin\theta_0 - \epsilon\cos\theta_0\sin\alpha t\right)(\cos\theta_0 - \epsilon\sin\theta_0\sin\alpha t),$$

$$\epsilon\, m\, a\, g\cos\theta_0 \approx -\alpha^2\,\epsilon\left(I + ma^2\right) - \epsilon\left(I + ma^2\right)\Omega^2\left[\frac{b - a\sin\theta_0}{a}\sin\theta_0 + \cos^2\theta_0\right]$$
$$+2\Omega\,\delta\left(I + ma^2\right)\frac{b - a\sin\theta_0}{a}\cos\theta_0, \tag{121}$$

$$\alpha^2 \approx -\frac{m\,a\,g\cos\theta_0}{I + ma^2} - \Omega^2\left[\frac{b - a\sin\theta_0}{a}\sin\theta_0 + \cos^2\theta_0\right] + 4\Omega^2\cos^2\theta_0. \tag{122}$$

For this case we also have Ω^2 given by eq. (102), so,

$$\alpha^2 \approx -\frac{m\,a\,g\cos\theta_0}{I + ma^2} - \frac{mag\sin^2\theta_0}{(I + ma^2)\cos\theta_0} + 3\frac{mag\sin\theta_0\cos\theta_0}{(I + ma^2)\,(b/a - \sin\theta_0)}$$
$$= \frac{mag}{(I + ma^2)\,(b/a - \sin\theta_0)\cos\theta_0}[3\sin\theta_0\cos^2\theta_0 - \sin^2\theta_0 - \cos^2\theta_0(b/a - \sin\theta_0)]. \tag{123}$$

A numerical calculation (https://kirkmcd.princeton.edu/examples/rim.xlsx) indicates that $\alpha^2 < 0$ for any angle θ_0 when $b/a > 1.88$.[13] In regulation basketball, b/a is very close to 2, so $\alpha^2 < 0$ for any θ_0, and the equilibrium with $\omega_1 = 0$ is unstable. That is, if a basketball starts to roll around the hoop, it quickly falls in or out.

We now turn to the equilibrium of a sphere whose center is at rest directly above some point on the hoop, with the sphere spinning about the vertical.

We consider possible, small nutations about this equilibrium as in eqs. (117)-(118), but here, $\theta_0 = 0 = \Omega$. Then, the right side of eq. (113) is of second order, so in the first approximation ω_1 is constant. Equation (115) now implies that,

$$-\,\delta\left(I + ma^2\right)\frac{b}{a}\cos\alpha t = \epsilon I\,\omega_1\,\cos\alpha t, \qquad \delta = -\epsilon\frac{I}{I + ma^2}\frac{a\omega_1}{b}\,, \tag{124}$$

and eq. (114) leads to,

$$\epsilon\, mag\sin\alpha t \approx -\alpha^2\,\epsilon\left(I + ma^2\right)\sin\alpha t + \delta\, I\,\omega_1\sin\alpha t,$$
$$mag \approx -\alpha^2\,\left(I + ma^2\right) - \frac{I}{I + ma^2}\frac{a\omega_1}{b}I\,\omega_1, \tag{125}$$
$$\alpha^2 \approx \left(\frac{I}{I + ma^2}\right)^2\omega_1^2\frac{a}{b} - \frac{m\,a\,g}{I + ma^2}\,. \tag{126}$$

This equilibrium stable for,

$$\omega_1 > \frac{I + ma^2}{I}\sqrt{\frac{gb}{a^2}\frac{ma^2}{I + ma^2}}\,. \tag{127}$$

[13]When stability is possible, it is most stable angle for $\theta_0 \approx 42°$.

For a basketball of radius $a = 12$ cm and a hoop with $b/a = 2$, the minimum "spin" ω_1 for stability of this equilibrium is only 2 Hz.[14]

In the limit of $b \to \infty$, the hoop becomes a long, straight wire for small ϕ, say along the x direction. Then, the horizontal vector $\hat{\mathbf{2}}$ is $\hat{\mathbf{x}}$, and the quantity $\phi(b - a\sin\theta \approx \phi\, b$ takes on the significance of the position x of the center of the sphere along the wire (for small ϕ). Furthermore, $\dot\phi(b - a\sin\theta) \to \dot x$ and $\ddot\phi(b - a\sin\theta) \to \ddot x$. Then, the component equations of motion (113)-(115) become,

$$\hat{\mathbf{1}}: \qquad a\,\dot\omega_1 = \dot\theta\,\dot x, \tag{128}$$

$$\hat{\mathbf{2}}: \qquad \left(I + ma^2\right)\ddot\theta = mag\sin\theta, \tag{129}$$

$$\hat{\mathbf{3}}: \qquad -\left(I + ma^2\right)\ddot x = aI\,\omega_1\,\dot\theta. \tag{130}$$

These are the equations of motion found on p. 214 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l20.pdf, but with $x \to -x$. *See also §424, p. 360 of Milne's* Vectorial Mechanics, *especially eqs. (7)-(9).*

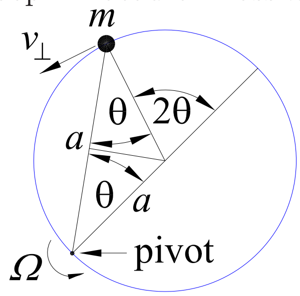
In particular, eq. (129) indicates that once θ is nonzero, further motion only increases θ (until the sphere loses contact with the wire).

A finite radius of curvature b of the (horizontal) wire leads to more intricate 2- and 3-components of the equations of motion, i.e., eqs. (114)-(115), which permit gyroscopic stabilization of the sphere for large enough "spin" ω_1 (at some values of θ_0).

[14] Gyroscopic stability of a basketball on a curved hoop occurs for smaller ω_1 than when balancing it on your finger (Prob. 1 above).

4. *This is Prob. 4.16, p. 62 of* D.F. Lawden, *Analytical Mechanics* (Allen & Unwin, 1972), http://kirkmcd.princeton.edu/examples/mechanics/lawden_72.pdf

A circular hoop of radius a rotates constant angular velocity Ω in a horizontal plane about a fixed point on the hoop. A bead of mass m silds freely on the hoop.



(a) The potential energy V of the bead can be taken as 0, while its kinetic energy is,

$$T = \frac{mv^2}{2} = \frac{m}{2}\left[\dot{r}^2 + r^2\left(\Omega + \dot{\theta}\right)^2\right], \tag{131}$$

$$r = 2a\cos\theta, \qquad \dot{r} = -2a\,\dot{\theta}\sin\theta, \tag{132}$$

$$T = \frac{m}{2}\left[4a^2\,\dot{\theta}^2\sin^2\theta + 4a^2\cos^2\theta\left(\dot{\theta}^2 + 2\,\dot{\theta}\,\Omega + \Omega^2\right)\right]$$
$$= 2ma^2\left(\dot{\theta}^2 + 2\,\dot{\theta}\,\Omega\cos^2\theta + \Omega^2\cos\theta^2\right). \tag{133}$$

The equation of motion via Lagrange's method is, with $\mathcal{L} = T - V = T$,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = 4ma\,\ddot{\theta} - 8ma\,\dot{\theta}\,\Omega\sin\theta\cos\theta$$
$$= \frac{\partial\mathcal{L}}{\partial\theta} = -8ma\,\dot{\theta}\,\Omega\sin\theta\cos\theta - 4ma\,\Omega^2\sin\theta\cos\theta, \tag{134}$$
$$\ddot{\theta} = -\Omega^2\sin\theta\cos\theta. \tag{135}$$

Steady motion exists for $\theta_0 = 0$, and small oscillations about this equilibrium have angular velocity $\omega = \Omega$.

(b) The Hamiltonian for this system is,

$$H = \dot{\theta}\,p_\theta - \mathcal{L}, \tag{136}$$

where,

$$p_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = 4ma^2\,\dot{\theta} + 4ma^2\,\Omega\cos^2\theta, \qquad \dot{\theta} = \frac{p_\theta}{4ma^2} - \Omega\cos^2\theta. \tag{137}$$

Then,

$$H = 4ma\left(\dot{\theta}^2 + \dot{\theta}\,\Omega\cos^2\theta\right) - 2ma^2\left(\dot{\theta}^2 + 2\,\dot{\theta}\,\Omega\cos^2\theta + \Omega^2\cos^2\theta\right) \tag{138}$$
$$= 2ma^2\left(\dot{\theta}^2 - \Omega^2\cos\theta^2\right) = 2ma^2\left(\frac{p_\theta^2}{(4ma^2)^2} - \frac{2p_\theta\,\Omega\cos^2\theta}{4ma^2} + \Omega^2\cos^4\theta - \Omega^2\cos^2\theta\right)$$
$$= \frac{p_\theta^2}{8ma^2} - p_\theta\,\Omega\cos^2\theta - 2ma^2\,\Omega^2\sin^2\theta\cos^2\theta = \frac{p_\theta^2}{8ma^2} - p_\theta\,\Omega\cos^2\theta - \frac{ma^2}{2}\,\Omega^2\sin^2 2\theta.$$

Hamilton's equations are,

$$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{4ma^2} - \Omega\cos^2\theta, \qquad \Rightarrow \qquad p_\theta = 4ma^2\,\dot\theta + 4ma^2\,\Omega\cos^2\theta, \tag{139}$$

and,

$$\begin{aligned}\dot p_\theta &= -\frac{\partial H}{\partial \theta} = -p_\theta\,\Omega\sin 2\theta + ma^2\,\Omega^2\sin 4\theta \\ &= -4ma^2\,\dot\theta\,\Omega\sin 2\theta - 4ma^2\,\Omega^2\sin 2\theta\cos^2\theta \\ &+4ma^2\,\Omega^2\sin 2\theta\cos 2\theta - 4ma^2\,\Omega^2\sin\theta\cos\theta \\ &= -4ma^2\,\dot\theta\,\Omega\sin 2\theta - 4ma^2\,\Omega^2\sin\theta\cos\theta,\end{aligned} \tag{140}$$

using Dwight 403.04, from which we can obtain,

$$\ddot\theta = \frac{\dot p_\theta}{4ma^2} + \dot\theta\,\Omega\sin 2\theta = -\Omega^2\sin\theta\cos\theta, \tag{141}$$

as in eq. (135), but rather more laboriously.

(c) The (unknown) constraint force has a component tangential to the hoop, so $\mathbf{F} = m\mathbf{a}$ is difficult to apply. We avoid use of the constraint force by considering a torque analysis about the center of the hoop in the rotating frame of the hoop. The angular momentum about this point is,[15]

$$L = mv_\perp = 2ma\dot\theta, \tag{142}$$

where the subscript $\perp$ indicates the component perpendicular to the radius from the center of the hoop to the mass m. The torque about the center of the hoop is due to the centrifugal force $F_C = m\,\Omega^2 r = 2ma\,\Omega^2\cos\theta$,[16]

$$\tau = -aF_{C,\perp} = -aF_C\sin\theta = -2ma\,\Omega^2\cos\theta\sin\theta. \tag{143}$$

Hence the torque equation of motion in the rotating frame, $\tau = dL/dt$, implies,

$$\ddot\theta = -\Omega^2\sin\theta\cos\theta. \tag{144}$$

as found in parts (a) and (b).

[15] While $v_\perp = a(2\dot\theta)$ in the rotating frame follows from the figure on the previous page, we also note that it can be deduced from the $r-$ and $\theta-$components of the velocity in the rotating frame, recalling eq. (132), $v_\perp = -\dot r\sin\theta + r\,\dot\theta\cos\theta = 2a\,\dot\theta(\sin^2\theta + \cos^2\theta) = 2a\dot\theta$.

[16] The constraint force, and the Coriolis force in the rotating frame, $2\mathbf{\Omega}\times\mathbf{v}_\perp$, are both along the radius from the center of the hoop to the mass, and so exert no torque about the latter.

5. **The Piano.**

A piano wire is struck by a sharp blow from a hammer, and a fairly pure note is produced.

Following Helmholtz, we suppose the force of the hammer blow can be described as,

$$F(x,t) = \begin{cases} F\,\delta(x-b)\sin\frac{2\pi t}{T} & (0 < t < T/2), \\ 0 & (\text{otherwise}). \end{cases} \tag{145}$$

That is, the force goes through one half period of s sinusoidal oscillation.

The force is applied at distance b from one end of a wire of length l and mass density ρ per unit length, which is fixed at both end and subject to a tension that makes the transverse wave velocity equal to c.

The transverse displacement $s(x,t)$ of the string can be written as a sum of spatial modes, $\sin(n\pi x/l)$, whose time dependence $\phi_n(t)$ is to be determined,

$$s(x,t) = \sum_n \phi_n(t) \sin\frac{n\pi x}{l}\,. \tag{146}$$

The equation of motion of the string is,

$$\rho\ddot{s} = c^2\rho s'' + F(x,t), \tag{147}$$

$$\sum_n \left(\ddot{\phi}_n + \frac{n^2\pi^2c^2}{l^2}\phi_n\right)\sin\frac{n\pi x}{l} = \frac{1}{\rho}F(x,t) = \frac{1}{\rho}\sum_n F_n(t)\sin\frac{n\pi x}{l}\,, \tag{148}$$

where the Fourier coefficients $F_n(t)$ are related by,

$$\int_0^l dx\,\sum_n F_n(t)\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l} = \frac{l}{2}F_m(t)$$

$$= \int_0^l dx\,F(x,t)\sin\frac{m\pi x}{l} = \begin{cases} F\,\sin\frac{2\pi t}{T}\sin\frac{m\pi b}{l} & (0 < t < T/2), \\ 0 & (\text{otherwise}). \end{cases} \tag{149}$$

Hence, the coefficient $\phi_n(t)$ obeys the differential equation of a forced, undamped oscillator,

$$\ddot{\phi}_n + \frac{n^2\pi^2c^2}{l^2}\phi_n = \frac{2F}{\rho l}\begin{cases} \sin\frac{2\pi t}{T}\sin\frac{n\pi b}{l} & (0 < t < T/2), \\ 0 & (\text{otherwise}). \end{cases} \tag{150}$$

Recalling the method of Green, discussed on p. 145 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l13.pdf, we have for ϕ_n at times $t > T/2$, noting that $\omega_1 = n\pi c/l$,

$$\phi_n(t > T/2) = \frac{l}{n\pi c}\frac{2F}{\rho l}\int_0^{T/2} dt' \, \sin\frac{2\pi t}{T}\sin\frac{n\pi b}{l}\sin\frac{n\pi c}{l}(t - t')$$
$$= \frac{2F}{n\pi c\rho}\sin\frac{n\pi b}{l}\int_0^{T/2} dt' \, \frac{1}{2}\left\{\cos\left[\left(\frac{2\pi}{T} + \frac{n\pi c}{l}\right)t' - \frac{n\pi}{l}t\right] - \cos\left[\left(\frac{2\pi}{T} - \frac{n\pi c}{l}\right)t' + \frac{n\pi}{l}t\right]\right\}$$
$$= \frac{F}{n\pi c\rho}\sin\frac{n\pi b}{l}\left\{\frac{\sin\left[\left(\frac{2\pi}{T} + \frac{n\pi c}{l}\right)\frac{T}{2} - \frac{n\pi}{l}t\right] + \sin\frac{n\pi ct}{l}}{\frac{2\pi}{T} + \frac{n\pi c}{l}} - \frac{\sin\left[\left(\frac{2\pi}{T} - \frac{n\pi c}{l}\right)\frac{T}{2} + \frac{n\pi}{l}t\right] - \sin\frac{n\pi ct}{l}}{\frac{2\pi}{T} - \frac{n\pi c}{l}}\right\}$$
$$= \frac{F}{n\pi c\rho}\sin\frac{n\pi b}{l}\left\{\frac{\sin\left[\pi - \frac{n\pi c}{l}\left(t - \frac{T}{2}\right)\right] + \sin\frac{n\pi ct}{l}}{\frac{2\pi}{T} + \frac{n\pi c}{l}} - \frac{\sin\left[\pi + \frac{n\pi c}{l}\left(t - \frac{T}{2}\right)\right] - \sin\frac{n\pi ct}{l}}{\frac{2\pi}{T} - \frac{n\pi c}{l}}\right\}$$
$$= \frac{F}{n\pi c\rho}\sin\frac{n\pi b}{l}\left\{\frac{\sin\frac{n\pi c}{l}\left(t - \frac{T}{2}\right) + \sin\frac{n\pi ct}{l}}{\frac{2\pi}{T} + \frac{n\pi c}{l}} + \frac{\sin\frac{n\pi c}{l}\left(t - \frac{T}{2}\right) + \sin\frac{n\pi ct}{l}}{\frac{2\pi}{T} - \frac{n\pi c}{l}}\right\}$$
$$= \frac{2F}{n\pi c\rho}\sin\frac{n\pi b}{l}\frac{\frac{4\pi}{T}\cos\frac{n\pi cT}{4l}\sin\frac{n\pi ct}{l}\left(t - \frac{T}{4}\right)}{\left(\frac{2\pi}{T}\right)^2 - \left(\frac{n\pi c}{l}\right)^2}. \tag{151}$$

The displacement is now given by,

$$s(x,t) = \frac{2FT}{\pi^2 c\,\rho}\sum_n \frac{1}{n(1 - (\frac{ncT}{2l})^2)}\sin\frac{n\pi b}{l}\cos\frac{n\pi cT}{4l}\sin\frac{n\pi x}{l}\sin\frac{n\pi c(t - T/4)}{l}. \tag{152}$$

If we take $b = l/2$, the midpoint, and $T = 2l/c$, the fundamental period, then,

$$s(x,t) = \frac{Fl}{\pi^2 T}\sum_n \frac{\sin n\pi}{n(1 - n^2)}\sin n\pi x \, \sin\frac{n\pi c(t - T/4)}{l}, \tag{153}$$

so all harmonics vanish except $n = 1$, since $\lim_{n\to 1}\frac{\sin n\pi}{1-n^2} = \frac{\pi\cos n\pi}{-2n} = \frac{\pi}{2}$.

Even if $T = 2l/c$ can't be achieved exactly in practice, the series converges quickly as the terms go as $1/n^3$ for large n.

In contemporary pianos, $b = l/8$, in which case (for $T = 2l/c$),

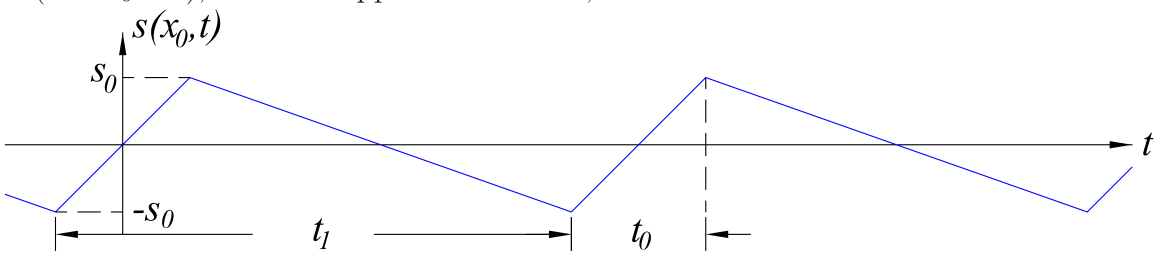
$$s(x,t) = \frac{2Fl}{\pi^2 T}\sum_n \frac{1}{n(1 - n^2)}\sin\frac{n\pi}{8}\cos\frac{n\pi}{2}\sin n\pi x \, \sin\frac{n\pi c(t - T/4)}{l}, \tag{154}$$

which includes harmonics $n = 1$, 2, 4, 6, 10, ...

Many harpsichords are built with $b = l/2$, which gives them a purer tone, although perhaps less interesting than that of a piano.

6. **The Violin.**

Following Helmholtz, we suppose the action of the bow of a violin is to force the string of length l into a transverse vibration at x_0, the point of application of the bow $(0 < x_0 < l)$, with the approximate form,



which is periodic with the period of the fundamental, free oscillation, $t_1 = 2l/c$, where c is the velocity of transverse waves on the stretched string. That is,

$$s(x_0, t) = 2s_0 \begin{cases} \frac{t}{t_0} & (0 < t < t_0/2), \\ -\frac{t-t_1/2}{t_1-t_0} & (t_0/2 < t < t_1 - t_0/2), \\ \frac{t-t_1}{t_0} & (t_1 - t_0/2 < t < t_1). \end{cases} \tag{155}$$

The rising motion occupies time $t_0 < t_1$ related by $x_0/l = t_0/t_1$.

As this waveform begins and ends each period (of duration t_1) with $s = 0$, its Fourier analysis has the form,

$$s(x_0, t) = \sum_n A_n \sin \frac{2n\pi t}{t_1}, \tag{156}$$

where the Fourier coefficients are related by,

$$A_n = \frac{2}{t_1} \int_0^{t_1} dt\, s(x_0, t) \sin \frac{2n\pi t}{t_1}$$

$$= \frac{4s_0}{t_0 t_1} \int_0^{t_0/2} dt\, t \sin \frac{2n\pi t}{t_1} - \frac{4s_0}{t_1(t_1 - t_0)} \int_{t_0/2}^{t_1 - t_0/2} dt \left(t - \frac{t_1}{2}\right) \sin \frac{2n\pi t}{t_1}$$

$$+ \frac{4s_0}{t_0 t_1} \int_{t_1 - t_0/2}^{t_1} dt\ (t - t_1) \sin \frac{2n\pi t}{t_1} = \frac{4s_0}{t_0 t_1} \left(\frac{t_1}{2n\pi}\right)^2 \left[\sin \frac{n\pi t_0}{t_1} - \frac{2n\pi}{t_1} \frac{t_0}{2} \cos \frac{n\pi t_0}{t_1}\right]$$

$$- \frac{4s_0}{t_1(t_1 - t_0)} \left(\frac{t_1}{2n\pi}\right)^2 \left\{\sin \frac{2n\pi}{t_1} \left(t_1 - \frac{t_0}{2}\right) - \sin \frac{2n\pi}{t_1} \frac{t_0}{2}\right.$$

$$\left. - \frac{2n\pi}{t_1} \left[\left(t_1 - \frac{t_0}{2}\right) \cos \frac{2n\pi}{t_1} \left(t_1 - \frac{t_0}{2}\right) - \frac{t_0}{2} \cos \frac{2n\pi}{t_1} \frac{t_0}{2}\right]\right\}$$

$$- \frac{2s_0}{(t_1 - t_0)} \frac{t_1}{2n\pi} \left[\cos \frac{2n\pi}{t_1} \left(t_1 - \frac{t_0}{2}\right) - \cos \frac{2n\pi}{t_1} \frac{t_0}{2}\right]$$

$$+ \frac{4s_0}{t_0 t_1} \left(\frac{t_1}{2n\pi}\right)^2 \left\{\sin 2n\pi - \sin \frac{2n\pi}{t_1} \left(t_1 - \frac{t_0}{2}\right)\right.$$

$$\left. - 2n\pi \cos 2n\pi + \frac{2n\pi}{t_1} \left(t_1 - \frac{t_0}{2}\right) \cos \frac{2n\pi}{t_1} \left(t_1 - \frac{t_0}{2}\right)\right\}$$

$$+ \frac{4s_0}{t_0} \frac{t_1}{2n\pi} \left[\cos 2n\pi - \cos \frac{2n\pi}{t_1} \left(t_1 - \frac{t_0}{2}\right)\right]$$

$$= \frac{4s_0}{t_0 t_1}\left(\frac{t_1}{2n\pi}\right)^2\left[\sin\frac{n\pi t_0}{t_1} - \frac{2n\pi}{t_1}\frac{t_0}{2}\cos\frac{n\pi t_0}{t_1}\right]$$
$$-\frac{4s_0}{t_1(t_1-t_0)}\left(\frac{t_1}{2n\pi}\right)^2\left\{-\sin\frac{2n\pi}{t_1}\frac{t_0}{2} - \sin\frac{2n\pi}{t_1}\frac{t_0}{2}\right.$$
$$\left.-\frac{2n\pi}{t_1}\left[\left(t_1 - \frac{t_0}{2}\right)\cos\frac{2n\pi}{t_1}\frac{t_0}{2} - \frac{t_0}{2}\cos\frac{2n\pi}{t_1}\frac{t_0}{2}\right]\right\}$$
$$-\frac{2s_0}{(t_1-t_0)}\frac{t_1}{2n\pi}\left[\cos\frac{2n\pi}{t_1}\frac{t_0}{2} - \cos\frac{2n\pi}{t_1}\frac{t_0}{2}\right]$$
$$+\frac{4s_0}{t_0 t_1}\left(\frac{t_1}{2n\pi}\right)^2\left\{\sin\frac{2n\pi}{t_1}\frac{t_0}{2} - 2n\pi + \frac{2n\pi}{t_1}\left(t_1 - \frac{t_0}{2}\right)\cos\frac{2n\pi}{t_1}\frac{t_0}{2}\right\}$$
$$+\frac{4s_0}{t_0}\frac{t_1}{2n\pi}\left(1 - \cos\frac{2n\pi}{t_1}\frac{t_0}{2}\right)$$
$$= \frac{4s_0}{t_1}\left(\frac{t_1}{2n\pi}\right)^2\sin\frac{n\pi t_0}{t_1}\left[\frac{1}{t_0} + \frac{2}{t_1-t_0} + \frac{1}{t_0}\right]$$
$$+2s_0\frac{t_1}{2n\pi}\cos\frac{n\pi t_0}{t_1}\left[-\frac{1}{t_1} + \frac{2}{t_1} + \frac{2}{t_0 t_1}\left(t_1 - \frac{t_0}{2}\right) - \frac{2}{t_0}\right] + \frac{4s_0}{t_1}\frac{t_1}{2n\pi}\left(-\frac{2n\pi}{2n\pi} + 1\right) \qquad (157)$$
$$= \frac{8s_0}{t_1}\left(\frac{t_1}{2n\pi}\right)^2\sin\frac{n\pi t_0}{t_1}\frac{t_1}{t_1-t_0} = \frac{2s_0 t_1^2}{n^2\pi^2 t_0(t_1-t_0)}\sin\frac{n\pi t_0}{t_1} = \frac{2s_0 t_1^2}{n^2\pi^2 t_0(t_1-t_0)}\sin\frac{n\pi x_0}{l}\,.$$

Thus, the Fourier analysis in time of the motion of the point of contact is,

$$s(x_0,t) = \frac{2s_0 t_1^2}{\pi^2 t_0(t_1-t_0)}\sum_n \frac{1}{n^2}\sin\frac{n\pi x_0}{l}\sin\frac{2n\pi t}{t_1}\,. \qquad (158)$$

In general, we expect the motion of the entire string to the analyzable as,

$$s(x,t) = \sum_n \sin\frac{n\pi x}{l}\left(A_n\cos\frac{2n\pi t}{t_1} + B_n\sin\frac{2n\pi t}{t_1}\right), \qquad (159)$$

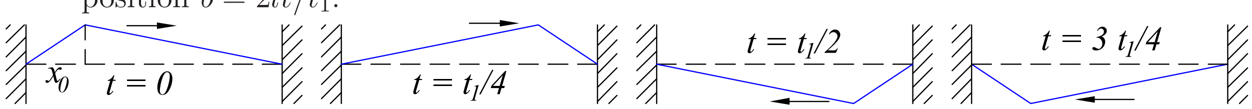
so it follows from eq. (15) that $A_n = 0$ and,

$$s(x,t) = \frac{2s_0 t_1^2}{\pi^2 t_0(t_1-t_0)}\sum_n \frac{1}{n^2}\sin\frac{n\pi x}{l}\sin\frac{2n\pi t}{t_1}\,. \qquad (160)$$

On p. 228 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l21.pdf, we saw that a string plucked at $x = b$ at time $t = 0$ has the Fourier analysis,

$$s(x,t) = \frac{2s_0 l^2}{\pi^2 b(l-b)}\sum_n \frac{1}{n^2}\sin\frac{n\pi x}{l}\sin\frac{n\pi b}{l}\,. \qquad (161)$$

Hence, at any time t the violin string looks like the initial form of a string plucked at position $b = 2lt/t_1$.

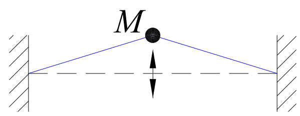


The crest of the wave motion moves along the string with velocity $c = 2l/t_1$. The "vibration" is better described as a traveling wave than a standing wave.

7. This problem is adapted from sec. 136, p. 204 of Lord Rayleigh *Theory of Sound*, 2[nd] ed. (Macmillan, 1894), http://kirkmcd.princeton.edu/examples/mechanics/rayleigh_theory_of_sound_1.pdf

A string of length l is fixed at both ends and stretched with tension T.

(a) A mass M is attached to the midpoint of the string.



Ignoring the mass of the string, the equation of motion of mass M, with transverse displacement s is,

$$M\ddot{s} \approx -2T\frac{s}{l/2} \qquad \ddot{s} = -\frac{4T}{lM}s \qquad s = s_0 \cos\Omega_0 t, \qquad \Omega_0 = 2\sqrt{\frac{T}{lM}}, \tag{162}$$

where T is the tension in the string.

(b) We next consider the string to have mass m, and suppose mass M is attached at distance $b < l$ from one end of the string.

We consider the system as consisting two strings, 1 on interval $0 < x < b$ and 2 on $b < x < l$, each with tension T. The equation of motion of mass M can now be written as,

$$M\ddot{s}_1(b) = M\ddot{s}_2(b) \approx T(s_2'(b) - s_1'(b)). \tag{163}$$

A normal mode of angular frequency Ω has the forms on strings 1 and 2 (fixed at $x = 0$ and l, and with wave velocity $c = \sqrt{T/\rho} = \sqrt{Tl/m}$),

$$s_1(x,t) = A_1 \sin\frac{\Omega x}{c}\cos\Omega t, \qquad s_2(x,t) = A_2 \sin\frac{\Omega(l-x)}{c}\cos\Omega t, \tag{164}$$

with the constraint that $s_1(b,t) = s_2(b,t)$, *i.e.*, $A_1 \sin(\Omega b/c) = A_2 \sin(\Omega(l-b)/c$. Hence, we can also write eq. (164) as,

$$s_1(x,t) = A\sin\frac{\Omega x}{c}\sin\frac{\Omega(l-b)}{c}\cos\Omega t, \quad s_2(x,t) = A\sin\frac{\Omega(l-x)}{c}\sin\frac{\Omega b}{c}\cos\Omega t. \tag{165}$$

Then, from the equation of motion (163) we have,

$$-\Omega^2 M\frac{\Omega b}{c}\sin\frac{\Omega(l-b)}{c} = -T\frac{\Omega}{c}\left(\cos\frac{\Omega(l-b)}{c}\sin\frac{\Omega b}{c} + \cos\frac{\Omega b}{c}\sin\frac{\Omega(l-b)}{c}\right), \tag{166}$$

$$\Omega \sin\frac{\Omega b}{c}\sin\frac{\Omega(l-b)}{c} = \frac{T}{Mc}\sin\frac{\Omega\, l}{c}. \tag{167}$$

(c) We now consider part (b) for the special case of $b = l/2$.

There are two classes of solutions:

i. Mass M does not move, so the point $x = b/2$ is a node of the standing wave functions, whose form is

$$s(x,t) = A\sin\frac{2n\pi x}{l}\cos\Omega t, \tag{168}$$

From the wave equation, $\ddot{s} = c^2 s''$, we have that,

$$\Omega = \frac{2n\pi c}{l}\,. \tag{169}$$

ii. For the modes where mass M moves, we have from eq. (167),

$$\Omega\,\sin^2\frac{\Omega l}{2c} = \frac{T}{Mc}\sin\frac{\Omega\,l}{c} = \frac{c\,m}{lM}2\sin\frac{\Omega l}{2c}\cos\frac{\Omega l}{2c}\,, \tag{170}$$

$$\frac{\Omega l}{2c}\tan\frac{\Omega l}{2c} = \frac{m}{M}\,, \tag{171}$$

recalling that the tension T is related by $c = \sqrt{Tl/m}$.

(d) If $M \ll m$, the lowest frequency obeys $\Omega l/2c = \pi/2 - \epsilon$ in eq. (171), which can be rewritten as,

$$\frac{M}{m}\frac{\Omega l}{2c}\sin\frac{\Omega l}{2c} = \cos\frac{\Omega l}{2c}\,, \qquad \frac{M}{m}\left(\frac{\pi}{2}-\epsilon\right)\cos\epsilon = \sin\epsilon, \qquad \frac{M}{m}\left(\frac{\pi}{2}-\epsilon\right) \approx \epsilon, \tag{172}$$

$$\epsilon \approx \frac{\frac{M}{m}\frac{\pi}{2}}{1+\frac{M}{m}} = \frac{\pi}{2}\frac{M}{M+m}\,, \qquad \frac{\Omega l}{2c} \approx \frac{\pi}{2}\left(1 - \frac{M}{m+M}\right) = \frac{\pi}{2}\frac{m}{m+M} \approx \frac{\pi}{2}\left(1-\frac{M}{m}\right), \tag{173}$$

$$\Omega \approx \frac{\pi c}{l}\left(1-\frac{M}{m}\right) = \Omega_1\left(1-\frac{M}{m}\right), \tag{174}$$

where $\Omega_1 = \pi c/l$ is the fundamental angular frequency when mass M is not present.

(e) If $m \ll M$, eq. (171) tells us that for the lowest-frequency mode,

$$\frac{m}{M} = \frac{\Omega l}{2c}\tan\frac{\Omega l}{2c} \approx \left(\frac{\Omega l}{2c}\right)^2, \qquad \Omega \approx \frac{2c}{l}\sqrt{\frac{m}{M}} = \frac{2}{l}\sqrt{\frac{Tl}{m}\frac{m}{M}} = 2\sqrt{\frac{T}{lM}} = \Omega_0, \tag{175}$$

Where Ω_0 is the angular frequency found in part (a) for M attached to the mid-point of a massless string.

We now seek a correction to eq. (175) of order m/M. For this, we approximate eq. (171) to 4$^{\text{th}}$ order in $\Omega l/2c$,

$$\frac{m}{M} \approx \frac{\Omega l}{2c}\frac{\frac{\Omega l}{2c} - \frac{1}{6}\left(\frac{\Omega l}{2c}\right)^3}{1-\frac{1}{2}\left(\frac{\Omega l}{2c}\right)^2} \approx \left(\frac{\Omega l}{2c}\right)^2\left(1-\frac{1}{6}\left(\frac{\Omega l}{2c}\right)^2\right)\left(1+\frac{1}{2}\left(\frac{\Omega l}{2c}\right)^2\right)$$

$$= \left(\frac{\Omega l}{2c}\right)^2\left(1+\frac{1}{3}\left(\frac{\Omega l}{2c}\right)^2\right), \tag{176}$$

$$\left(\frac{\Omega l}{2c}\right)^4 + 3\left(\frac{\Omega l}{2c}\right)^2 - 3\frac{m}{M} \approx 0, \tag{177}$$

$$\left(\frac{\Omega l}{2c}\right)^2 \approx \frac{-3 \pm 3\sqrt{1+\frac{4m}{3M}}}{2} \approx -\frac{3}{2} + \frac{3}{2}\left[1 + \frac{2m}{3M} - \frac{1}{8}\left(\frac{4m}{3M}\right)^2\right] = \frac{m}{M} - \frac{1}{3}\left(\frac{m}{M}\right)^2, \tag{178}$$

$$\frac{\Omega l}{2c} \approx \sqrt{\frac{m}{M} - \frac{1}{3}\left(\frac{m}{M}\right)^2} \approx \sqrt{\frac{m}{M}}\left(1 - \frac{1}{6}\frac{m}{M}\right), \tag{179}$$

$$\Omega \approx \frac{2c}{l}\sqrt{\frac{m}{M}}\left(1 - \frac{1}{6}\frac{m}{M}\right) = \Omega_0\left(1 - \frac{1}{6}\frac{m}{M}\right). \tag{180}$$

We can also proceed from the first form of eq. (179) to write,

$$\Omega \approx \frac{2c}{l}\sqrt{\frac{m}{M}\frac{1}{1+\frac{1}{3}\frac{m}{M}}} = 2\sqrt{\frac{c^2 m}{l^2(M+m/3)}} = 2\sqrt{\frac{T}{l(M+m/3)}}. \tag{181}$$

so that (for this mode) the mass of the string appears as a correction $m/3$ to mass M.

In practice, the vibration of a loaded string is not, in general, a single normal mode, so the result (181) is not what is observed. See, for example, http://kirkmcd.princeton.edu/examples/mechanics/sears_ajp_37_645_69.pdf.

This phenomenon can be related to renormalization-group techniques, as discussed in http://kirkmcd.princeton.edu/examples/mechanics/nunes_ajp_62_423_94.pdf.

8. A uniform bar of mass m has rest length l_0. One end is fixed and the other end is attached to a mass M.

   Longitudinal vibrations of such a bar were discussed on pp.233-234 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l21.pdf. The wave equation for longitudinal displacements $s(x,t)$ was found to be,

$$\ddot{s} = \frac{AY}{\rho} s'', \tag{182}$$

where A is the cross sectional area of the bar, Y is its Young's modulus, and $\rho = m/l_0$ is its linear (rest) mass density. It was also noted that the (longitudinal) spring constant of the bar is $k = AY/l_0$. That is, the wave equation (182) can also be written as,

$$\ddot{s} = \frac{kl_0^2}{m} s''. \tag{183}$$

For standing waves of angular frequency ω with longitudinal displacement $s = f(x)\, e^{i\omega t}$, we have $\ddot{s} = -\omega^2 s$ and $s'' = f'' s$, such that,

$$-\omega^2 f = \frac{kl_0^2}{m} f'' m \qquad f'' = -\frac{\omega^2 m}{kl_0^2} f, \tag{184}$$

$$f = A \sin \Omega x + B \cos \Omega x, \qquad \text{where} \qquad \Omega = \frac{\omega}{l_0}\sqrt{\frac{m}{k}}\,. \tag{185}$$

In the present problem, we consider the bar to extend over $0 < x < l_0$ when at rest, with the end at $x = 0$ fixed. Hence, $B = 0$ in eq. (185), and $f = A \sin \Omega x$.

At $x = l_0$, where mass M is attached, its equation of motion is, for the standing wave $s = A \sin \Omega x\, e^{i\omega t}$,

$$M\ddot{s}(l_0, t) = -\omega^2 M s(l_0, t) = F. \tag{186}$$

The spring force F on mass M is not simply $-ks(l_0, t)$ because the vibrating bar does not, in general, stretch uniformly. So, we consider a segment of the bar $l_0 - dx < x < l_0$ when at rest, where dx is so small that when the bar is vibrating, the stretch over this segment, $s(l_0, t) - s(l_0 - dx, t) = s'(l_0, t)\, dx$, is uniform.

The internal tension in this segment is the same as when the entire bar is uniformly stretched, by amount $(l_0/dx) s'(l_0, t)\, dx = l_0\, s'(l_0, t)$, and so the spring force in the segment $l_0 - dx < x < l_0$ next to mass M is $F = -kl_0\, s'(l_0, t)$.

The equation of motion (186) for mass M can now be written as,

$$M\ddot{s}(l_0, t) = -\omega^2 M A \sin(\Omega l_0)\, e^{i\omega t} = F = -kl_0\, s'(l_0, t) = -kl_0 A \Omega \cos(\Omega l_0)\, e^{i\omega t}, \tag{187}$$

$$\cot(\Omega l_0) = \frac{M\omega^2}{kl_0 \Omega} = \frac{M}{m} \Omega l_0. \tag{188}$$

For the lowest frequency Ω (which is very small compared to ω),

$$\frac{M}{m}\Omega l_0 = \cot(\Omega l_0) \approx \frac{1 - \frac{\Omega^2 l_0^2}{2}}{\Omega l_0 - \frac{\omega^3 l_0^3}{6}} \approx \frac{(1 - \frac{\Omega^2 l_0^2}{2})(1 + \frac{\omega^2 l_0^2}{6})}{\Omega l_0} \approx \frac{1 - \frac{\Omega^2 l_0^2}{3}}{\Omega l_0}, \tag{189}$$

$$\frac{M}{m}\Omega^2 l_0^2 \approx 1 - \frac{\Omega^2 l_0^2}{3}, \qquad \Omega l_0 \approx \sqrt{\frac{1}{\frac{M}{m} + \frac{1}{3}}}, \qquad \Omega = \frac{\omega}{l_0}\sqrt{\frac{m}{k}} \approx \frac{1}{l_0}\sqrt{\frac{1}{\frac{M}{m} + \frac{1}{3}}}. \tag{190}$$

Hence,

$$\omega \approx \sqrt{\frac{k}{M + m/3}}, \tag{191}$$

as found in Prob. 5, Set 1, http://kirkmcd.princeton.edu/examples/ph205set1.pdf.