Princeton University

Ph205

Mechanics

Problem Set 10

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(1988)

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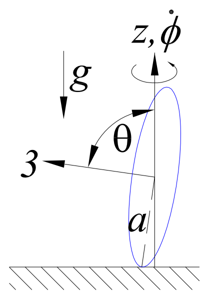
1. **Spinning Coin Revisited.**

It is possible to spin a coin on a horizontal table about a vertical diameter, with its center at rest. But, if the angular velocity becomes too low, the coin falls over and takes on the motion considered in Prob. 6, Set 9.

Consider a thin, uniform disk of mass m and radius a that spins without friction on a horizontal table (such that its center moves only vertically). Use Euler angles θ, ϕ and ψ, in the manner of Fig. 47, p. 110 of L.D. Landau and E.M. Lifshitz, *Mechanics*, $3^{\rm rd}$ ed. (Pergamon, 1976), http://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics.pdf, and Lagrange's method to analyze the motion.



(a) Show that the steady precession rate about the vertical is,

$$\dot\phi_{\rm steady} = \frac{\omega_3 \pm \sqrt{\omega_3^2 + 4g\cos^2\theta/a\sin\theta}}{\cos\theta}, \qquad (1)$$

where θ is the angle between the symmetry axis 3 of the disk to the vertical, and ω_3 is the angular velocity about that axis.

This relation becomes invalid at $\theta = 90^\circ$, when the coin is "on edge".

(b) Show that in this case the two possible classes of steady motion are,

- ω_3 arbitrary, $\dot\phi = 0, \Leftrightarrow$ rolling and slipping.
- $\omega_3 = 0$, $\dot\phi$ arbitrary, $\Leftrightarrow$ spinning on edge.

For the second case (spinning on edge), use a small-angle approximation, $\theta = \pi/2 + \epsilon$, to show that the motion is stable if $\dot\phi > 2\sqrt{g/a}$; otherwise the coin falls over.

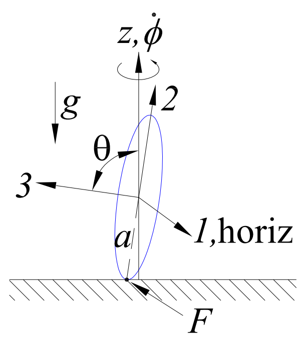
2. **Rolling Disk Revisited.**

Consider arbitrary motion of a disk that rolls without slipping on a horizontal plane. Experts will note that this example has 5 coordinates and 2 nonholonomic constraints. But, try it without resort to Lagrange multipliers.

The thin, uniform disk has mass m and radius a. Use a coordinate system that is similar to, but not quite the same as that of Euler:

- $\hat{\mathbf{z}}$ is vertical.
- Principal axis $\hat{\mathbf{1}}$ is always horizontal.
- Principal axis $\hat{\mathbf{2}}$ lies in a vertical plane that includes the center of the disk.
- Principal axis $\hat{\mathbf{3}} = \hat{\mathbf{1}} \times \hat{\mathbf{2}}$ is the symmetry axis of the disk. Axes $\hat{\mathbf{2}}$ and $\hat{\mathbf{3}}$ lie in the same vertical plane.

The axes $(1, 2, 3)$ are principal axes, but they are not body axes (that are fixed with respect to the rotating disk).



Also define,

- θ = angle between $\hat{\mathbf{2}}$ and $\hat{\mathbf{z}}$.
- $\dot{\phi}$ = angular velocity of the disk (and of the $(1, 2, 3)$ axes) about the vertical.
- $\mathbf{F}$ = the (as yet unknown) force on the disk at the point of contact with the horizontal surface.

The "elementary" equations of motion are,

$$\mathbf{F}_{\text{total}} = m\frac{d\mathbf{v}_{\text{cm}}}{dt}, \qquad \boldsymbol{\tau}_{\text{cm}} = \frac{d\mathbf{L}_{\text{cm}}}{dt}. \tag{2}$$

The constraint of rolling without slipping can be written in terms of velocities (as a time-dependent version of Chasles' theorem),

$$\mathbf{v}_{\text{contact}} = 0 = \mathbf{v}_{\text{cm}} + \boldsymbol{\omega} \times \mathbf{a}, \tag{3}$$

where,

- $\mathbf{a} = -a\,\hat{\mathbf{2}}$ = vector from the center of mass to the point of contact.
- $\boldsymbol{\omega}_{(1,2,3)} = -\dot{\theta}\,\hat{\mathbf{1}} + \dot{\phi}\,\hat{\mathbf{z}}$ = angular velocity of the axes $(1, 2, 3)$.
- $\boldsymbol{\omega} = \boldsymbol{\omega}_{(1,2,3)} + \dot{\psi}\,\hat{\mathbf{3}}$ = total angular velocity of the disk.
- $\dot{\psi}$ = (spin) angular velocity of the disk relative to the $(1, 2, 3)$ axes.

Then, $\omega_3 = \omega_{(1,2,3)_3} + \dot{\psi}$.

Eliminate $\mathbf{F}$ and $\mathbf{v}_{\rm cm}$ to arrive at the equation of motion,

$$\frac{ma^2}{4}\frac{d}{dt}\left(-\dot{\theta}\,\hat{\mathbf{1}} + \dot{\phi}\sin\theta\,\hat{\mathbf{2}} + 2\omega_3\,\hat{\mathbf{3}}\right) = mga\cos\theta\,\hat{\mathbf{1}} - ma^2\,\hat{\mathbf{2}} \times \frac{d}{dt}\left(\omega_3\,\hat{\mathbf{1}} + \dot{\theta}\,\hat{\mathbf{3}}\right). \tag{4}$$

Note that for axis $\hat{\mathbf{i}}$,

$$\frac{d\hat{\mathbf{i}}}{dt} = \boldsymbol{\omega}_{(1,2,3)} \times \hat{\mathbf{i}}. \tag{5}$$

Show that for steady motion ($\dot{\phi}$, ω_3 constant, $\dot{\theta} = 0$),

$$\dot{\phi}^2 \sin\theta\cos\theta - 6\omega_3\dot{\phi}\sin\theta - \frac{4g}{a}\cos\theta = 0. \tag{6}$$

At $\theta = \pi/2$ the disk is "on edge". Here, we are interested in the rolling motion $\dot{\phi} = 0$ but ω_3 arbitrary. Is this motion stable?

To answer this, consider $\theta = \pi/2 + \epsilon$ for small ϵ, small $\dot{\phi}$ and arbitrary ω_3. Ignore 2nd-order terms in the equation of motion, such as ϵ^2 and $\epsilon\,\dot{\theta}$, to show that the:

- $\hat{\mathbf{1}}$ terms $\Rightarrow \ddot{\epsilon} \approx -\frac{4}{5}\left(3\omega_3^2 - \frac{g}{a}\right)\epsilon$.
- $\hat{\mathbf{2}}$ terms $\Rightarrow \dot{\phi} \approx 2\,\epsilon\,\omega_3\epsilon$.
- $\hat{\mathbf{3}}$ terms $\Rightarrow \dot{\omega}_3 \approx 0$ (ω_3 constant).

Hence, the rolling "on edge" is stable if $\omega_3 > \sqrt{g/3a}$; otherwise the disk falls over into (generally unstable) motion of the form considered in Prob. 5, Set 9.

3. **Marble Rolling on a Turntable.** (Is this how you lost your marbles?)

A marble (uniform sphere of mass m and radius a) rolls without slipping on a horizontal turntable that rotates with constant angular velocity $\mathbf{\Omega}$ about the symmetry axis of the turntable.

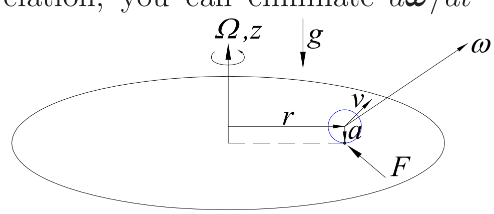
Use the vectorial approach of Prob. 2 to analyze the motion by "elementary" methods in the lab frame.

(a) What is the constraint between $\mathbf{a}$, $\mathbf{r}$, $\mathbf{v}$, $\mathbf{\Omega}$ and $\boldsymbol{\omega}$ where,

- $\mathbf{a}$ = vector from the center of the marble to the point of contact with the turntable.
- $\mathbf{r}$ = vector perpendicular to the symmetry axis of the turntable to the center of the marble.
- $\mathbf{v}$ = velocity of the center (of mass of) the marble.
- $\boldsymbol{\omega}$ = total angular velocity of the marble in the lab frame.

(b) Consider arbitrary motion of the marble (rolling without slipping on the turntable). Note that by differentiating the constraint relation, you can eliminate $d\boldsymbol{\omega}/dt$ from the equations of motion, leading to,

$$\frac{I + ma^2}{I}\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{\Omega} \times \mathbf{r}), \qquad (7)$$



with Ω constant. Show that this leads to,

$$\mathbf{v} = \frac{I}{I + ma^2}\mathbf{\Omega} \times \boldsymbol{\rho}, \quad \text{where} \quad \mathbf{r} + \mathbf{R} + \boldsymbol{\rho} \quad \text{and} \quad \mathbf{R} = \mathbf{r}_0 + \frac{I + ma^2}{I}\frac{\mathbf{\Omega} \times \mathbf{v}_0}{\Omega}, \qquad (8)$$

and $\mathbf{r}_0$ and $\mathbf{v}_0$ are the initial position of velocity of the center of the marble.

This is motion in a circle of radius ρ about the axis parallel to $\mathbf{a}$ at distance $\mathbf{R}$ from the axis of the turntable.

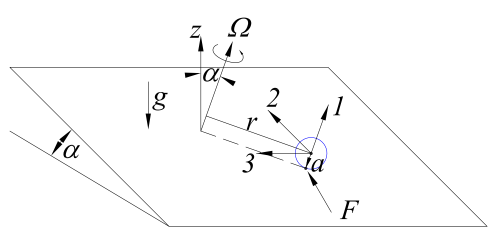
Ignore a possible "spin" angular velocity of the marble about the axis $\mathbf{a}$, which "spin" would be independent of Ω, and find the total angular velocity $\boldsymbol{\omega}$ of the marble.

You should find that the angular velocity component about the vertical is $\Omega I/(I + ma^2) = 2\Omega/7$ *for a uniform sphere..*

(c) Suppose that the plane of the turntable makes angle α to the horizontal.

Note that the normal force of the turntable on the marble is not necessarily $mg\cos\alpha$.

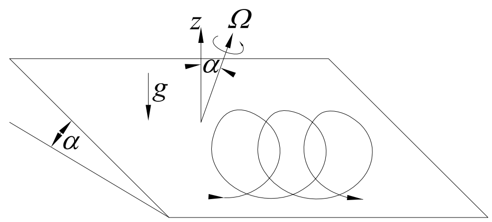
Consider axes $\hat{\mathbf{1}}$ perpendicular to the tilted turntable, $\hat{\mathbf{2}}$ pointing up the slope, and $\hat{\mathbf{3}}$ horizontal.



Deduce the equation of motion, slightly modified from that in part (b). Show that a solution is $\mathbf{v} = \mathbf{v}_{(b)} + \mathbf{v}_{\rm drift}$ where $\mathbf{v}_{(b)}$ is that found in part (b) (in the $(1, 2, 3)$ coordinate system) and,

$$\mathbf{v}_d = \frac{mga^2 \sin\alpha}{I\Omega}\,\hat{\mathbf{3}}. = \text{constant vector} \tag{9}$$

Hence, the motion involves a horizontal drift, as sketched in the figure below.

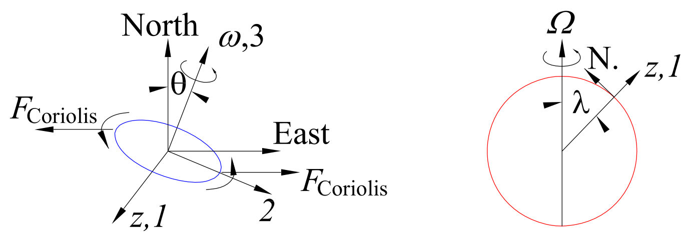


Why not try it?

4. **Gyrocompass.**

A gyrocompass is a spinning flywheel whose axis $\boldsymbol{\omega}$ of rotation is constrained to lie in a horizontal plane at the surface of the Earth.

If we analyze the motion in a frame fixed to the surface of the (spinning) Earth, the Coriolis force must be taken into account. When $\boldsymbol{\omega}$ makes angle θ to the North, as shown in the figure, the left side of the flywheel is moving up, and the Coriolis force on it is to the West. Similarly, the right side of the flywheel is moving down, and the Coriolis force on it is to the East. Hence, there is a net torque on the flywheel that tends to restore θ to zero, *i.e.*, to the North.



(a) Suppose the flywheel is a hoop of mass m and radius a. Calculate the total Coriolis torque about the center of the hoop to show that,

$$\tau_1 = ma^2\,\omega\,\Omega\sin\lambda\sin\theta, \qquad \tau_2 = -ma^2\,\omega\,\Omega\sin\lambda\cos\theta, \qquad \tau_3 = 0, \tag{10}$$

where $\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}}$ are principal axes (but not body axes), with $\hat{\mathbf{1}}$ vertical upwards (*i.e.*, $\hat{\mathbf{1}} = \hat{\mathbf{z}}$), $\hat{\mathbf{2}}$ horizontal, and $\hat{\mathbf{3}}$ along the symmetry axis of the hoop; Ω is the angular velocity of the Earth, and λ is the colatitude of the gyrocompass.

Show that this torque causes the gyrocompass to make small oscillations in θ with angular frequency $\sqrt{2\,\omega\,\Omega\sin\lambda}$.

(b) Analyze the motion in an inertial frame, where the torque equation about the center of the hoop is $\boldsymbol{\tau} = d\mathbf{L}/dt$, where $\mathbf{L}$ is the angular momentum. In this frame, the torque is only due to the constraint forces on the axle of the gyrocompass, which keep the axle in the horizontal plane with respect to the Earth, but which do not make the gyro point North.

Note that $\mathbf{L} = \mathsf{I}\cdot\boldsymbol{\omega}_{\rm total}$, where the total angular velocity has three pieces,

- Rotation about the gyro axle (axis $\hat{\mathbf{3}}$) with angular velocity $\boldsymbol{\omega}$.
- Rotation at angular velocity $\dot{\theta}$ about the local vertical axis ($\hat{\mathbf{z}} = \hat{\mathbf{1}}$) with respect to the Earth.
- Rotation at angular velocity Ω of the Earth about its axis.

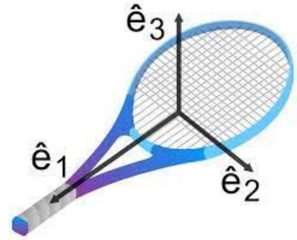
The principal axes $\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}}$ introduced in part (a) rotate with angular velocity $\boldsymbol{\omega}_{123} = \dot{\boldsymbol{\theta}} + \boldsymbol{\omega}$.

Show that the components of the torque equation for with $\tau_i = 0$ (in the inertial frame) lead to $\dot{\omega} = 0$ and $\ddot{\theta} + (2\,\lambda\,\Omega\sin\lambda)\,\theta = 0$, which leads again to the result of part (a).

In practice, a motor is required to keep ω constant. And, the gyrocompass must have a mechanism to find the horizontal plane even when its supports are tipping, *as on an airplane or ship. This requires a "plumb bob", and a mechanism to defeat the effect of a possibly oscillation point of support...*

5. **The Tennis Racquet Theorem.**

Consider a rigid body whose principal moments of inertia are $I_1 < I_2 < I_3$. As discussed in sec. 37 of Landau's *Mechanics*, free rotation with angular velocity $\boldsymbol{\omega}$ pointing close to axis 2 is "unstable".



Examine the special case where the kinetic energy has the form $T = L^2/2I_2$, and $\mathbf{L}$ is the angular momentum about the center of mass. Use expressions for T and $\mathbf{L}$ to show that,

$$\omega_1^2 = \frac{I_3 - I_2}{I_3 - I_1}\frac{L^2 - I_2^2\omega_2^2}{I_1 I_2}, \qquad \omega_3^2 = \frac{I_2 - I_1}{I_3 - I_1}\frac{L^2 - I_2^2\omega_2^2}{I_2 I_3}, \tag{11}$$

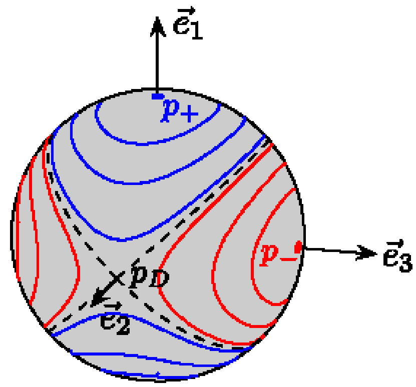
and that Euler's equations lead to,

$$\omega_1 = \omega_{1,\max}\,\mathrm{sech}[k\,\omega_{2,\max}\,(t - t_0)], \tag{12}$$

$$\omega_2 = \omega_{2,\max}\tanh[k\,\omega_{2,\max}\,(t - t_0)], \qquad \omega_{2,\max} = \frac{L}{I_2}, \qquad k = \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}}, \tag{13}$$

$$\omega_3 = \omega_{3,\max}\,\mathrm{sech}[k\,\omega_{2,\max}\,(t - t_0)]. \tag{14}$$

As $t \to \infty$, $\omega_1,\ \omega_3 \to 0$ while $\omega_2 \to \omega_{2,\max}$, and the final rotation is about axis 2. Thus, for this special case of motion along "separating polhodes", a kind of stability occurs. That is, the motion consider in this problem is along the dashed lines in the figure below.



In practice, the special case is hard to achieve, since for any slight perturbation of the kinetic energy T away from $L^2/2I_2$, $\boldsymbol{\omega}$ will move towards axis **2** along a path close to one of the separating polhohdes, then "bounce away" from the $\hat{\mathbf{2}}$ axis and move towards axis $-\hat{\mathbf{2}}$ along a path close to the other separating polhode, "bounce away" from this axis, and repeat the cycle.... To a viewer of the spinning tennis racquet, this cycle seems "unstable" because the axis of rotation migrates between $\hat{\mathbf{2}}$ and $\hat{-\mathbf{2}}$ every half cycle, although in the mathematical sense it is a "stable" orbit.

We infer from this problem that the cycle time for trajectories very close to the separating polhodes is very long, approaching infinity in the limit considered here. The long period of such cycles contributes to the impression by the "casual" observer that the motion is "unstable".

One of many YouTube videos on this theme: https://www.youtube.com/watch?v=1VPfZ_XzisU

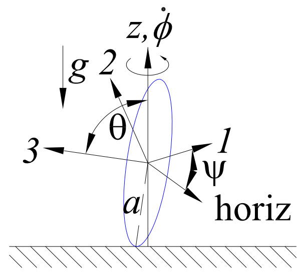
Solutions

1. **Spinning Coin Revisited.**

This appears as Prob. 11-58, p. 353 of W. Chester, *Mechanics* (Allen & Unwin, 1979), http://kirkmcd.princeton.edu/examples/mechanics/chester_mechanics_79.pdf

We consider a coin spinning without friction on a horizontal table. Unlike Prob. 6, Set 9, the instantaneous axis is not necessarily the diameter in the vertical plane.

We use Lagrange's method to find the motion in terms of three Euler angles, θ = angle of the symmetry axis of the coin to the vertical, ϕ = azimuthal angle of the horizontal diameter of the coin, and ψ = angle to the horizontal of body axis 1, where body axes 1 and 2 are in the plane of the disk and axis 3 is the symmetry axis.



(a) As deduced in eq. (35.2), p. 111 of L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed. (Pergamon, 1976), http://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics.pdf, the kinetic energy of rotation about the center of the disk is,

$$T_{\rm rot} = \frac{I_1}{2}\left(\dot\theta^2 + \dot\phi^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot\phi\cos\theta + \dot\psi\right)^2, \tag{15}$$

where for a thin, uniform disk of mass m and radius a, $I_1 = I_2 = ma^2/4$ and $I_3 = ma^2/2$.

The center of the disk is at height $z = a\sin\theta$ above the table, so the kinetic energy of the motion of the center of mass of the disk is,

$$T_{\rm cm} = \frac{m\dot z^2}{2} = \frac{ma^2\dot\theta^2\cos^2\theta}{2}, \tag{16}$$

and the potential energy can be written as,

$$V = mga\sin\theta. \tag{17}$$

The energy $E = T + V = T_{\rm cm} + T_{\rm rot} + V$ is conserved, and as the Lagrangian $\mathcal{L} = T - V$ does not depend on either ϕ or ψ, the generalized momenta P_ϕ and P_ψ are also conserved,

$$P_\phi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \frac{\partial T_{\rm rot}}{\partial\dot\phi} = \left(I_1\sin^2\theta + I_3\cos^2\theta\right)\dot\phi + I_3\,\dot\psi\,\cos\theta, \tag{18}$$

$$P_\psi = \frac{\partial\mathcal{L}}{\partial\dot\psi} = \frac{\partial T_{\rm rot}}{\partial\dot\psi} = I_3\left(\dot\phi\,\cos\theta + \dot\psi\right). \tag{19}$$

Note that the total angular momentum vector is (from eq. (35.1), p. 111 of the above link),

$$\boldsymbol{\omega} = \left(\dot\phi \sin\theta\sin\psi + \dot\theta\cos\psi, \dot\phi \sin\theta\cos\psi - \dot\theta\sin\psi, \dot\phi \cos\theta + \dot\psi\right), \tag{20}$$

so we recognize that the conserved, generalized momentum P_ψ is the angular momentum about axis 3,

$$P_\psi = I_3\left(\dot\phi \cos\theta + \dot\psi\right) = I_3\,\omega_3 = L_3. \tag{21}$$

The forces on the spinning coin are only vertical, so the angular momentum L_z about the vertical axis is also conserved. Since angle ϕ is the azimuth about the vertical axis, we anticipate that the conserved, generalized momentum P_ϕ is L_z. To verify this, we consider a moment when angle $\psi = 0$, axis 1 is horizontal, and axis 2 is in a vertical plane containing the center of the coin. Then, the total angular momentum in the $(1,2,3)$ system is $\boldsymbol{\omega} = \left(\dot\theta, \dot\phi \sin\theta, \dot\phi \cos\theta + \dot\psi\right)$, the angular momentum is,

$$\mathbf{L} = \mathsf{I}\cdot\boldsymbol{\omega} = \left(I_1\dot\theta, I_2\,\dot\phi \sin\theta, I_3\left(\dot\phi \cos\theta + \dot\psi\right)\right), \tag{22}$$

and,

$$L_z = L_2\sin\theta + L_3\cos\theta = I_1\,\dot\phi \sin^2\theta + I_3\left(\dot\phi \cos^2\theta + \dot\psi \cos\theta\right) = P_\phi. \tag{23}$$

Of the equations of motion, only that for coordinate θ remains to be discussed,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} = I_1\,\ddot\theta = \frac{\partial\mathcal{L}}{\partial\theta} = I_1\,\dot\phi^2 \sin\theta\cos\theta - I_3\left(\dot\phi \cos\theta + \dot\psi\right)\dot\phi \sin\theta - mga\cos\theta$$

$$= \sin\theta\left(I_1\,\dot\phi^2 \cos\theta - I_3\,\omega_3\,\dot\phi - \frac{mga\cos\theta}{\sin\theta}\right) = \frac{ma^2\sin\theta}{4}\left(\dot\phi^2 \cos\theta - 2\omega_3\,\dot\phi - \frac{4g\cos\theta}{a\sin\theta}\right). \tag{24}$$

For steady motion, $\ddot\theta = 0$, and,

$$\dot\phi_{\rm steady} = \frac{\omega_3 \pm \sqrt{\omega_3^2 + 4g\cos^2\theta/a\sin\theta}}{\cos\theta}. \tag{25}$$

This relation becomes invalid at $\theta = 90°$, when the coin is "on edge".

For $\theta = 0$, eq. (24) provides no constraint on the steady motion, so $\dot\phi = \omega_3$ (in this case) is arbitrary (in the idealization of zero friction).

(b) For steady motion at $\theta = 90°$, eq. (24) reduces to $I_3\,\omega_3\,\dot\phi = 0$, which implies that,

- Either ω_3 arbitrary, $\dot\phi = 0, \Leftrightarrow$ rolling and slipping,
- Or $\omega_3 = 0$, $\dot\phi$ arbitrary, $\Leftrightarrow$ spinning on edge.

To discuss the stability of the second case, spinning on edge, we consider a small departure, $\theta = \pi/2+\epsilon$ from the steady motion. Then, $\ddot\theta = \ddot\epsilon$, $\cos\theta \approx -\epsilon$, $\sin\theta \approx 1$, and the equation of motion (24) becomes,

$$I_1\,\ddot\epsilon \approx -I_1\,\dot\phi^2\,\epsilon - I_3\,\omega_3\,\dot\phi + mga\epsilon = -\left(I_1\,\dot\phi^2 - mga\right)\epsilon - L_3\,\dot\phi, \tag{26}$$

which is stable (simple harmonic motion in ϵ with constant $\dot\phi$) for,

$$\dot\phi > \sqrt{\frac{mga}{I_1}} = 2\sqrt{\frac{g}{a}}. \tag{27}$$

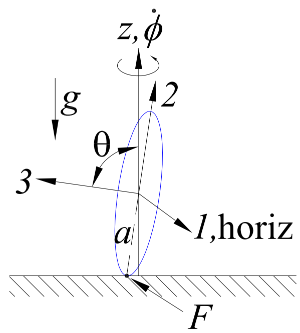
2. **Rolling Disk Revisited.**

We consider arbitrary motion of a thin, uniform disk of mass m and radius a that rolls without slipping on a horizontal plane.

We use a coordinate system that is similar to, but not quite the same as that of Euler:

- $\hat{\mathbf{z}}$ is vertical.
- Principal axis $\hat{\mathbf{1}}$ is always horizontal.
- Principal axis $\hat{\mathbf{2}}$ lies in a vertical plane that includes the center of the disk.
- Principal axis $\hat{\mathbf{3}} = \hat{\mathbf{1}} \times \hat{\mathbf{2}}$ is the symmetry axis of the disk.

The axes $(1, 2, 3)$ are principal axes, but they are not body axes (that are fixed with respect to the rotating disk).



We also define,

- θ = angle between $\hat{\mathbf{3}}$ and $\hat{\mathbf{z}}$.
- $\dot{\phi}$ = angular velocity of the disk, and of the $(1, 2, 3)$ axes, about the vertical.
- $\mathbf{F}$ = the force on the disk at the point of contact with the horizontal surface.

The "elementary" equations of motion are,

$$\mathbf{F}_{\rm total} = \mathbf{F} - mg\,\hat{\mathbf{z}} = m\frac{d\mathbf{v}_{\rm cm}}{dt}\,, \qquad \boldsymbol{\tau}_{\rm cm} = \mathbf{a} \times \mathbf{F} = \frac{d\mathbf{L}_{\rm cm}}{dt}\,. \tag{28}$$

The constraint of rolling without slipping can be written in terms of velocities (as a time-dependent version of Chasles' theorem),

$$\mathbf{v}_{\rm contact} = 0 = \mathbf{v}_{\rm cm} + \boldsymbol{\omega} \times \mathbf{a}, \tag{29}$$

where,

- $\mathbf{a} = -a\,\hat{\mathbf{2}}$ = vector from the center of mass to the point of contact.
- $\boldsymbol{\omega}_{(1,2,3)} = \dot{\theta}\,\hat{\mathbf{1}} + \dot{\phi}\,\hat{\mathbf{z}}$ = angular velocity of the axes $(1, 2, 3)$.
- $\dot{\psi}$ = (spin) angular velocity of the disk relative to the $(1, 2, 3)$ axes.
- $\boldsymbol{\omega} = \boldsymbol{\omega}_{(1,2,3)} + \dot{\psi}\,\hat{\mathbf{3}}$ = total angular velocity of the disk.

We also have that the principal moments of inertial about the center of the disk are $I_1 = I_2 = I_3/2 = ma^2/4$, and,

$$\hat{\mathbf{z}} = \sin\theta\,\hat{\mathbf{2}} + \cos\theta\,\hat{\mathbf{3}}, \tag{30}$$

$$\boldsymbol{\omega} = \dot\theta\,\hat{\mathbf{1}} + \dot\phi\,\hat{\mathbf{z}} + \dot\psi\,\hat{\mathbf{3}} = \dot\theta\,\hat{\mathbf{1}} + \dot\phi\sin\theta\,\hat{\mathbf{2}} + \left(\dot\psi + \dot\phi\cos\theta\right)\,\hat{\mathbf{3}}, \tag{31}$$

$$\mathbf{L}_{\rm cm} = \mathsf{I}_{\rm cm}\cdot\boldsymbol{\omega} = \left[I_1\dot\theta\,\hat{\mathbf{1}} + I_1\dot\phi\sin\theta\,\hat{\mathbf{2}} + I_3\left(\dot\psi + \dot\phi\cos\theta\right)\,\hat{\mathbf{3}}\right]. \tag{32}$$

We note that $\omega_3 = \dot\psi + \dot\phi\cos\theta$.

From the rolling constraint (29), we have that,

$$\mathbf{v}_{\rm cm} = -\boldsymbol{\omega}\times\mathbf{a} = a\boldsymbol{\omega}\times\hat{\mathbf{2}}, \qquad \frac{d\mathbf{v}_{\rm cm}}{dt} = a\frac{d}{dt}(\boldsymbol{\omega}\times\hat{\mathbf{2}}). \tag{33}$$

The equation of motion of the center of mass can now be rewritten as,

$$\mathbf{F} = mg\,\hat{\mathbf{z}} + m\frac{d\mathbf{v}_{\rm cm}}{dt} = mg(\sin\theta\,\hat{\mathbf{2}} + \cos\theta\,\hat{\mathbf{3}}) + ma\frac{d}{dt}(\boldsymbol{\omega}\times\hat{\mathbf{2}}), \tag{34}$$

and the torque equation about the center of mass can be rewritten as,

$$\begin{aligned}\boldsymbol{\tau}_{\rm cm} = \mathbf{a}\times\mathbf{F} &= -a\,\hat{\mathbf{2}}\times\left[mg(\sin\theta\,\hat{\mathbf{2}} + \cos\theta\,\hat{\mathbf{3}}) + ma\frac{d}{dt}(\boldsymbol{\omega}\times\hat{\mathbf{2}})\right]\\ &= -mga\cos\theta\,\hat{\mathbf{1}} + ma^2\,\hat{\mathbf{2}}\times\frac{d}{dt}\left(\omega_3\,\hat{\mathbf{1}} - \dot\theta\,\hat{\mathbf{3}}\right)\\ &= \frac{d\mathbf{L}_{\rm cm}}{dt} = \frac{d}{dt}\left(I_1\dot\theta\,\hat{\mathbf{1}} + I_1\dot\phi\sin\theta\,\hat{\mathbf{2}} + I_3\omega_3\,\hat{\mathbf{3}}\right).\end{aligned} \tag{35}$$

Note that for axis $\hat{\mathbf{i}}$,

$$\frac{d\,\hat{\mathbf{i}}}{dt} = \boldsymbol{\omega}_{(1,2,3)}\times\hat{\mathbf{i}} = \left(\dot\theta\,\hat{\mathbf{1}} + \dot\phi\,\hat{\mathbf{z}}\right)\times\hat{\mathbf{i}} = \left(\dot\theta\,\hat{\mathbf{1}} + \dot\phi\sin\theta\,\hat{\mathbf{2}} + \dot\phi\cos\theta\,\hat{\mathbf{3}}\right)\times\hat{\mathbf{i}}, \tag{36}$$

$$\frac{d\,\hat{\mathbf{1}}}{dt} = \dot\phi\cos\theta\,\hat{\mathbf{2}} - \dot\phi\sin\theta\,\hat{\mathbf{3}}, \qquad \frac{d\,\hat{\mathbf{2}}}{dt} = -\dot\phi\cos\theta\,\hat{\mathbf{1}} + \dot\theta\,\hat{\mathbf{3}}, \qquad \frac{d\,\hat{\mathbf{3}}}{dt} = \dot\phi\sin\theta\,\hat{\mathbf{1}} - \dot\theta\,\hat{\mathbf{2}}. \tag{37}$$

For steady motion, $\dot\phi$, ω_3 constant, $\dot\theta = 0$, we have,

$$\frac{d\,\hat{\mathbf{1}}}{dt} = \dot\phi\cos\theta\,\hat{\mathbf{2}} - \dot\phi\sin\theta\,\hat{\mathbf{3}}, \qquad \frac{d\,\hat{\mathbf{2}}}{dt} = -\dot\phi\cos\theta\,\hat{\mathbf{1}}, \qquad \frac{d\,\hat{\mathbf{3}}}{dt} = \dot\phi\sin\theta\,\hat{\mathbf{1}}, \tag{38}$$

and only the $\hat{\mathbf{1}}$-component of eq. (35) is nonzero,

$$-\,mga\cos\theta - ma^2\omega_3\,\dot\phi\sin\theta = \left(-I_1\dot\phi^2\sin\theta\cos\theta + I_3\,\omega_3\,\dot\phi\sin\theta\right) \tag{39}$$

$$\dot\phi^2\sin\theta\cos\theta - 6\omega_3\dot\phi\sin\theta - \frac{4g}{a}\cos\theta = 0. \tag{40}$$

At $\theta = \pi/2$ the disk is "on edge". Here, we are interested in the rolling motion $\dot\phi = 0$ but ω_3 arbitrary. Is this motion stable?

To answer this, we consider $\theta = \pi/2 + \epsilon$ for small ϵ, small $\dot\phi$ and arbitrary ω_3. Then, $\dot\theta = \dot\epsilon$, $\ddot\theta = \ddot\epsilon$, $\cos\theta \approx -\epsilon$, $\sin\theta \approx 1$, and on ignoring small terms like $\epsilon\,\dot\phi$, and we have,

$$\frac{d\,\hat{\mathbf{1}}}{dt} \approx -\epsilon\,\dot\phi\,\hat{\mathbf{2}} - \dot\phi\,\hat{\mathbf{3}}, \qquad \frac{d\,\hat{\mathbf{2}}}{dt} \approx \epsilon\,\dot\phi\,\hat{\mathbf{1}} + \dot\epsilon\,\hat{\mathbf{3}}, \qquad \frac{d\,\hat{\mathbf{3}}}{dt} \approx \dot\phi\,\hat{\mathbf{1}} - \dot\epsilon\,\hat{\mathbf{2}}, \tag{41}$$

and eq. (35) becomes, (again ignoring $2^{\rm nd}$-order terms),

$$\begin{aligned}
\boldsymbol{\tau} &\approx \epsilon\, mga\,\hat{\mathbf{1}} + \hat{\mathbf{2}} \times ma^2 \frac{d}{dt}\left(\omega_3\,\hat{\mathbf{1}} - \dot\epsilon\,\hat{\mathbf{3}}\right) \\
&\approx \epsilon\, mga\,\hat{\mathbf{1}} + \hat{\mathbf{2}} \times ma^2 \left(\dot\omega_3\,\hat{\mathbf{1}} - \ddot\epsilon\,\hat{\mathbf{3}} - \epsilon\,\omega_3\,\dot\phi\,\hat{\mathbf{2}} - \omega_3\,\dot\phi\,\hat{\mathbf{3}} - \dot\epsilon\,\dot\phi\,\hat{\mathbf{1}}\right) \\
&= \epsilon\, mga\,\hat{\mathbf{1}} + ma^2 \left(-\dot\omega_3\,\hat{\mathbf{3}} - \ddot\epsilon\,\hat{\mathbf{1}} - \omega_3\,\dot\phi\,\hat{\mathbf{1}} + \dot\epsilon\,\dot\phi\,\hat{\mathbf{3}}\right) \\
&= \left(\epsilon\, mga - ma^2(\ddot\epsilon + \omega_3\,\dot\phi)\right)\,\hat{\mathbf{1}} - ma^2\left(\dot\omega_3 - \dot\epsilon\,\dot\phi\right)\,\hat{\mathbf{3}} \\
&= \frac{d\mathbf{L}}{dt} \approx \frac{d}{dt}\left(I_1\,\dot\epsilon\,\hat{\mathbf{1}} + I_1\,\dot\phi\,\hat{\mathbf{2}} + I_3\,\omega_3\,\hat{\mathbf{3}}\right) \\
&\approx I_1\,\ddot\epsilon\,\hat{\mathbf{1}} + I_1\,\ddot\phi\,\hat{\mathbf{2}} + I_3\,\dot\omega_3\,\hat{\mathbf{3}} - I_1\,\dot\epsilon\,\dot\phi\,\hat{\mathbf{3}} + I_1\,\dot\phi\,\dot\epsilon\;\hat{\mathbf{3}} + I_3\,\omega_3\,\dot\phi\,\hat{\mathbf{1}} - I_3\,\dot\epsilon\,\omega_3\,\hat{\mathbf{2}} \\
&= \left(I_1\,\ddot\epsilon + I_3\,\omega_3\,\dot\phi\right)\,\hat{\mathbf{1}} + \left(I_1\,\ddot\phi - I_3\,\dot\epsilon\,\omega_3\right)\,\hat{\mathbf{2}} + I_3\,\dot\omega_3\,\hat{\mathbf{3}}.
\end{aligned} \tag{42}$$

Then,

- $\hat{\mathbf{3}}$ terms $\Rightarrow$ $(ma^2 + I_3)\,\dot\omega_3 \approx ma^2\,\dot\epsilon\,\dot\phi \approx 0, \qquad \Rightarrow \qquad \omega_3 \approx$ constant, since $\epsilon\,\dot\phi$ is of $2^{\rm nd}$ order.
- $\hat{\mathbf{2}}$ terms $\Rightarrow I_1\,\ddot\phi \approx I_3\,\dot\epsilon\,\omega_3 \qquad \Rightarrow \qquad \dot\phi = I_3\,\epsilon\,\omega_3/I_1 = 2\,\epsilon\,\omega_3.$
- $\hat{\mathbf{1}}$ terms $\Rightarrow (ma^2{+}I_1)\,\ddot\epsilon \approx -(ma^2{+}I_3)\,\omega_3\dot\phi{+}\epsilon\, mga \approx -\epsilon(I_3(ma^2{+}I_3)\omega_3^2/I_1{-}mga).$

Hence, the rolling "on edge" is stable if $\omega_3 > \sqrt{I_1 mga/I_3(ma^2 + I_3)} = \sqrt{g/3a}$; otherwise the disk falls over into (generally unstable) motion of the form considered in Prob. 5, Set 9.

This problem is Ex. 11.6.5, p. 339 of W. Chester, *Mechanics* (Allen & Unwin, 1979),
http://kirkmcd.princeton.edu/examples/mechanics/chester_mechanics_79.pdf
Additional aspects of this problem are discussed in
http://kirkmcd.princeton.edu/examples/rollingdisk.pdf

3. **Marble Rolling on a Turntable.**

This problem first appeared on pp. 280-283 of S. Earnshaw, *Dynamics*, 3rd ed. (Cambridge, 1844), http://kirkmcd.princeton.edu/examples/mechanics/earnshaw_44.pdf
See also, http://kirkmcd.princeton.edu/examples/mechanics/weltner_ajp_47_984_79.pdf
http://kirkmcd.princeton.edu/examples/mechanics/weckesser_ajp_65_736_97.pdf
http://kirkmcd.princeton.edu/examples/mechanics/borisov_ejp_39_065001_18.pdf

A marble (uniform sphere of mass m and radius a) rolls without slipping on a horizontal turntable that rotates with constant angular velocity $\mathbf{\Omega}$ about the symmetry axis of the turntable.

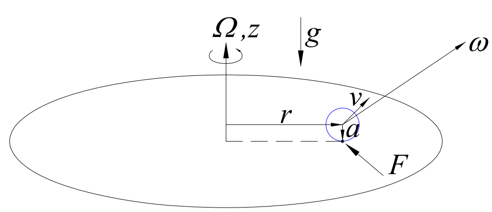
We make a vectorial analysis in the lab frame.

(a) The marble rolls without slipping, so the (nonholonomic) rolling constraint (and its time derivative) can be written as,

$$\mathbf{v}_{\rm contact} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{a} = \mathbf{\Omega} \times \mathbf{r}, \qquad \frac{d\mathbf{v}_{\rm contact}}{dt} = \frac{d\mathbf{v}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{a} = \mathbf{\Omega} \times \frac{d\mathbf{r}}{dt} = \mathbf{\Omega} \times \mathbf{v}, \tag{43}$$

where,

- $\mathbf{a} = -a\,\hat{\mathbf{z}}$ = vector from the center of the marble to the point of contact with the turntable.
- $\mathbf{r}$ = vector perpendicular to the symmetry axis of the turntable to the center of the marble.
- $\mathbf{v} = d\mathbf{r}/dt$ = velocity of the center (of mass of) the marble.
- $\boldsymbol{\omega}$ = total angular velocity of the marble in the lab frame.



(b) The equations of motion of the marble are,

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{g} + \mathbf{F}_{\rm contact}, \qquad I\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\tau}_{\rm cm} = \mathbf{a} \times \mathbf{F}_{\rm contact} = \mathbf{a} \times \left(m\frac{d\mathbf{v}}{dt} - m\mathbf{g}\right), \tag{44}$$

where the moment of inertia about the center of the (uniform) marble is $I = 2ma^2/5$.

In the first part of this problem, the turntable is horizontal, so $\mathbf{a}$ (and $-\hat{\mathbf{z}}$) is parallel to $\mathbf{g}$, and the torque equation simplifies to,

$$I\frac{d\boldsymbol{\omega}}{dt} = m\mathbf{a} \times \frac{d\mathbf{v}}{dt}\,. \tag{45}$$

To combine eqs. (43) and (45), we write,

$$\mathbf{a} \times I\frac{d\boldsymbol{\omega}}{dt} = \mathbf{a} \times \left(m\mathbf{a} \times \frac{d\mathbf{v}}{dt}\right) = m\left(\mathbf{a} \cdot \frac{d\mathbf{v}}{dt}\right)\mathbf{a} - ma^2\frac{d\mathbf{v}}{dt} = -ma^2\frac{d\mathbf{v}}{dt}$$

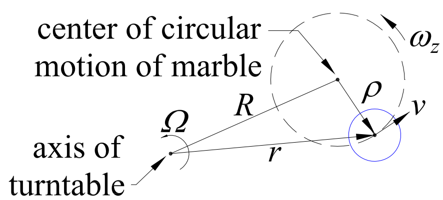
$$= I\left(\frac{d\mathbf{v}}{dt} - \boldsymbol{\Omega} \times \mathbf{v}\right), \tag{46}$$

$$\frac{d\mathbf{v}}{dt} = \frac{I}{I + ma^2}\boldsymbol{\Omega} \times \mathbf{v} = \frac{I}{I + ma^2}\boldsymbol{\Omega} \times \frac{d\mathbf{r}}{dt}. \tag{47}$$

The acceleration $d\mathbf{v}/dt$ is constant in magnitude and perpendicular to the velocity $\mathbf{v}$, which corresponds to circular motion of the center of the marble.
We integrate eq. (47) to find,

$$\mathbf{v} = \frac{I}{I + ma^2}\boldsymbol{\Omega} \times (\mathbf{r} - \mathbf{R}) = \frac{I}{I + ma^2}\boldsymbol{\Omega} \times \boldsymbol{\rho}, \qquad v = \frac{I\Omega\,\rho}{I + ma^2}, \tag{48}$$

where $\mathbf{R}$ is the (constant) vector from the center of the turntable to the center of the circle (on the turntable) in which the marble moves in the lab frame, and $\boldsymbol{\rho} = \mathbf{r} - \mathbf{R}$ is the vector from the center of the circle to the point of contact of the marble.



From eq. (48) we have, since $\boldsymbol{\Omega}$ is perpendicular to $\boldsymbol{\rho}$,

$$\boldsymbol{\Omega} \times \mathbf{v} = \frac{I}{I + ma^2}\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\rho}) = -\frac{I}{I + ma^2}\Omega^2\,\boldsymbol{\rho}, \qquad \boldsymbol{\rho} = \frac{I + ma^2}{I\Omega^2}\mathbf{v} \times \boldsymbol{\Omega}. \tag{49}$$

$$\mathbf{R} = \mathbf{r} - \boldsymbol{\rho} = \mathbf{r} - \frac{I + ma^2}{I\Omega^2}\mathbf{v} \times \boldsymbol{\Omega}. = \mathbf{r}_0 + \frac{I + ma^2}{I\Omega^2}\boldsymbol{\Omega} \times \mathbf{v}_0. \tag{50}$$

If we ignore a possible "spin" angular velocity about the z-axis, then the angular velocity component ω_z is just that due to the motion of the marble in the circle of radius ρ,

$$\omega_z = \frac{v}{\rho} = \frac{I\,\Omega}{I + ma^2} \qquad (\omega_{z,\mathrm{spin}} = 0), \tag{51}$$

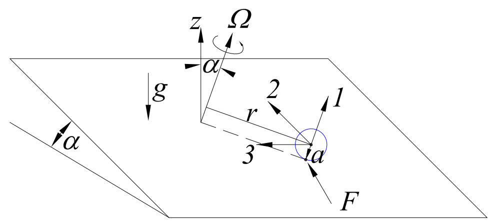
and so for a solid, uniform marble, $\omega_z = 2\,\Omega/7$.
From the rolling constraint (43), we can now write,

$$\mathbf{a} \times (\boldsymbol{\omega} \times \mathbf{a}) = a^2\boldsymbol{\omega} - (\mathbf{a} \cdot \boldsymbol{\omega})\mathbf{a} = a^2\boldsymbol{\omega} - a^2\omega_z\,\hat{\mathbf{z}}$$

$$= \mathbf{a} \times (\boldsymbol{\Omega} \times \mathbf{r} - \mathbf{v}) = a\Omega\,\mathbf{r} - \mathbf{a} \times \left(\frac{I}{I + ma^2}\boldsymbol{\Omega} \times \boldsymbol{\rho}\right) = a\Omega\,\mathbf{r} - \frac{a\Omega I}{I + ma^2}\,\boldsymbol{\rho}, \tag{52}$$

$$\boldsymbol{\omega} = \omega_z\,\hat{\mathbf{z}} + \frac{\Omega}{a}\left(\mathbf{r} - \frac{I}{I + ma^2}\,\boldsymbol{\rho}\right) = \omega_z\,\hat{\mathbf{z}} + \frac{\Omega}{a}\frac{ma^2\mathbf{r} + I\mathbf{R}}{I + ma^2}. \tag{53}$$

(c) We now suppose that the plane of the turntable makes angle α to the horizontal. We consider principal axes $\hat{\mathbf{1}}$ perpendicular to the tilted turntable, $\hat{\mathbf{2}}$ pointing up the slope, and $\hat{\mathbf{3}}$ horizontal.



Then, the vertical axis is $\hat{\mathbf{z}} = \cos\alpha\,\hat{\mathbf{1}} + \sin\alpha\,\hat{\mathbf{2}}$, and the vector from the center of the marble to the point of contact with the tilted turntable is $\mathbf{a} = -a\,\hat{\mathbf{1}}$.

The rolling constraint can still be written as eq. (43), and the equations of motion are again given by eq. (44). However, since $\mathbf{a}$ is no longer (anti)parallel to $\mathbf{g}$, eq. (45) becomes,

$$I\frac{d\boldsymbol{\omega}}{dt} = m\mathbf{a}\times\frac{d\mathbf{v}}{dt} - m\mathbf{a}\times\mathbf{g} = m\mathbf{a}\times\frac{d\mathbf{v}}{dt} + magsin\alpha\,\hat{\mathbf{3}}. \qquad (54)$$

We combine the second of eq. (43) with (54) to find,

$$\mathbf{a}\times I\frac{d\boldsymbol{\omega}}{dt} = \mathbf{a}\times\left(m\mathbf{a}\times\frac{d\mathbf{v}}{dt} + mag\sin\alpha\,\hat{\mathbf{3}}\right) = -ma^2\frac{d\mathbf{v}}{dt} + ma^2 g\sin\alpha\,\hat{\mathbf{2}}$$

$$= I\left(\frac{d\mathbf{v}}{dt} - \boldsymbol{\Omega}\times\mathbf{v}\right), \qquad (55)$$

$$\frac{d\mathbf{v}}{dt} = \frac{I}{I+ma^2}\boldsymbol{\Omega}\times\mathbf{v} + \frac{ma^2 g\sin\alpha}{I+ma^2}\,\hat{\mathbf{2}}. \qquad (56)$$

We try a solution of the form $\mathbf{v} = \mathbf{v}(t) + \mathbf{v}_{\rm drift}$ for constant drift velocity $\mathbf{v}_{\rm drift}$.

$$\frac{d\mathbf{v}(t)}{dt} = \frac{I}{I+ma^2}\boldsymbol{\Omega}\times\mathbf{v}(t) + \frac{I}{I+ma^2}\boldsymbol{\Omega}\times\mathbf{v}_{\rm drift} + \frac{ma^2 g\sin\alpha}{I+ma^2}\,\hat{\mathbf{2}}, \qquad (57)$$

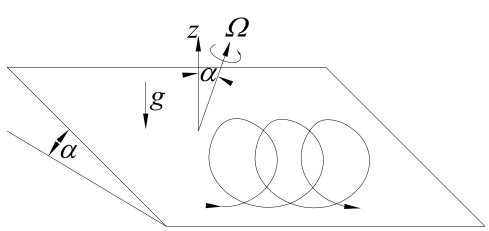
$$\mathbf{v}(t) = \frac{I}{I+ma^2}\boldsymbol{\Omega}\times(\mathbf{r}-\mathbf{R}) = \mathbf{v}_{(b)}, \qquad \frac{I}{I+ma^2}\boldsymbol{\Omega}\times\mathbf{v}_{\rm drift} = -\frac{ma^2 g\sin\alpha}{I+ma^2}\,\hat{\mathbf{2}}, \quad (58)$$

where $\mathbf{v}_{(b)}$ is the form found in eq. (48) above. Finally, noting that $\boldsymbol{\Omega} = \Omega\,\hat{\mathbf{1}}$,

$$\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{v}_{\rm drift}) = -\Omega^2\mathbf{v}_{\rm drift} = -\boldsymbol{\Omega}\times\frac{ma^2 g\sin\alpha}{I}\,\hat{\mathbf{2}} = -\frac{ma^2 g\Omega\sin\alpha}{I}\,\hat{\mathbf{3}}, \qquad (59)$$

$$\mathbf{v}_{\rm drift} = \frac{ma^2 g\sin\alpha}{I\Omega}\,\hat{\mathbf{3}}, \qquad \mathbf{r} = \mathbf{R} + \mathbf{v}_{\rm drift}t + \boldsymbol{\rho}. \qquad (60)$$

Hence, the motion involves a horizontal drift, as sketched in the figure below.

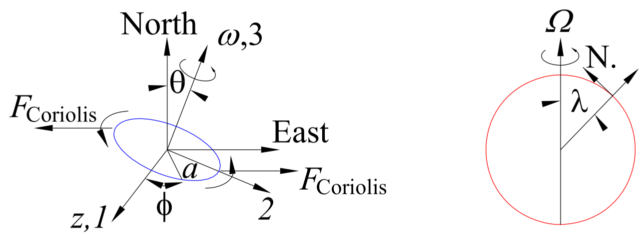


This like the motion of a charged particle in crossed, uniform **E** *and* **B** *fields (just as the circular motion in part (b) is like that of a charged particle in a uniform magnetic field).*

4. **Gyrocompass.**

A gyrocompass is a spinning flywheel whose axis $\boldsymbol{\omega}$ of rotation is constrained to lie in a horizontal plane at the surface of the Earth.

If we analyze the motion in a frame fixed to the surface of the (spinning) Earth, the Coriolis force must be taken into account. When $\boldsymbol{\omega}$ makes angle θ to the North, as shown in the figure, the left side of the flywheel is moving up, and the Coriolis force on it. is to the West. Similarly, the right side of the flywheel is moving down, and the Coriolis force on it is to the East. Hence, there is a net torque on the flywheel that tends to restore θ to zero, *i.e.*, to the North.



(a) We first analyze the motion in a frame fixed to the surface of the Earth, which latter rotates about its axis with angular velocity Ω with respect to the "fixed stars". We suppose the flywheel is a hoop of mass m and radius a. It is subject to torques about its center of mass due to the Coriolis (of order Ω, and the centrifugal force (of order Ω^2 which latter we neglect here.

In the frame fixed to the Earth, the flywheel has angular velocity ω about its symmetry axis, $\hat{\mathbf{3}}$, which axis is constrained to lie a the horizontal plane containing the center of the wheel. The symmetry axis makes (variable) angle θ to the local North direction (in the horizontal plane).

We also define principal axes $\hat{\mathbf{1}}$ to be vertical (also called $\hat{z}$, and $\hat{2}$ in the horizontal plane.

Then, an element $d\phi$ of the hoop at angle ϕ from the $\hat{\mathbf{1}}$ axis in the 1-2 plane has velocity relative to the 123 axes,

$$\mathbf{v}_{\rm rel} = a\omega(-\sin\phi\,\hat{\mathbf{1}} + \cos\phi\,\hat{\mathbf{2}}), \tag{61}$$

while these axes have angular velocity $-\dot{\theta}\,\hat{\mathbf{1}}$ relative to the coordinate system fixed to the Earth. In calculating the Coriolis force and torque, we neglect the small velocity of the hoop due to small $\dot{\theta}$.

The (constant) angular velocity $\mathbf{\Omega}$ of the Earth is,

$$\mathbf{\Omega} = \Omega(\cos\lambda\,\hat{\mathbf{z}} + \sin\lambda\,\hat{\mathbf{N}}) = \Omega(\cos\lambda\,\hat{\mathbf{1}} - \sin\lambda\sin\theta\,\hat{\mathbf{2}} + \sin\lambda\cos\theta\,\hat{\mathbf{3}}), \tag{62}$$

where $\hat{\mathbf{N}} = -\sin\theta\,\hat{\mathbf{2}} + \cos\theta\,\hat{\mathbf{3}}$ points to the local North. The Coriolis force on the mass element $m\,d\phi/2\pi$, at position $a(\cos\phi\,\hat{\mathbf{1}} + \sin\phi\,\hat{\mathbf{2}})$, is (neglecting the small velocity associated with $\dot{\theta}$),

$$d\mathbf{F} = -2\,dm\,\mathbf{\Omega}\times\mathbf{v}_{\rm rel} \tag{63}$$

$$= \frac{m\,d\phi}{\pi} a\,\omega\,\Omega[\sin\lambda\cos\theta\cos\phi\,\hat{\mathbf{1}} + \sin\lambda\cos\theta\sin\phi\,\hat{\mathbf{2}} + (\sin\lambda\sin\theta\sin\phi - \cos\lambda\cos\phi)\,\hat{\mathbf{3}}].$$

The torque about the center of the wheel on the mass element is,

$$\begin{aligned} d\boldsymbol{\tau} &= a(\cos\phi\,\hat{\mathbf{1}} + \sin\phi\,\hat{\mathbf{2}}) \times d\mathbf{F} \\ &= \frac{m\,d\phi}{\pi} a^2\,\omega\,\Omega[\sin\lambda\sin\theta\sin^2\phi - \cos\lambda\cos\phi\sin\phi)\,\hat{\mathbf{1}} \\ &\qquad + \sin\lambda\cos\theta\sin\phi - \sin\lambda\cos\theta\cos^2\phi\,\hat{\mathbf{2}} \\ &\qquad +(\sin\lambda\cos\theta\sin\phi\cos\phi - \sin\lambda\cos\theta\cos\phi\sin\phi)\,\hat{\mathbf{3}}]. \end{aligned} \tag{64}$$

Integrating over $d\phi$, we find the total Coriolis torque to be,

$$\boldsymbol{\tau} = ma^2\,\omega\,\Omega(\sin\lambda\sin\theta\,\hat{\mathbf{1}} - \sin\lambda\cos\theta\,\hat{\mathbf{2}}) \tag{65}$$

Torque component τ_2 tends to rotate the plane of the wheel out of the vertical, which action is compensated by the constraint mechanism of the gyrocompass. Torque component 1 leads to the equation of motion,

$$\tau_1 = ma^2\,\omega\,\Omega\sin\lambda\sin\theta = \frac{dL_1}{dt} = \frac{d}{dt}(-I_1\,\dot{\theta}) = -\frac{ma^2}{2}\ddot{\theta}, \tag{66}$$

which implies simple harmonic motion for small θ with angular frequency $\sqrt{2\,\omega\,\Omega\sin\lambda}$.

(b) We now analyze the motion in an inertial frame, where the torque about the center of the gyrocompass is only due to the constraint forces on the axle of the gyrocompass, which keep the axle in the horizontal plane with respect to the Earth (but which do not make the gyro point North). That is, $\boldsymbol{\tau} = \tau_2\,\hat{\mathbf{2}}$ in this frame (ignoring the small torque component τ_3 needed to keep the spin angular velocity $\omega\,\hat{\mathbf{3}}$ constant).

This is discussed in Ex. 11.6.4, p. 337 of W. Chester, *Mechanics* (Allen & Unwin, 1979), http://kirkmcd.princeton.edu/examples/mechanics/chester_mechanics_79.pdf

Here, the angular momentum is $\mathbf{L} = \mathsf{I}\cdot\boldsymbol{\omega}_{\rm total}$, where the total angular velocity has three pieces,

- Rotation about the gyro axle (axis $\hat{\mathbf{3}}$) with angular velocity $\boldsymbol{\omega}$.
- Rotation at angular velocity $-\dot{\theta}$ about the local vertical axis ($\hat{\mathbf{z}} = \hat{\mathbf{1}}$) with respect to the Earth.
- Rotation at (constant) angular velocity $\boldsymbol{\Omega}$ of the Earth about its axis, given in eq. (62) above.

That is,

$$\boldsymbol{\omega}_{\rm total} = \omega\,\hat{\mathbf{3}} - \dot{\theta}\,\hat{\mathbf{1}} + \boldsymbol{\Omega} = \omega\,\hat{\mathbf{3}} + \boldsymbol{\omega}_{123}, \qquad \frac{d\hat{\mathbf{i}}}{dt} = \boldsymbol{\omega}_{123} \times \hat{\mathbf{i}}, \tag{67}$$

where the principal axes $\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}}$ introduced in part (a) rotate with angular velocity,

$$\boldsymbol{\omega}_{123} = -\dot{\theta}\,\hat{\mathbf{1}} + \boldsymbol{\Omega} = (-\dot{\theta} + \Omega\cos\lambda)\,\hat{\mathbf{1}} - \Omega\sin\lambda\sin\theta\,\hat{\mathbf{2}} + \Omega\sin\lambda\cos\theta\,\hat{\mathbf{3}}. \tag{68}$$

The angular momentum is,

$$\mathbf{L} = \mathsf{I} \cdot \boldsymbol{\omega}_{\text{total}} = I_1(-\dot{\theta} + \Omega\cos\lambda)\,\hat{\mathbf{1}} - I_2\Omega\sin\lambda\sin\theta\,\hat{\mathbf{2}} + I_3(\omega + \Omega\sin\lambda\cos\theta)\,\hat{\mathbf{3}}. \tag{69}$$

The torque equation is now,

$$\begin{aligned}\boldsymbol{\tau} = \tau_2\,\hat{\mathbf{2}} = \frac{d\mathbf{L}}{dt} &= \frac{\partial\mathbf{L}}{\partial t} + \boldsymbol{\omega}_{123} \times \mathbf{L} \\ = -I_1\,\ddot{\theta}\,\hat{\mathbf{1}} &- I_2\,\Omega\,\dot{\theta}\sin\lambda\cos\theta\,\hat{\mathbf{2}} + I_3(\dot{\omega} - \Omega\,\dot{\theta}\sin\lambda\sin\theta)\,\hat{\mathbf{3}} \\ &+I_1(-\dot{\theta} + \Omega\cos\lambda)(\Omega\sin\lambda\sin\theta\,\hat{\mathbf{3}} + \Omega\sin\lambda\cos\theta\,\hat{\mathbf{2}}) \\ &+I_2(-\Omega\sin\lambda\sin\theta)((-\dot{\theta} + \Omega\cos\lambda)\,\hat{\mathbf{3}} - \Omega\sin\lambda\cos\theta\,\hat{\mathbf{1}}) \\ &+I_3(\omega + \Omega\sin\lambda\cos\theta)((\dot{\theta} - \Omega\cos\lambda)\,\hat{\mathbf{2}} - \Omega\sin\lambda\sin\theta\,\hat{\mathbf{1}}).\end{aligned} \tag{70}$$

The 1-component of eq. (70) is, ignoring terms in the very small quantity Ω^2,

$$0 = -I_1\,\ddot{\theta} - I_3\,\omega\,\Omega\sin\lambda\sin\theta, \tag{71}$$

$$\ddot{\theta} = -\frac{I_3}{I_1}\omega\,\Omega\sin\lambda\sin\theta \approx -(2\,\omega\,\Omega\sin\lambda)\theta, \tag{72}$$

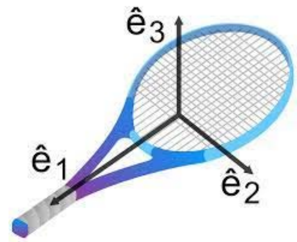
so small oscillations in θ have angular frequency $\sqrt{2\,\omega\,\Omega\sin\lambda}$ as found in part (a). The 3-component of eq. (70) is,

$$0 = I_3(\dot{\omega} - \Omega\,\dot{\theta}\sin\lambda\sin\theta) + I_2\Omega\,\dot{\theta}\sin\lambda\sin\theta \approx I_3\,\dot{\omega}, \qquad \omega \approx \text{constant}, \tag{73}$$

noting that $\dot{\theta} \ll \omega$, such that terms in $\omega\,\dot{\theta}$ are negligible.

5. **The Tennis Racquet Theorem.**

Consider a rigid body whose principal moments of inertia are $I_1 < I_2 < I_3$. We have claimed that free rotation with angular velocity $\boldsymbol{\omega}$ pointing close to axis 2 is "unstable".



For a kind of exception to this behavior, we consider the special case where the kinetic energy has the form $T = L^2/2I_2$, and $\mathbf{L}$ is the angular momentum about the center of mass. In general,

$$T = \frac{I_1\,\omega_1^2}{2} + \frac{I_2\,\omega_2^2}{2} + \frac{I_3\,\omega_3^2}{2}\,, \tag{74}$$

$$\mathbf{L} = I_1\omega_1\,\hat{\mathbf{1}} + I_2\omega_2\,\hat{\mathbf{2}} + I_3\omega_3\,\hat{\mathbf{3}}, \qquad L^2 = I_1^2\,\omega_1^2 + I_2^2\,\omega_2^2 + I_3^2\,\omega_3^2, \tag{75}$$

so for the special case, $2I_2T = I_1I_2\,\omega_1^2 + I_2^2\,\omega_2^2 + I_2I_3\,\omega_3^2 = L^2$. From eq. (75), we can also write $I_3^2\,\omega_3^2 = L^2 - I_1^2\,\omega_1^2 - I_2^2\,\omega_2^2$, which leads to,

$$2I_2I_3T = I_1I_2I_3\,\omega_1^2 + I_2^2I_2\,\omega_2^2 + I_2(L^2 - I_1^2\,\omega_1^2 - I_2^2\,\omega_2^2) = I_3L^2, \tag{76}$$

$$\omega_1^2 I_1I_2(I_3 - I_2) = (I_3 - I_2)(L^2 - I_2^2\,\omega_2^2), \tag{77}$$

$$\omega_1^2 = \frac{I_3 - I_2}{I_3 - I_1}\frac{L^2 - I_2^2\omega_2^2}{I_1I_2}\,, \qquad \text{and with } 1 \leftrightarrow 3, \qquad \omega_3^2 = \frac{I_2 - I_1}{I_3 - I_1}\frac{L^2 - I_2^2\omega_2^2}{I_2I_3}\,. \tag{78}$$

Then, Euler's equation for $\dot{\omega}_2$ in torque-free motion leads to,

$$\dot{\omega}_2 = -\frac{I_1 - I_3}{I_2}\omega_1\omega_3 = -\frac{I_1 - I_3}{I_2}\sqrt{\frac{I_3 - I_2}{I_3 - I_1}\frac{I_2 - I_1}{I_3 - I_1}}\frac{L^2 - I_2^2\omega_2^2}{I_2\sqrt{I_1I_3}}$$

$$= \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1I_3}}\left(\frac{L^2}{I_2^2} - \omega_2^2\right), \tag{79}$$

$$\frac{d\,\omega_2}{\omega_{2,\max}^2 - \omega_2^2} = k\,dt, \qquad \text{with} \qquad k = \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1I_3}}\,, \qquad \omega_{2,\max} = \frac{L}{I_2}\,, \tag{80}$$

$$\frac{1}{\omega_{2,\max}}\tanh^{-1}\frac{\omega_2}{\omega_{2,\max}} = k(t - t_0), \qquad \omega_2 = \omega_{2,\max}\tanh[k\,\omega_{2,\max}\,(t - t_0)], \tag{81}$$

using Dwight 140.1, http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf

Then, from eq. (78),

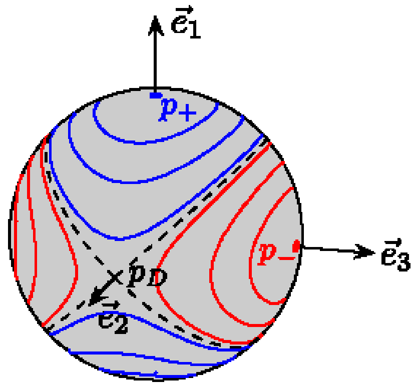
$$\omega_1^2 = \frac{I_3 - I_2}{I_3 - I_1}\frac{I_2}{I_1}(\omega_{2,\max}^2 - \omega_2^2) = \frac{I_3 - I_2}{I_3 - I_1}\frac{I_2}{I_1}\omega_{2,\max}^2\,\mathrm{sech}^2[k\,\omega_{2,\max}\,(t - t_0)], \tag{82}$$

$$\omega_1 = \omega_{2,\max}\sqrt{\frac{I_2}{I_1}\frac{I_3 - I_2}{I_3 - I_1}}\mathrm{sech}[k\,\omega_{2,\max}\,(t - t_0)] = \omega_{1,\max}\,\mathrm{sech}[k\,\omega_{2,\max}\,(t - t_0)]. \tag{83}$$

And, exchanging indices 1 and 3,

$$\omega_3 = \omega_{2,\max}\sqrt{\frac{I_2}{I_3}\frac{I_2 - I_1}{I_3 - I_1}}\text{sech}[k\,\omega_{2,\max}\,(t - t_0)] = \omega_{3,\max}\,\text{sech}[k\,\omega_{2,\max}\,(t - t_0)]. \qquad (84)$$

As $t \to \infty$, ω_1, $\omega_3 \to 0$, while $\omega_2 \to \omega_{2,\max}$, and the final rotation is about axis 2. Thus, for this special case of motion along "separating polhodes", a kind of stability occurs. That is, the motion consider in this problem is along the dashed lines in the figure below (from http://kirkmcd.princeton.edu/examples/mechanics/vandamme_physica_d338_17_17.pdf).



In practice, the special case is hard to achieve, since for any slight perturbation of the kinetic energy T away from $L^2/2I_2$, $\boldsymbol{\omega}$ will move towards axis **2** along a path close to one of the separating polhodes, then "bounce away" from the $\hat{\mathbf{2}}$ axis and move towards axis $-\hat{\mathbf{2}}$ along a path close to the other separating polhode, "bounce away" from this axis, and repeat the cycle.... To a viewer of the spinning tennis racquet, this cycle seems "unstable" because the axis of rotation migrates between $\hat{\mathbf{2}}$ and $-\hat{\mathbf{2}}$ every half cycle, although in the mathematical sense it is a "stable" orbit.

We infer from this problem that the cycle time for trajectories very close to the separating polhodes is very long, approaching infinity in the limit considered here. The long period of such cycles contributes to the impression by the "casual" observer that the motion is "unstable".

The tennis-racquet theorem was first deduced by L. Poinsot, *Théorie Nouvelle de la Rotation des Corps* (Bachlier, 1851),

http://kirkmcd.princeton.edu/examples/mechanics/poinsot_motion_34.pdf

http://kirkmcd.princeton.edu/examples/mechanics/poinsot_motion_34_english.pdf