Princeton University

Ph205
Mechanics
Problem Set 1

Kirk T. McDonald

(1988)

kirkmcd@princeton.edu

http://kirkmcd.princeton.edu/examples/

References:

V. Barger and M. Olsson, *Classical Mechanics: A Modern Perspective* (McGraw-Hill), 1973), http://kirkmcd.princeton.edu/examples/EM/barger_73.pdf

L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed. (Pergamon, 1976), http://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics_76.pdf

H. Goldstein, C.P. Poole and H. Safko, *Classical Mechanics*, 3rd ed. (Addison-Wesley, 2002), http://kirkmcd.princeton.edu/examples/mechanics/goldstein_3ed.pdf

E.J. Routh, *Elementary Rigid Dynamics*, 7th ed. (Macmillan, 1905), http://kirkmcd.princeton.edu/examples/mechanics/routh_elementary_rigid_dynamics.pdf
Advanced Rigid Dynamics, 6th ed. (Macmillan, 1905), http://kirkmcd.princeton.edu/examples/mechanics/routh_advanced_rigid_dynamics.pdf

E.T. Whittaker, *A Treatise on Analytical Dynamics of Particles and Rigid Bodies* (Cambridge U. Press, 1904, 1917, 1927, 1937), http://kirkmcd.princeton.edu/examples/mechanics/whittaker_dynamics_17.pdf

J.H. Jeans, *The Mathematical Theory of Electricity and Magnetism* (Cambridge U. Press, 1908), http://kirkmcd.princeton.edu/examples/EM/jeans_electricity.pdf

H. Lamb, *Dynamics* (Cambridge U. Press, 1914), http://kirkmcd.princeton.edu/examples/mechanics/lamb_dynamics_14.pdf
Higher Mechanics (Cambridge U. Press, 1920), http://kirkmcd.princeton.edu/examples/mechanics/lamb_higher_mechanics.pdf

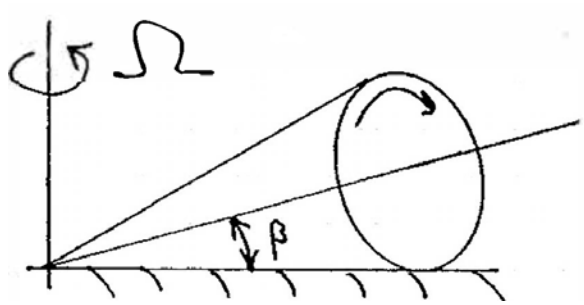
A. Sommerfeld, *Mechanik* (Leipzig, 1943), English translation: *Mechanics* (Academic Press, 1952), http://kirkmcd.princeton.edu/examples/mechanics/sommerfeld_mechanics_52.pdf

E.A. Milne, *Vectorial Mechanics* (Metheun; Interscience Publishers, 1948), http://kirkmcd.princeton.edu/examples/mechanics/milne_mechanics.pdf

L.A. Pars, *Introduction to Dynamics* (Cambridge U. Press, 1953), http://kirkmcd.princeton.edu/examples/mechanics/pars_dynamics_53.pdf

K.R. Symon, *Mechanics* (Addison-Wesley, 1971), 3rd ed. http://kirkmcd.princeton.edu/examples/mechanics/symon_71.pdf

1. A cone of half angle β rolls without slipping on a horizontal plane. The angular velocity about an axis perpendicular to the plane and through the point of the cone is $\mathbf{\Omega}$. What is the angular velocity $\boldsymbol{\omega}$ of the cone about its instantaneous axis of rotation (which passes through the point of the cone? Where is the instantaneous axis?

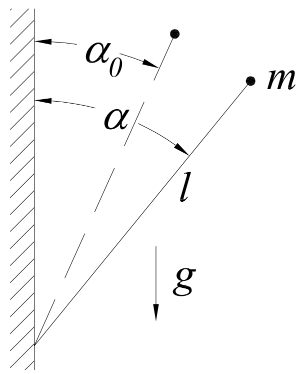


2. *Variant of prob. 2, p. 28 of Barger and Olsson.* What is the maximum deceleration of a car, traveling on the Earth's surface, such that all four of its wheels stay in contact with the ground and roll without slipping?

   The car has total mass M. The center of mass of the car is at height h above the road, and is at horizontal distances d_F and d_R from vertical axes through the front and rear wheels, respectively (and between them). These wheels have masses m_F and m_R, radii r_F and r_R, and moments of inertia $I_F = k_F m_F r_F^2$ and $I_R = k_R m_R r_R^2$. The coefficients of static friction of the wheels and the road are μ_F and μ_R.

   A related problem about maximal acceleration is at
   http://kirkmcd.princeton.edu/examples/rocketcar.pdf

3. A mass m is attached at one end of a massless rigid rod of length l, and the rod is supported at its other end by a frictionless pivot, as shown below. The rod is released from rest at angle $\alpha_0 < \pi/2$ to the vertical. At what angle α does the force in the rod change from compression to tension?



4. (a) A perfectly flexible cable has length L. Initially, length l_0 of the cable hangs at rest over the edge of the a table, and the rest of the cable is along a line perpendicular to the edge of the table. Neglecting friction, and assuming that the part of the cable not on the table is vertical, what is the length l hanging over the edge after time t since the release of the cable?

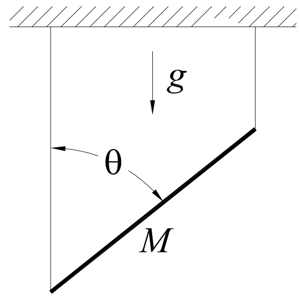
   (b) Suppose the cable is bunched up near the edge of the table, again with length l_0 hanging of the edge initially. As the cable falls, to a good approximation only part hanging over the edge is in motion, again assuming that part to be vertical. Find the velocity of that part of the cable as a function of time. Verify that mechanical energy, KE + PE, is not conserved. Where has the "missing energy" gone?

5. A spring with spring constant k has mass m. A block of mass M is attached to one end of the spring, and its other end is fixed. What is the period of oscillation, ignoring friction?

   Hint: Consider the energy.

   Later in the course we will consider waves on a massive spring, and find that the simple result of this problem is an excellent approximation to the lowest-frequency mode. See Prob. 8 of http://kirkmcd.princeton.edu/examples/ph205set11.pdf.

6. A thin, uniform rod of mass M is suspended from above by two vertical strings, as sketched below, such that the rod makes angle θ to the vertical. What is the tension in the longer string before, and immediately after, the shorter string is cut?



7. A sphere of cross sectional area A moves through air with speed v. Suppose that the air has density ρ, and that all air molecules have the same mass and the same speed s. Assume that all collisions between air molecules and the sphere are completely inelastic, but the resulting change in the mass of the sphere is negligible.

   (a) If $s \ll v$, show that the drag force on the sphere is $F_{\rm drag} = \rho A v^2$.

   (b) If $s \gg v$, show that $F_{\rm drag} \approx \rho A v s$.

   As a simplification, suppose 1/2 of the molecules move in the same direction as the sphere, and the other 1/2 move towards it, head on.

   (c) Repeat parts (a) and (b) for arbitrary s, supposing that the molecular directions are isotropic in the lab frame. Show that for $s \le v$, $F_{\rm drag} = \rho A(v^2 + 4s^2/3 - s^4/15v^2)$, and that for $s \ge v$, $F_{\rm drag} = \rho A(4vs/3 + 4v^3/15s)$.

8. A spherical raindrop falls vertically due to gravity (g = constant here), but experiences a drag force $\mathbf{F} = -kr^2\mathbf{v}$, where r is the radius of the drop and $\mathbf{v}$ is its velocity.

   (a) If the radius r is time independent, find the velocity $v(t)$ supposing $v(0) = 0$.
   How does v behave for small time? That is, if $v(t) \approx gt(1 - \epsilon)$, what is ϵ?
   What is the terminal velocity of the drop (at large times)?

   (b) Suppose at time $t = 0$ the drop has radius r_0 and (vertical) velocity v_0. It then enters a cloud and gains mass according to $dm/dt = \alpha r^2 \propto$ surface area, while the density ρ of the drop remains constant. Then, what is $v(t)$?
   As a special case, show that as $r_0 \to 0$ and $v_0 \to 0$, then,

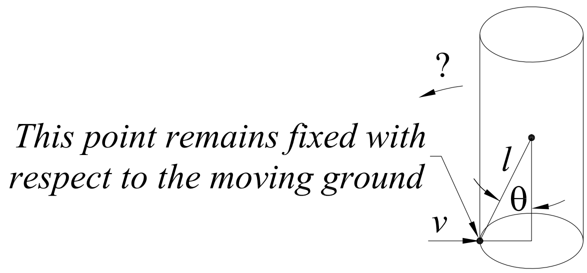
$$v \to \frac{gt}{4 + 3k/\alpha} < \frac{gt}{4}. \tag{1}$$

   In reality, the bottom of a falling raindrop is flattened by the air drag force. See, J.E. McDonald, The Shape of Falling Raindrops, Sci. Am. **190***(2), 64 (1954),* (http://kirkmcd.princeton.edu/examples/fluids/mcdonald_sa_190_2_64_54.pdf

9. **Greek-Temple Seismograph**

Following a major earthquake near Naples in 1857, R. Mallet suggested that a measure of the horizontal velocity of the ground during an earthquake could be deduced from the maximum height of cylindrical columns that remained standing.[1]

Deduce the minimum horizontal velocity v needed to overturn a solid, vertical, cylindrical column whose diagonal has length $2l$ and makes angle θ to the vertical, assuming that a point on the base remains fixed with respect to the moving ground, as sketched below.



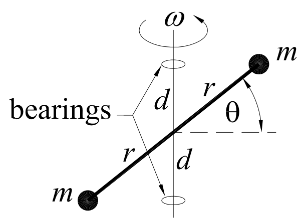
If the cylinder is too squat (large θ), it can lose contact with the ground during its motion. Supposing that the velocity v is the minimum value found above, deduce a condition on the angle θ such that the cylinder always remains in contact with the ground as it falls over.

This problem is taken from secs. 174-175 of E.J. Routh, The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies, *7th ed. (Macmillan, 1905),* http://kirkmcd.princeton.edu/examples/mechanics/routh_elementary_rigid_dynamics.pdf

[1]R. Mallet, *The First Principles of Observational Seismology* (Chapman and Hall, 1862), Vol. 1, Chap. 16, http://kirkmcd.princeton.edu/examples/mechanics/mallet_chap16.pdf

10. **Unbalanced Tire**

Suppose a tire is "balanced" except for two masses m each at distance r from the geometric center of the tire along a line which makes angle $90° - \theta$ to the axle.



A gas-station attendant using an (obsolete) static balance would claim that this tire is balanced.

In this problem, ignore all other mass of the tire except the $2m$. Suppose the tire is not rolling, but is in a special setup (the "balancing" apparatus) with the axle vertical and the center of mass fixed. The tire rotates about the vertical (z) axis with constant angular velocity ω, which is large enough that you may ignore effects of gravity.

(a) What is the angular momentum $\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i$, and the torque $\boldsymbol{\tau} = d\mathbf{L}/dt$?

(b) Suppose the axle is supported by two bearings, each at distance d from the center of mass, along the axle. What is the (vector) force exerted by each bearing (ignoring gravity)?

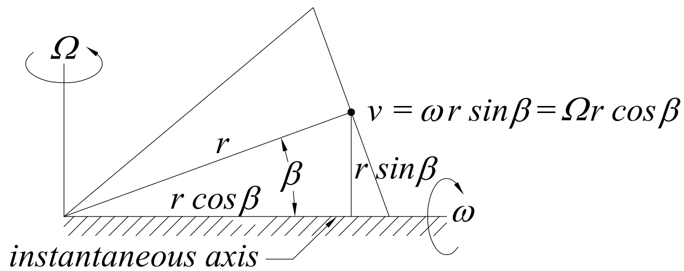
(c) Suppose that at some moment the tire broke free from the bearings (but its center of mass remains fixed, ignoring gravity). Describe the subsequent motion of the tire. What is its period?

(d) Suppose instead that the wheel bolts were loose so that angle θ could vary, but the tire is still forced to rotate about the vertical axis with constant angular velocity ω. Show that the tire oscillates (wobbles) about $\theta = 0$.

What is the angular frequency Ω of oscillation, supposing that θ is small?

Solutions

1. The instantaneous axis is the line of contact of the cone with the plane (just as the instantaneous axis of a wheel that rolls without slipping on a plane is the point/line of contact of the wheel with the plane).



The velocity of a point on the axis of the cone at distance r from its tip is into the page, with

$$v = \omega r \sin\beta = \Omega r \cos\beta, \qquad \omega = \Omega \cot\beta. \tag{2}$$

2. The car moves in the $+x$ direction. The vertical axis is in the $+y$ direction.

The horizontal forces of static friction of the road on the wheels are $\mathbf{F}_F = -F_F\,\hat{\mathbf{x}}$ and $\mathbf{F}_R = -F_R\,\hat{\mathbf{x}}$. The total horizontal force on the (four-wheeled) car is related to the deceleration a by,

$$F_{\rm tot} = 2F_F + 2F_R = Ma, \tag{3}$$

The magnitudes of the frictional forces are bounded by,

$$F_F \le \mu_F N_F, \qquad \text{and} \qquad F_R \le \mu_R N_R, \tag{4}$$

where N_F and N_R are the (upward) normal forces on the front and rear wheels. Of course, the total upward normal force balances the downward force of gravity on the car,

$$2N_F + 2N_R = Mg, \tag{5}$$

where g is the acceleration due to gravity at the Earth's surface, and all four wheels of the car are in contact with the road.

For four all wheels to stay in contact with the road, the total torque about its center of mass must be zero,[2,3]

$$\tau_{\rm cm} = 0 = 2N_F d_F - 2N_R d_R - 2(F_F + F_R)h, \tag{6}$$

$$N_F d_F - N_R d_R = \frac{Mah}{2}\,, \tag{7}$$

using eq. (3).

The normal forces are determined by eqs. (5) and (7),

$$N_F = M\frac{d_R g + ah}{2(d_F + d_R)}\,, \qquad \text{and} \qquad N_R = M\frac{d_F g - ah}{2(d_F + d_R)}\,. \tag{8}$$

These must both be positive for the wheels to be in contact with the road, which implies a limit on the (positive) deceleration a,

$$a \le \frac{d_F}{h} g. \tag{9}$$

We obtain another limit on a by combining eqs.(3), (4) and (8),

$$a = \frac{F_F + F_R}{2M} \le \frac{ah(\mu_F - \mu_R) + g(\mu_F d_R + \mu_R d_F)}{4(d_F + d_R)}\,, \tag{10}$$

$$a \le \frac{\mu_F d_R + \mu_R d_F}{4(d_F + d_R) + h(\mu_R - \mu_F)} g. \tag{11}$$

[2]The center of mass of the car is accelerating, so we can make a torque analysis about the center of mass using only the forces F_i and N_i. However, if we are to make analyses about, say, either the point of contact of the front or rear wheel with the road, we must include torques associated with "coordinate forces".

For discussion of this issue, see http://kirkmcd.princeton.edu/examples/torque.pdf

[3]The brakes apply torques to the wheels via internal forces on the wheels. By Newton's third law, there exist equal and opposite forces and torques of the wheels on the brakes, so the total internal torque associated with the brakes is zero.

If $\mu_F = \mu_R = \mu$, then,

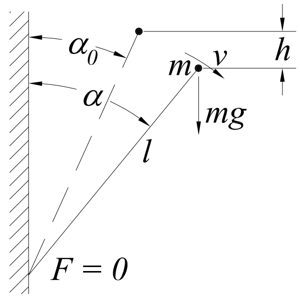
$$a \leq \frac{\mu}{4} g. \tag{12}$$

This limit is often stronger than that of eq. (9).

If either of conditions (4) are not satisfied, wheels will skid.

According to eqs. (4) and (8), the maximal braking force without skidding can be larger on the front wheels that the rear, so cars are built with stronger braked on the front wheels. When the book of Barger and Olsson was written (1973), disc brakes were more expensive than (now larger obsolete) drum brakes, and often used only on front wheels. Nowadays, all wheels on most cars have disc brakes, with larger pads (stronger braking) on the front wheels.

3. When the force in the rod changes from compression to tension, it is momentarily zero. Then, the only force on mass m is $m\mathbf{g}$ due to gravity, and the motion is instantaneously uniform circular motion at velocity $\mathbf{v}$.



The mass has fallen by vertical height,

$$h = l(\cos\alpha_0 - \cos\alpha), \tag{13}$$

so by conservation of energy,

$$\frac{mv^2}{2} = mgh. \tag{14}$$

For the uniform circular motion, the centripetal force is,

$$mg\cos\alpha = \frac{mv^2}{l} = \frac{2mgh}{l} = 2mg(\cos\alpha_0 - \cos\alpha), \tag{15}$$

and hence,

$$\cos\alpha = \frac{2}{3}\cos\alpha_0. \tag{16}$$

4. (a) The equation of motion for the cable of length L, with length l hanging vertically over the edge of the table can be written as,

$$F = m\frac{l}{L}g = m\ddot{l}, \tag{17}$$

assuming the part of the cable on the table is along a line perpendicular to the edge of the table.
The solution has the form,

$$l(t) = A\,e^{\sqrt{g/L}\,t} + B,e^{-\sqrt{g/L}\,t}, \tag{18}$$

with initial conditions,

$$l(0) = l_0, \qquad \dot{l}(0) = 0. \tag{19}$$

Hence,

$$A + B = l_0, \qquad A - B = 0, \qquad A = \frac{l_0}{2} = -B, \tag{20}$$

and

$$l(t) = l_0 \cosh\sqrt{\frac{g}{L}}\,t. \tag{21}$$

(b) If instead the part of the cable on the table is bunched up so that it remains essentially at rest at all times, the equation of motion can be written as,

$$F = \rho l g = \frac{d}{dt}(\rho v l) = \rho\frac{d(l\dot{l})}{dt} \qquad (l\dot{l})\frac{d(l\dot{l})}{dt} = l^2\dot{l}g, \tag{22}$$

where $\rho = m/L$ and $v = \dot{l}$ is the velocity of the part of the cable hanging over the edge of the table, assuming that part to be vertical at all times. This can be integrated to find,

$$\frac{(l\dot{l})^2}{2} = \frac{(l^3 - l_0^3)g}{3}\,. \tag{23}$$

The kinetic energy is

$$\mathrm{KE} = \frac{\rho\dot{l}^2}{2} = \rho g\frac{l^3 - l_0^3}{3l^2} = \rho g\frac{l - l_0}{3}\left(1 + \frac{l_0^2}{l^2}\right), \tag{24}$$

while the potential energy, defined to be zero initially, is,

$$\mathrm{PE} = \rho g\frac{l_0 - l}{2}. \tag{25}$$

Since KE + PE $\neq 0$, mechanical energy is not conserved.

In this problem, during each small time interval, a small portion of the chain is yanked from being at rest to being at velocity v, which implies a kind of inelastic collision has taken place.

If the initial length l_0 of the chain over the edge of the table is very small, then the equation of motion is approximately,

$$\dot{l}^2 \approx \frac{2gl}{3}, \qquad \frac{1}{\sqrt{l}}\frac{dl}{dt} \approx \sqrt{\frac{2g}{3}}, \tag{26}$$

which integrates to,

$$2\sqrt{l} \approx \sqrt{\frac{2g}{3}}t, \qquad l \approx \frac{gt^2}{6}, \qquad \ddot{l} \approx \frac{g}{3} \tag{27}$$

The kinetic energy of the chain is KE $= \rho l\dot{l}^2/2$, and its gravitational potential energy can be written as $V = -\rho g l^2/2$. If mechanical energy were conserved, we would have $l\dot{l}^2 - gl^2 = -gl_0^2$, and for small l_0, $\dot{l}^2 \approx gl$, *i.e.*., $\ddot{l} \approx g/2$.

This problem was first discussed by Cayley (1857),
http://kirkmcd.princeton.edu/examples/mechanics/cayley_prsl_8_506_57.pdf
See also http://kirkmcd.princeton.edu/examples/string.pdf

5. We assume that the spring of mass m and rest length x_0 stretched uniformly when the attached mass M is moved from x_0 to x.

   The potential energy of the stretched spring is then,

$$\mathrm{PE} = \frac{1}{2}k(x - x_0)^2, \tag{28}$$

   where k is the spring constant.

   If mass M is let free from a stretched position of the spring and has velocity v when at position x, the kinetic energy of the system is,

$$\mathrm{KE} = \frac{1}{2}Mv^2 + \frac{1}{2}\int_0^{x_0} \frac{m}{x_0}\,dx' \left(\frac{x'}{x_0}v\right)^2 = \frac{1}{2}\left(M + \frac{m}{3}\right)v^2 \equiv \frac{1}{2}M'v^2. \tag{29}$$

   If the spring were massless, the angular frequency of oscillation of mass M would be,

$$\omega = \sqrt{\frac{k}{M}}\,. \tag{30}$$

   We make a small leap to infer that in case of a massive spring, the frequency of oscillation is,

$$\omega = \sqrt{\frac{k}{M'}} = \sqrt{\frac{k}{M + m/3}}\,. \tag{31}$$

   If there were no attached mass M, we infer that the frequency of oscillation of the massive spring along would be,

$$\omega = \sqrt{\frac{3k}{m}}\,. \tag{32}$$

6. When both strings are intact, the tension in each is the same, $Mg/2$, such that they carry the weight of the rod, and that the torque about its center of mass is zero.

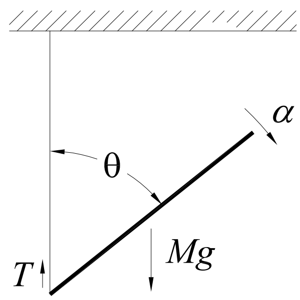
   Just after the shorter string is cut, the tension in the longer string is the unknown T, whose direction is still upwards. Then, the total force on the rod is $Mg - T$, also vertical, so the acceleration of the center of mass of the rod is vertical, with magnitude,

$$a = g - \frac{T}{M} . \tag{33}$$

   The initial angular acceleration of the rod about its lowest point is,

$$\alpha = \frac{a}{(l/2)\sin\theta} , \tag{34}$$

   where l is the length of the rod. This is also the initial angular acceleration of the rod about its center of mass.



   The torque equation for this angular acceleration is,[4]

$$\tau = T\frac{l}{2}\sin\theta = I_{\rm cm}\alpha = \frac{Ml^2}{12}\frac{g - T/M}{(l/2)\sin\theta} , \tag{36}$$

   recalling that the moment of inertia of the rod about its center of mass is $I_{\rm cm} = Ml^2/12$, and hence,

$$3T\sin^2\theta = Mg - T, \qquad T = \frac{Mg}{1 + 3\sin^2\theta} . \tag{37}$$

   If the rod was initially horizontal, $\theta = \pi/2$, the tension in one string just after cutting other is $Mg/4$, while if the rod was initially vertical, $\theta = 0$ or π, the tension in the longer string just after cutting the shorter one is Mg.

---

[4]As the center of mass of the system is accelerating, it is prudent to make the torque analysis about the center of mass to avoid consideration of possible "fictitious" forces associated with accelerated coordinate systems.

For example, the lowest point on the rod (where the longer string is attached) is still at rest immediately after the other string is cut, so it is tempting to make an analysis about this point as well. Recalling that the moment of inertia of a rod about either end is $I = ml^2/3$, we would then find (ignoring any "fictitious" torques),

$$\tau = Mg\frac{l}{2}\sin\theta = I\alpha = \frac{Ml^2}{3}\frac{g - T/M}{(l/2)\sin\theta} , \qquad \frac{3}{4}Mg\sin^2\theta = Mg - T, \qquad T = Mg\left(1 - \frac{3}{4}\sin^2\theta\right) . \tag{35}$$

This happens to agree with eq. (37) for $\theta = 0$, $\pi/2$ and π, but not in general.

7. (a) If the velocity v of the sphere is large compared to the velocity a of the air molecules, the latter can be regarded as at rest.

In time Δt the sphere, of cross sectional area A, sweeps through volume $\Delta V = Av\Delta t$, and accumulates mass $\Delta m = \rho\Delta V = \rho Av\Delta t$, supposing the collisions with the air molecules were completely inelastic, and ρ is the mass density of air.

The molecules that stuck to the moving sphere took on momentum $\Delta P = \Delta mv = \rho Av^2\Delta t$, so the reaction/drag force on the moving sphere would be

$$F_{\text{drag}} = \frac{\Delta P}{\Delta t} = \rho Av^2. \tag{38}$$

(b) If the speed s of the molecules is large compared to v of the moving sphere, and half the molecules move towards the sphere, during time Δt the mass of molecules swept up by the front rear of the sphere is $\Delta m_{\text{front}} = \rho A(v+s)\Delta t/2$, while the mass of molecules that overtake the rear of the sphere is $\Delta m_{\text{front}} = \rho A(s-v)\Delta t/2$.

The change in velocity of the molecules that stuck to the front of the sphere is from $-s$ to v, so $\Delta v = v - (-s) = v + s$, and the change in momentum of these molecules is $\Delta P_{\text{front}} = \rho A(v+s)^2\Delta t/2$.

The change in velocity of the molecules that stuck to the rear of the sphere is from s to v, so $\Delta v = v - s$, and the change in momentum of these molecules is $\Delta P_{\text{rear}} = -\rho A(s-v)^2\Delta t/2$.

The total change in momentum is $\Delta P = 2\rho Avs\Delta t$, so the corresponding drag force on the moving sphere is,

$$F_{\text{drag}} = \frac{\Delta P}{\Delta t} = 2\rho Avs. \tag{39}$$

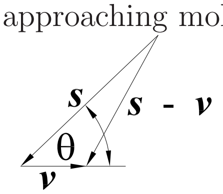
(c) We now suppose that the air molecules all have the same speed s and their directions are isotropic.

In our analysis we first suppose all the molecules has a single velocity $\mathbf{s}$, and later average of the direction of $\mathbf{s}$.

To compute the number of these molecules that stick to the moving sphere during time Δt, we go to a frame in which the sphere is at rest, and the velocity of the molecules is $\mathbf{s} - \mathbf{v}$. The mass density of the molecules in this frame is still ρ (supposing that both v and s are small compared to the speed of light, so that we can neglect effects of special relativity). So, the mass density swept up in time Δt is,

$$\Delta m = \rho A\,|\mathbf{s} - \mathbf{v}|\,\Delta t = \rho A\sqrt{v^2 + s^2 + 2vs\cos\theta}\,\Delta t, \tag{40}$$

where the direction of the approaching molecules makes angle θ to $\mathbf{v}$.



After averaging over directions, the drag force will be in the direction of **v**, so we are interested in the change of momentum (in the lab frame) of the stuck molecules in that direction, namely,

$$\Delta P_v = \Delta m(v - (-s\cos\theta)) = \Delta m(v + s\cos\theta)$$
$$= \rho A(v + s\cos\theta)\sqrt{v^2 + s^2 + 2vs\cos\theta}\,\Delta t. \tag{41}$$

To find the drag force, we average the momentum change (41) over θ and divide by Δt. For isotropic directions of the molecules their numbers vary with θ as $(1/2)d\cos\theta$, so the drag force is,

$$F_{\rm drag} = \int_{-1}^{1} \frac{d\cos\theta}{2}\rho A(v + s\cos\theta)\sqrt{v^2 + s^2 + 2vs\cos\theta}$$
$$= \frac{\rho A}{2}\left[\frac{(v^2 + s^2 + 2vs\cos\theta)^{3/2}}{3s}\right.$$
$$\left. + \frac{2s}{4v^2s^2}\left(\frac{(v^2 + s^2 + 2vs\cos\theta)^{5/2}}{5} - \frac{(v^2 + s^2)(v^2 + s^2 + 2vs\cos\theta)^{3/2}}{3}\right)\right]_{-1}^{1}, \tag{42}$$

using Dwight 193.01 and 193.11, http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf.
To go further, we must consider the cases $s > v$ and $s < v$ separately. After some algebra, we find,

$$F_{\rm drag} = \frac{4\rho Avs}{3} + \frac{4\rho Av^3}{15s} \qquad (s > v), \tag{43}$$
$$= \rho Av^2 + \frac{2\rho As^2}{3} - \frac{\rho As^4}{15v^2} \qquad (s < v) \tag{44}$$

If $v = s$, both eq. (43) and (44) yield $F_{\rm drag} = 4\rho Av^2/15$.

8. (a) For a drag force $F = -kr^2 v$ on a spherical raindrop of constant radius r and vertical velocity v, the equation of motion can be written as,

$$m\frac{dv}{dt} = mg - kr^2 r, \qquad \frac{dv}{dt} = g - \frac{kr^2}{m}v. \tag{45}$$

This first-order, linear differential equation has the solution,

$$v(t) = \frac{mg}{kr^2} + C\, e^{-kr^2 t/m}. \tag{46}$$

For $v(0)$, we have,

$$v(t) = \frac{mg}{kr^2}\left(1 - C\, e^{-kr^2 t/m}\right). \tag{47}$$

For small t,

$$v(t \ll m/kr^2) \approx \frac{mg}{kr^2}\frac{kr^2 t}{m} = gt, \tag{48}$$

which is just free fall.

For large t, the drop falls with the terminal velocity,

$$v_{\text{terminal}} = \frac{mg}{kr^2} = \frac{4\pi\rho g r}{3k}, \tag{49}$$

where ρ is the mass density of the drop, such that $m = 4\pi\rho r^3/3$.

(b) For a raindrop inside a cloud, a possible model of the growth of the drop is that $dm/dt = \alpha r^2$. Noting that $m = 4\pi\rho r^3/3$, we have that,

$$\frac{dm}{dt} = 4\pi\rho r^2\frac{dr}{dt}\left(= \frac{3m}{r}\frac{dr}{dt}\right) = \alpha r^2, \qquad \frac{dr}{dt} = \frac{\alpha}{4\pi\rho} \equiv \beta, \qquad r(t) = r_0 + \beta t. \tag{50}$$

We can write the equation of motion of the variable-mass drop as,

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt} = mg - kr^2. \tag{51}$$

Hence,

$$\frac{dv}{dt} = g - \frac{v}{m}\frac{dm}{dt} - \frac{kr^2 v}{m} = g - \frac{3v}{r}\frac{dr}{dt} - \frac{3kv}{4\pi\rho r}, \tag{52}$$

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = g - \frac{v}{m}\frac{dm}{dt} - \frac{kr^2 v}{m} = g - \frac{3v}{r}\frac{dr}{dt} - \frac{3kv}{4\pi\rho r}, \tag{53}$$

$$\frac{dv}{dr} = \frac{g}{\beta} - \frac{3v}{r} - \frac{3kv}{4\pi\beta\rho r} = \frac{g}{\beta} - \frac{3v}{r}\left(1 + \frac{k}{\alpha}\right). \tag{54}$$

We try a solution of the form $v = Ar^i + Br^j$, for which,

$$\frac{dv}{dr} = \frac{iAr^i}{r} + jBr^{j-1} = \frac{i}{r}v + (j-i)Br^{j-1}. \tag{55}$$

This works (a small miracle) for,

$$i = -3\left(1 + \frac{k}{\alpha}\right) \equiv -\gamma, \qquad j = 1, \qquad (1+\gamma)B = \frac{g}{\beta}. \tag{56}$$

So far, we have that,

$$v(r) = \frac{A}{r^\gamma} + \frac{gr}{\beta(1+\gamma)}. \tag{57}$$

We also have that,

$$v(r_0) = v_0 = \frac{A}{r_0^\gamma} + \frac{gr_0}{\beta(1+\gamma)}, \qquad A = r_0^\gamma\left(v_0 - \frac{gr_0}{\beta(1+\gamma)}\right). \tag{58}$$

Finally, we note that $r = r_0 + \beta t$ to write,

$$v(t) = \left(\frac{r_0}{r_0 + \beta t}\right)^\gamma \left(v_0 - \frac{gr_0}{\beta(1+\gamma)}\right) + \frac{g(r_0 + \beta t)}{\beta(1+\gamma)}. \tag{59}$$

There is no terminal velocity in this model, so long as the drop is inside the cloud. For small r_0 and small v_0,

$$v(t) \approx \frac{gt}{(1+\gamma)} = \frac{gt}{4 + 3k/\alpha} < \frac{gt}{4}. \tag{60}$$

9. The effect on the vertical cylinder of a sudden horizontal velocity v of the ground is equivalent to the case of the ground remaining at rest and the center of mass of the cylinder being given a sudden horizontal velocity v. In either view, a large impulsive force acts at the point on the base of the cylinder that remains fixed relative to the ground. This force exerts no torque about the fixed point, so the change ΔL in angular momentum of the cylinder during the impulse equals that associated with the sudden horizontal velocity of the center of mass (in the frame where the ground remains at rest). Hence,

$$\Delta L = mvl\cos\theta = I_p\,\omega_0, \qquad \text{and so} \qquad \omega_0 = \frac{mvl\cos\theta}{I_p}\,, \tag{61}$$

where m is the mass of the cylinder, I_p is its moment of inertial about the fixed point, and ω_0 is the initial angular velocity just after the impulse. Recalling that the moment of inertia of a thin disc about an a diameter is $mr^2/4$, the moment of inertia I_p is, using the parallel axis theorem,

$$I_p = \int_0^h \frac{m\,dy}{h}\left(\frac{r^2}{4} + r^2 + y^2\right) = \frac{m}{4}\left(5r^2 + \frac{4h^2}{3}\right) = \frac{ml^2}{12}(15\sin^2\theta + 16\cos^2\theta)$$
$$= \frac{ml^2}{12}(15 + \cos^2\theta), \tag{62}$$

noting that $r = l\sin\theta$ and $h = 2l\cos\theta$.

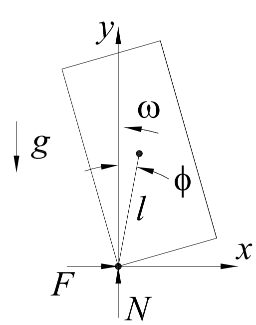
The column will fall over if the initial kinetic energy $I_p\,\omega_0^2/2$ just after the earthquake is sufficient that the center of mass of the column can rise from $h/2 = l\cos\theta$ to l. Hence, the minimum velocity of the ground needed to topple the column is related by,

$$mgl(1-\cos\theta) = \frac{I_p\,\omega_{0,\min}^2}{2} = \frac{m^2 v_{\min}^2 l^2\cos^2\theta}{2I_p}\,, \tag{63}$$

and so,

$$v_{\min}^2 = \frac{2gI_p(1-\cos\theta)}{ml\cos^2\theta} = \frac{gl}{6}\,\frac{(1-\cos\theta)(15+\cos^2\theta)}{\cos^2\theta}\,. \tag{64}$$

As the column rotates about the fixed point with angular velocity $\omega(\phi)$, where ϕ is the angle of the diagonal to the vertical, it will lose contact with the ground if the normal force N goes to zero.



Referring to the figure above, the y-equation of motion of the center of mass of the cylinder is,

$$F_y = m\ddot{y} = m\frac{d^2}{dt^2}(l\cos\phi) = ml\frac{d}{dt}(\omega\sin\phi) = ml(\dot{\omega}\sin\phi - \omega^2\cos\phi) = N - mg, \quad (65)$$

noting that $d\phi/dt \equiv \dot{\phi} = -\omega$. The normal force goes to zero if there is an angle ϕ such that,

$$g = l(\omega^2\cos\phi - \dot{\omega}\sin\phi). \quad (66)$$

As the column rotates (about the z-axis), conservation of energy relates ω and ϕ according to,

$$\omega^2 = \omega_0^2 - 2\frac{mgl}{I_p}(\cos\phi - \cos\theta). \quad (67)$$

If we restrict our attention to the case that the velocity of the ground is the minimum value (64), then using eq. (63) in (67) yields,

$$\omega^2 = \frac{2mgl}{I_p}(1 - \cos\phi). \quad (68)$$

Taking the time derivative of eq. (68) we find that,

$$\dot{\omega} = -\frac{mgl}{I_p}\sin\phi, \quad (69)$$

and the condition (66) becomes,

$$\frac{I_p}{mgl^2} = \frac{15 + \cos^2\theta}{12} = 2\cos\phi(1 - \cos\phi) + \sin^2\phi = -3\cos^2\phi + 2\cos\phi + 1, \quad (70)$$

or,

$$3\cos^2\phi - 2\cos\phi + \frac{3 + \cos^2\theta}{12} = 0. \quad (71)$$

Thus, the column will loose contact with the ground if,

$$\cos\phi = \frac{2 \pm \sqrt{4 - 12(3 + \cos^2\theta)/12}}{6} = \frac{2 \pm \sin\theta}{6}. \quad (72)$$

As the column rotates, angle ϕ is less than θ (and $\phi_0 = \theta$) so that $\cos\phi \geq \cos\theta$. If the column is to remain in contact with the ground at all times, we must have that neither solution (72) is greater than $\cos\theta$, *i.e.*,

$$\cos\theta > \frac{2 + \sin\theta}{6}. \quad (73)$$

The critical angle is roughly 61.3°, so if the height of the column is more than 1.83 times its diameter it will remain in contact with the ground at all times while it falls over after an earthquake that is minimally capable of causing this.

10. (a) The angular momentum of our model of the unbalanced tire is, for a time when the positions $\mathbf{r}_i$ of the two masses m are $\pm\mathbf{r} = \pm r(\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{z}})$, and $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$,

$$\mathbf{L} = \sum_{i=1,2} \mathbf{r}_i \times \mathbf{p}_i = \sum_i \mathbf{r}_i \times m(\boldsymbol{\omega} \times \mathbf{r}_i) = m\sum_i [r_i^2\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\mathbf{r}_i)\mathbf{r}_i]$$
$$= 2mr^2\omega[\hat{\mathbf{z}} - \sin\theta)(\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{z}})] = 2mr^2\omega\cos\theta(\cos\hat{\mathbf{z}} - \sin\theta\,\hat{\mathbf{x}}). \qquad (74)$$

Note that $\mathbf{L}\cdot\mathbf{r} = 0$, *i.e.*, the angular momentum $\mathbf{L}$ is perpendicular to the line along which the two masses m lie. Also, the magnitude of $\mathbf{L}$ is $L = 2mr^2\omega\cos\theta$. This motion requires a torque,

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \frac{d}{dt}\sum_i \mathbf{r}_i \times m\frac{d\mathbf{r}_i}{dt} = m\sum_i \mathbf{r}_i \times \frac{d^2\mathbf{r}_i}{dt^2}. \qquad (75)$$

This example involves rigid-body rotation about the z-axis at angular velocity ω, so for any vector $\mathbf{A}$ associated with that body,

$$\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega} \times \mathbf{A}, \qquad (76)$$

and

$$\frac{d^2\mathbf{A}}{dt^2} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{A}) = (\boldsymbol{\omega}\cdot\mathbf{A})\boldsymbol{\omega} - \omega^2\mathbf{A}). \qquad (77)$$

Hence,

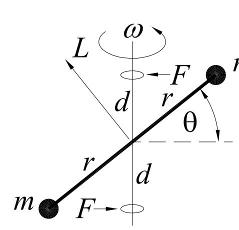
$$\frac{d^2\mathbf{r}_i}{dt^2} = (\boldsymbol{\omega}\cdot\mathbf{r}_i)\boldsymbol{\omega} - \omega^2\mathbf{r}_i, \qquad \mathbf{r}_i \times \frac{d^2\mathbf{r}_i}{dt^2} = (\boldsymbol{\omega}\cdot\mathbf{r}_i)\mathbf{r}_i \times \boldsymbol{\omega} = -\omega^2 r^2\sin\theta\cos\theta\,\hat{\mathbf{y}}, \qquad (78)$$

and the required torque is,

$$\boldsymbol{\tau} = -2m\omega^2 r^2\sin\theta\cos\theta\,\hat{\mathbf{y}}, \qquad (79)$$

(b) The torque $\boldsymbol{\tau}$ is supplied by the equal-and-opposite forces $\mathbf{F}_j$ applied by two bearings along the z-axis at positions $\pm d\,\hat{\mathbf{z}}$. The force $\mathbf{F}_u$ on the upper bearing is in the $-x$ direction, and has magnitude,

$$F_u = \frac{m\omega^2 r^2}{d}\sin\theta\cos\theta. \qquad (80)$$



(c) If the tire broke free of the bearings, and we ignore gravity, then the center of mass remains fixed, as does the angular momentum $\mathbf{L}$. Then, the tire would rotate about the constant angular momentum vector (which is perpendicular to the line joining the two mass) with angular velocity $\omega' = L/I = L/2mr^2 = \omega\cos\theta$.

(d) If angle θ is free to vary while the angular velocity $\boldsymbol{\omega}$ about the vertical axis is maintained, we note that, in the rotating frame, each mass experiences a centrifugal force in the horizontal direction of magnitude $m\omega^2 r cos\theta$, which is just $m\omega^2 r$ for small θ. For small oscillations, each mass experiences an effective "gravity" $g_{\rm eff} = \omega^2 r$ direction horizontally outward, and so oscillates like a simple pendulum about $\theta = 0$ with angular frequency $\Omega = \sqrt{g_{\rm eff}/r} = \omega$.