Supplemental Notes on Complex Numbers, Complex Impedance, RLC Circuits, and Resonance

Complex numbers

Complex numbers are expressions of the form

$$z = a + ib,$$

where both a and b are real numbers, and $i = \sqrt{-1}$. Here a is called the *real part* of z, denoted by $a = \mathrm{Re}\,(z)$, and b the *imaginary part* of z, $b = \mathrm{Im}\,(z)$. A complex number is thus specified by two real numbers, a and b, and therefore it is convenient to think of it as a two-dimensional vector, plotting the real part on the x-axis, and the imaginary part on the y-axis. This is called the *complex plane*.

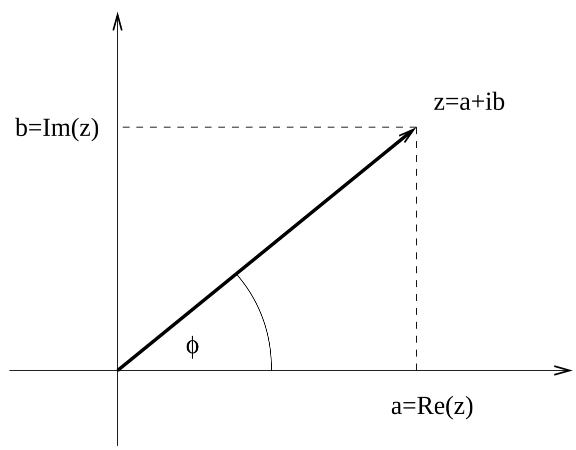


Figure 1: The complex plane

One can manipulate complex numbers like real numbers. For instance, we can add $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$:

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2).$$

Just like with vectors, we just have to add the components, which are here the real and imaginary parts. We can also multiply complex numbers:

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_2 b_1 + a_1 b_2),$$

where we used that $i^2 = -1$.

A useful notion is the one of the *complex conjugate* of z, denoted by z^*. It is obtained by multiplying the imaginary part by (-1), which means the we are reflecting the vector in the complex plane across the x-axis. I.e., if $z = a + ib$, then $z^* = a - ib$. If we now calculate

$$zz^* = (a + ib)(a - ib) = a^2 + iab - iab + b^2 = a^2 + b^2,$$

we see that zz^* is a positive real number, and $\sqrt{zz^*}$ is just the length of the vector z in the complex plane.

To calculate the quotient of two complex numbers, we multiply both the denominator and the numerator with the complex conjugate of the denominator:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)} \\ &= \frac{a_1a_2 + ia_2b_1 - ia_1b_2 + b_1b_2}{a_2^2 + b_2^2} = \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2}. \end{aligned}$$

Example: We calculate $1/i$. The complex conjugate of i is $-i$, hence we get

$$\frac{1}{i} = \frac{(1)(-i)}{(i)(-i)} = \frac{-i}{1} = -i.$$

Thus $1/i = -i$.

Let's check this result. i times it's inverse must of course be 1. We have

$$i\,\frac{1}{i} = 1 = i(-i) = -i^2 = -(-1) = 1.$$

It works out!

Since a complex number can be thought of as a two-dimensional vector, it can be specified either by its components (the real and imaginary parts), or by its length and the angle it makes with the x-axis (see figure 1). We already saw that the length, denoted by $|z|$, is given by $|z| = \sqrt{zz^*}$. From figure 1, we see that the angle is given by

$$\tan\phi = \frac{\mathrm{Im}\,(z)}{\mathrm{Re}\,(z)}.$$

In particular,

$$z = |z|\left(\cos\phi + i\sin\phi\right).$$

This can be conveniently rewritten, making use of Euler's formula:

$$e^{i\phi} = \cos\phi + i\sin\phi. \tag{1}$$

This formula can be derived by a Taylor expansion of both the exponential and the sine and cosine. It tells as the the complex number $e^{i\phi}$ is a vector of length 1 that makes an angle ϕ with the x-axis. Hence we see that any complex number z can be written as

$$z = |z|e^{i\phi},$$

$|z|$ being the magnitude of z and ϕ being the angle between z and the x-axis.

Example: Let's calculate the magnitude and angle of $z = -1 + i$. The magnitude is $\sqrt{zz^*}$, and $zz^* = (-1+i)(-1-i) = 2$. The angle is determined by $\tan\phi =$

$\text{Im}\,(z)/\text{Re}\,(z) = 1/(-1) = -1$. There are many ϕ's satisfying this equation, namely $\phi = -\pi/4 + n\pi$ for any integer n. Since we know that z has negative real part and positive imaginary part, ϕ must lie between $\pi/2$ and π, hence $\phi = 3\pi/4$.

This representation is very useful, for instance for multiplying or dividing two complex numbers: if $z_1 = |z_1|e^{i\phi_1}$ and $z_2 = |z_2|e^{i\phi_2}$, then

$$z_1 z_2 = |z_1||z_2|e^{i(\phi_1+\phi_2)} \quad , \qquad \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{i(\phi_1-\phi_2)}.$$

In this way we can also easily take roots of complex numbers:

$$\sqrt{z} = \sqrt{|z|e^{i\phi}} = \sqrt{|z|}\, e^{i\phi/2}.$$

Example: We calculate $\sqrt{i}$. We first write $i = e^{i\pi/2}$, which can be seen from Euler's formula. Hence

$$\sqrt{i} = \left(e^{i\pi/2}\right)^{1/2} = e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4) = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}.$$

We know that $\sqrt{i}^2$ must be equal to i. Let's check that our calculation was correct:

$$\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + 2i\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{i^2}{2} = \frac{1}{2} + i - \frac{1}{2} = i.$$

Complex Impedance

Alternating currents can be described by two numbers, their magnitude and their phase. In this respect, they are just like complex numbers. It is convenient to think of an alternating current as a two-dimensional vector (called a *phasor* in the book) that has a given magnitude $|I_0|$ and rotates around the origin at a given frequency $f = \omega/(2\pi)$. The observed current is the x-component of this vector, which oscillates in time. The same applies to an alternating voltage $\mathcal{E}(t)$. We can think of it as a complex quantity, denoted by $\widetilde{\mathcal{E}}(t)$,

$$\widetilde{\mathcal{E}}(t) = \mathcal{E}_0 e^{i\omega t},$$

where $\mathcal{E}_0$ is the *magnitude* of the voltage $\mathcal{E}$, which is a positive (and in particular real) number. (For simplicity, we assume that at time $t = 0$ the voltage is maximal. If this is not the case, $\mathcal{E}_0$ itself would contain a non-zero phase.) The physically observed voltage is the real part of $\widetilde{\mathcal{E}}(t)$,

$$\mathcal{E}(t) = \mathrm{Re}\,(\widetilde{\mathcal{E}}(t)) = \mathcal{E}_0 \cos(\omega t).$$

This alternating voltage leads to current that has the same frequency, but may be *out of phase*. Hence we can write

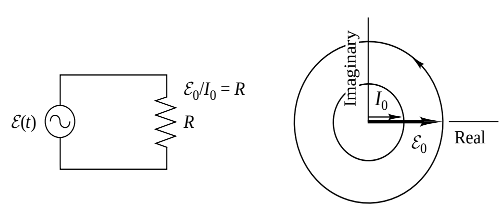
$$\widetilde{I}(t) = I_0 e^{i\omega t} \quad , \qquad \text{with } I_0 = |I_0| e^{-i\delta},$$

where δ is the phase difference between the voltage $\mathcal{E}$ and the current I. The observed current is the real part,

$$I(t) = \mathrm{Re}\,(\widetilde{I}(t)) = \mathrm{Re}\,(|I_0| e^{i(\omega t - \delta)}) = |I_0| \cos(\omega t - \delta).$$

To keep track of the phases, it is sometimes easier to work with the complex quantities and take the real part only at the end of the calculation. To illustrate this, consider the following three simple circuits.

Example 1: *Circuit with resistance.*



In this case, $\mathcal{E}(t)$ and $I(t)$ are in phase with one another. The current flows through the resistor the instant the voltage is applied. If $\mathcal{E}(t) = \mathcal{E}_0 \cos(\omega t)$, then

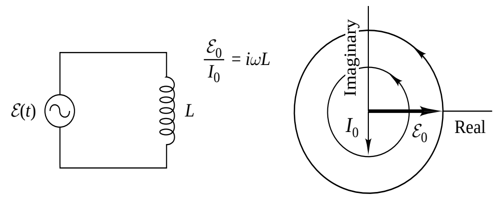
$$I(t) = \frac{1}{R}\mathcal{E}(t) = \frac{\mathcal{E}_0}{R}\cos(\omega t).$$

The vectors (or phasors) point in the same direction. In the complex notation, this just means that

$$\widetilde{\mathcal{E}}(t) = \mathcal{E}_0 e^{i\omega t} = R\,\widetilde{I}(t).$$

Hence Ohm's law holds also for the complex quantities.

Example 2: *Circuit with inductance.*



The potential drop across the inductor, V_L, is given by $V_L = L\,dI/dt$. By Kirchhoff's rule, this potential drop equals the voltage supplied by the battery, $V_L = \mathcal{E}(t)$. Hence we see that

$$\frac{dI}{dt} = \frac{\mathcal{E}_0}{L}\cos(\omega t),$$

and therefore

$$I(t) = \frac{\mathcal{E}_0}{\omega L}\sin(\omega t) = \frac{\mathcal{E}_0}{\omega L}\cos(\omega t - \pi/2). \tag{2}$$

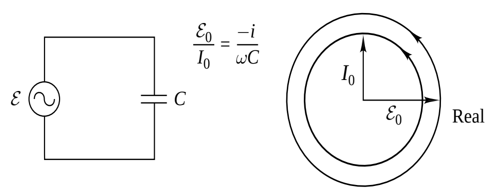
We see that the current lags in phase 90° behind the emf. The reason is that, if you suddenly apply an external voltage, the current takes a while to build up, because the induced magnetic field opposes the buildup according to Lenz's law. So, in an inductor the voltage leads the current.

In the complex notation, this means that, if $\widetilde{\mathcal{E}}(t) = \mathcal{E}_0 e^{i\omega t}$, then

$$\widetilde{I}(t) = \frac{\mathcal{E}_0}{\omega L} e^{i(\omega t - \pi/2)} = \frac{\mathcal{E}_0}{i\omega L} e^{i\omega t} = \frac{1}{i\omega L}\,\widetilde{\mathcal{E}}(t),$$

where we used that $e^{-i\pi/2} = -i = 1/i$. Note that this looks just like Ohm's law, with R replaced by $i\omega L$, which is imaginary!

Example 3: *Circuit with capacitance.*



The voltage drop across the capacitor is $V_C = Q/C$, where Q is the charge on the upper plate of the capacitor. Again by Kirchhoff's rule, $V_C = \mathcal{E}(t)$. Since $I(t) = dQ/dt$, we obtain

$$I(t) = \frac{dQ}{dt} = \frac{d}{dt}C\mathcal{E}(t) = -C\mathcal{E}_0\omega\sin(\omega t) = C\mathcal{E}_0\omega\cos(\omega t + \pi/2). \tag{3}$$

That is, the current leads by 90°, which is to say that the current must flow into a capacitor first before it can charge up a voltage different from zero, and hence the voltage lags behind the current.

In the complex notation, this means that, if $\widetilde{\mathcal{E}}(t) = \mathcal{E}_0 e^{i\omega t}$, then

$$\widetilde{I}(t) = C\omega\mathcal{E}_0 e^{i(\omega t+\pi/2)} = i\omega C\mathcal{E}_0 e^{i\omega t} = i\omega C\,\widetilde{\mathcal{E}}(t),$$

where we used that $e^{i\pi/2} = i$. Again, this looks like Ohm's law, this time with $1/(i\omega C)$ instead of R.

In all of these three cases, we could think of the alternating voltage as complex, $\widetilde{\mathcal{E}}(t) = \mathcal{E}_0 e^{i\omega t}$, and the corresponding complex current is then just obtained by a *general form of Ohm's law*:

$$\widetilde{\mathcal{E}}(t) = Z\widetilde{I}(t),$$

or, equivalently, after dividing by $e^{i\omega t}$,

$$\mathcal{E}_0 = ZI_0.$$

Here Z is the *impedance*, which is a complex number!

In Example 1, $Z = R$, which is real, while in Examples 2 and 3 it is imaginary, namely $Z = i\omega L$ and $Z = 1/(i\omega C)$, respectively. In general, we can decompose Z into its magnitude and its phase, i.e.,

$$Z = |Z|e^{i\delta}.$$

Since

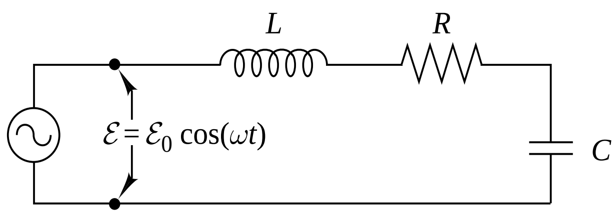
$$I_0 = \frac{\mathcal{E}_0}{Z} = \frac{\mathcal{E}_0}{|Z|}e^{-i\delta},$$

we see that the magnitude of I is given by $|I_0| = \mathcal{E}_0/|Z|$, and the phase difference between the current and the voltage is δ.

Exactly the same procedure works for more complex circuits. For instance, if the elements are in series, we just have to add the impedances, and apply the general form of Ohm's law. Why this works is explained in the next section.

RLC circuits

The starting point of this is an RLC circuit such as the one shown (note that for a series arrangement the order of the parts around the loop doesn't affect the equations).



Applying Kirchhoff's rule (which says that when you follow a path around the circuit, the voltages across the various components must sum to zero), we see that

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = \mathcal{E}_0 \cos(\omega t).$$

Substituting $I = dQ/dt$, we obtain

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = \mathcal{E}_0 \cos(\omega t). \tag{4}$$

This is a differential equation that is not easy to solve. The purpose of this section is to show you how to do it.

I turns out to be mathematically simpler to represent the driving voltage on the circuit not as the real function $\mathcal{E}_0 \cos(\omega t)$, but as the complex function $\mathcal{E}_0 e^{i\omega t}$, keeping in mind that we will ultimately want to take a real part of the solution $Q(t)$. Thus we want to solve the equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = \mathcal{E}_0 e^{i\omega t}. \tag{5}$$

Once we find the complex solution $Q(t)$ to this equation, the real part of $Q(t)$ will automatically be a solution to (4), simply because the equation is linear. Hence it is enough to find a solution to (5) and take its real part at the end. This has the advantage that we can deal with exponential functions instead of using complicated trigonometrical identities.

Because we know that exponential functions have derivatives that still contain the exponential function, we can try

$$Q(t) = Q_0 e^{i\omega t} \tag{6}$$

as a solution to (5). Thus we substitute it into equation (5) and find that

$$-LQ_0\omega^2 e^{i\omega t} + iR\omega Q_0 e^{i\omega t} + \frac{Q_0}{C} e^{i\omega t} = \mathcal{E}_0 e^{i\omega t}.$$

This equation clearly holds true all the time, too, because we can divide out the factor $e^{i\omega t}$ and satisfy the equation by satisfying

$$-LQ_0\omega^2 + iR\omega B_0 + \frac{Q_0}{C} = \mathcal{E}_0.$$

In fact, solving for Q_0 we find

$$Q_0 = \frac{\mathcal{E}_0}{-L\omega^2 + iR\omega + 1/C}. \tag{7}$$

Thus, with this value of Q_0, we have found that (6) solves the equation (5). There are other solutions, however, depending on the initial conditions. We can add to our solution any solution of the *homogeneous* equation (i.e., the one where $\mathcal{E}_0$ is zero), and still have a solution. These additional parts decay exponentially in time, however (see Tipler 29-5). The only part that survives in the long run is the *steady state* part of the solution, which is $Q(t) = Q_0 e^{i\omega t}$.

We are not usually very interested in Q but rather in $\widetilde{I}(t) = dQ/dt$. (We put the tilde on top to emphasize that this is a complex quantity, and we have to take the real part at the end.) So we differentiate equation (6) and find the current to be

$$\widetilde{I}(t) = \frac{dQ}{dt} = i\omega Q_0 e^{i\omega t} \equiv I_0 e^{i\omega t},$$

i.e., $I_0 = i\omega Q_0$. Inserting the value for Q_0 we obtained in (7), we can get an equation for $\mathcal{E}_0/I_0$, namely

$$\frac{\mathcal{E}_0}{I_0} = R + i\omega L + \frac{1}{i\omega C}. \tag{8}$$

The right side of this expression is just the sum of the impedances of the three elements of the circuit, each of which we have derived in the previous section, namely R, $i\omega L$, and $1/(i\omega C)$. Hence we see that for an arrangement in series, we just have to add the impedances of the elements, and then use the generalized form of Ohm's law, $\mathcal{E}_0 = ZI_0$, to obtain the current. Once we know this, we can forget about solving complicated differential equations, but rather we can obtain all the physical quantities using simple arithmetics with complex numbers.

In general, for AC (alternating-current) circuits, we think of the terms on the right side of equation (8) as forming a complex number, which consists of a real **resistance** R and imaginary terms called **reactances** defined by $X = \omega L - 1/\omega C$. These go together to make up an **impedance** Z, which can be represented by the simple linear relation

$$Z = \frac{\mathcal{E}_0}{I_0} = R + iX.$$

Since Z is complex, so, in general, are $\mathcal{E}_0$ and I_0.

We can then work out values for Z, $\mathcal{E}_0$, and I_0, assuming them to be complex numbers. At the end, we can simply take the real part of I_0 and $\mathcal{E}_0$ as the actual physical quantity we observe. When we do this, we obtain a cosine function that contains the correct amplitude and phase information.

In this way we can derive the amplitude and phase relationships for current and voltage in any arbitrary combinations of resistors, capacitors, and inductors. We can also see easily the phase relations between the voltages and currents in the individual components of our circuit.

Example. Let's look at the RC low-pass filter of Example 29-8 in Tipler/Mosca. We have a capacitor and a resistor in series with an AC generator with voltage magnitude $V_{\rm app}$, and are asked for the magnitude of the potential drop, $V_{\rm out}$, across the capacitor.

What is the total current in the system? The total impedance is just $Z = R + Z_C$, with $Z_C = 1/(i\omega C)$ the impedance of the capacitor. And, by Ohm's law,

$$I_0 = \frac{V_{\rm app}}{Z} = \frac{V_{\rm app}}{R + Z_C}.$$

The potential drop across the capacitor is, again by the generalized version of Ohm's law, the impedance of the capacitor times the current:

$$V_{\rm out} = Z_C I_0.$$

Substituting the value for I_0 we thus obtain

$$V_{\rm out} = V_{\rm app} \frac{Z_C}{R + Z_C}.$$

Its magnitude is given by

$$|V_{\rm out}| = V_{\rm app} \left| \frac{1/(i\omega C)}{R + 1/(i\omega C)} \right| = \frac{V_{\rm app}}{\sqrt{1 + (\omega RC)^2}}.$$

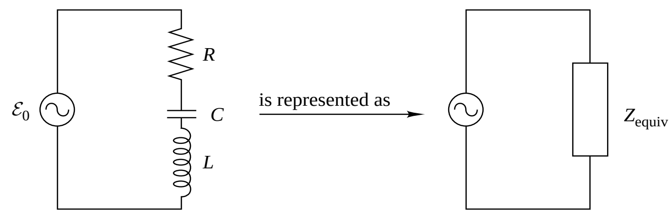
Similar considerations also work if the elements are not in series, but in parallel. In this case, we have to add not the impedances of the individual elements, but their inverse. To see why this works, look at the circuit in Figure 29-24 on page 956 of Tipler/Mosca, where there is a resistor, a capacitor and an inductor in parallel. The potential drop across all three elements is the same. We can write it in complex form, $\widetilde{\mathcal{E}} = \mathcal{E}_0 e^{i\omega t}$. By the generalized version of Ohm's law, the currents through R, C and L are then given by $\widetilde{I}_R = \widetilde{\mathcal{E}}/R$, $\widetilde{I}_C = \widetilde{\mathcal{E}}(i\omega C)$ and $\widetilde{I}_L = \mathcal{E}/(i\omega L)$, respectively. The total current is thus

$$\widetilde{I} = \widetilde{I}_R + \widetilde{I}_C + \widetilde{I}_L = \widetilde{\mathcal{E}} \left(\frac{1}{R} + i\omega C + \frac{1}{i\omega L} \right).$$

The term in brackets can be identified as $1/Z$, with Z the total (equivalent) impedance, which is exactly the sum of the inverse of the three impedances. We obtain for its magnitude

$$\left| \frac{1}{Z} \right| = \frac{1}{|Z|} = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L} \right)^2}.$$

We can now state general principles for solving problems with complex numbers:



First work out $Z_{\rm equiv}$, from

$$\begin{aligned} \text{Series}: &\qquad Z_{\rm equiv} = Z_1 + Z_2 + \cdots, \\ \text{Parallel}: &\qquad \frac{1}{Z_{\rm equiv}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots \end{aligned}$$

where the Z_i are complex numbers made up from

$$Z_R = R \quad \text{(real)}, \qquad Z_C = \frac{1}{i\omega C} \quad \text{(imaginary)}, \quad \text{and} \quad Z_L = i\omega L \quad \text{(imaginary)}.$$

Then write the complex number $Z_{\rm equiv} = R + iX$ as

$$Z_{\rm equiv} = |Z| e^{i\delta},$$

where

$$|Z| = \sqrt{R^2 + X^2} \quad \text{and} \quad \tan\delta = \frac{X}{R}.$$

With this approach, we can work out Z for a complicated circuit by using complex numbers and then, at the end, find the magnitude and phase.

As an example of how to do this, let's consider again the series RLC circuit. In this case,

$$Z = R + \frac{1}{i\omega C} + i\omega L = R + i\left[\omega L - \frac{1}{\omega C}\right],$$

which we want to write as $Z = |Z|e^{i\delta}$. We obtain

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

and

$$\tan\delta = \frac{\omega L - 1/\omega C}{R}.$$

The maximal current is then obtained as $|I_0| = \mathcal{E}_0/|Z|$, and the phase difference between $\mathcal{E}(t)$ and $I(t)$ is δ.

Resonance

Now let's look again at the RCL circuit. We have $I_0 = \mathcal{E}_0/Z$, with

$$Z = R + iX = R + i\left(\omega L - \frac{1}{\omega C}\right).$$

We can see by inspection of this equation that, at a very low frequency, $1/\omega C$ becomes very large, and the current drops to the low value as given by (3) and leads the voltage. At very high frequencies, the ωL becomes largest, and the current has the low value as given by (2) and lags behind the voltage.

At **resonance** the magnitude of the current becomes a maximum, and $|Z|$ becomes a minimum. Since

$$|Z| = \sqrt{R^2 + X^2}$$

this is the case, for a given resistance R, if $X = 0$, i.e.,

$$\omega L = \frac{1}{\omega C} \qquad , \quad \text{or} \qquad \omega = \omega_0 \equiv \frac{1}{\sqrt{LC}}.$$

In this case the reactances of capacitor and inductor cancel each other, and the current is in phase with the voltage.

The magnitude of the current is given by

$$|I_0| = \frac{\mathcal{E}_0}{|Z|} = \frac{\omega \mathcal{E}_0}{L\sqrt{(\omega^2 R^2/L^2) + (\omega^2 - \omega_0^2)^2}},$$

and the phase between current and voltage is given by

$$\tan\delta = \left[\frac{L\left(\omega^2 - \omega_0^2\right)}{\omega R}\right].$$

These two quantities are plotted, as a function of ω, in figure 2 below.

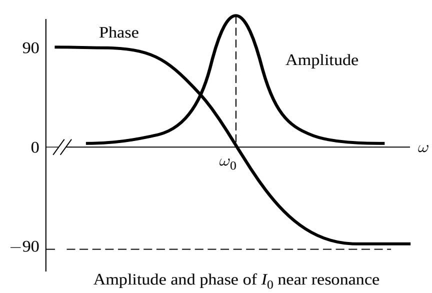


Figure 2: Amplitude and phase of I_0 near resonance

Power and quality factor

We know that current times voltage gives power in watts. However, suppose that we want to know the power dissipation in a reactive or imaginary part of a circuit. Let there be a current $\widetilde{I}(t) = I_0 e^{i\omega t}$ flowing through the circuit element. According to our generalization, the *voltage drop* $\widetilde{V}(t) = V_0 e^{i\omega t}$ across that element will be $\widetilde{V} = \widetilde{I}Z$. If Z is determined by a reactive element, then $Z = iX$, and $\widetilde{V}$ will be equal to $i\widetilde{I}X$. The voltage will be out of phase with the current. But we must think of actual real quantities, so the power must be $\mathrm{Re}\,(\widetilde{I}) \times \mathrm{Re}\,(\widetilde{V})$. What this represents is a situation where the current has a real component that is proportional to, say, $\cos(\omega t)$, but where the voltage will be represented by a real component that is out of phase, namely, by $\sin(\omega t)$. But on the average, $\sin(\omega t)\cos(\omega t)$ is always zero, i.e.,

$$\int_0^\infty \sin(\omega t)\cos(\omega t) dt = 0,$$

leading to the result that there is no average power dissipated in reactive elements. Of course, energy does indeed flow into and out of such a circuit element during each cycle.

On the other hand, since current and voltage are in phase in resistive or real elements, the power average will be an average of $\cos^2(\omega t)$ or $\sin^2(\omega t)$ which you know to be 1/2. So the power dissipated in a resistor is given by

$$P = \frac{1}{2}|\widetilde{I}|^2 R = \frac{1}{2}|I_0|^2 R = \frac{1}{2} I_0^* I_0 R.$$

We can write the power in these circuits in other ways. For example, since $R = \mathrm{Re}\,(Z)$ and $I_0^* I_0$ is always real, we have

$$I_0^* I_0 R = \mathrm{Re}\,(I_0^* I_0 Z) = \mathrm{Re}\,(I_0^* V_0) = |I_0||V_0|\cos\delta,$$

where δ is the phase difference between the current and the voltage. This leads to

$$P = \frac{1}{2}\mathrm{Re}\ (I_0^* V_0) = \frac{1}{2}\mathrm{Re}\ (I_0 V_0^*) = \frac{1}{2}|I_0||V_0|\cos\delta.$$

Using the foregoing equations you can also show that

$$P = \frac{1}{2} V_0^* V_0 \,\mathrm{Re}\left(\frac{1}{Z}\right).$$

Because of the factor of 1/2 that is present in all of these equations, one frequently absorbs that factor into the current or voltage by defining an I_{rms} or V_{rms} which is just $1/\sqrt{2}$ times less than the $|I_0|$ or $|V_0|$ we have used to represent the amplitude or maximum value of our complex vectors. I.e.,

$$I_{\mathrm{rms}} = \frac{1}{\sqrt{2}}|I_0| \quad , \qquad V_{\mathrm{rms}} = \frac{1}{\sqrt{2}}|V_0| \ .$$

Therefore, if you were to look at $V_{\text{rms}} = 120$ volts from a household circuit on your laboratory oscilloscope, you would see that it would actually reach a peak level of 169.7 volts at the maximum of the sine curve on the scope.

Expressing the average power in these quantities, we obtain

$$P = I_{\text{rms}}^2 R = V_{\text{rms}}^2 \frac{R}{|Z|^2} = I_{\text{rms}} V_{\text{rms}} \cos \delta. \tag{9}$$

The quantity $\cos \delta$ is called the **power factor**. At resonance, $\delta = 0$, and the power factor is 1.

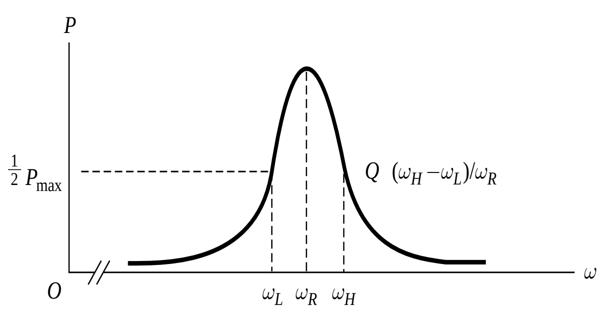
For the RLC circuit in series, we obtained above $Z = R + i\omega L - i/(\omega C)$, and hence

$$|Z|^2 = R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = R^2 + \frac{L^2}{\omega^2}\left(\omega^2 - \omega_0\right)^2,$$

where $\omega_0 = 1/\sqrt{LC}$ is the resonance frequency. Inserting this expression into (9) we obtain for the power as a function of the frequency ω

$$P = \frac{V_{\text{rms}}^2 R \omega^2}{L^2(\omega^2 - \omega_0^2)^2 + \omega^2 R^2}.$$

The **quality factor** of a resonant circuit is called Q and can be defined as the ratio of the resonance frequency to the difference between the two frequencies above and below resonance at which the power dissipated in the resonant circuit is 1/2 as much as that exactly at resonance:



It must remain as a project for those interested to show that

$$Q = \frac{\omega_0}{\omega_H - \omega_L} = \frac{\omega_0 L}{R}$$

and, using the last expression above, to show that in the case of a damped oscillator it will oscillate for Q cycles before its amplitude has diminished to less than about 5% of the original. The Q for a coil or cavity that is slightly lossy can also be defined as the ratio of the average energy stored in the cavity to the energy dissipated in a cycle, but this involves calculations that are a little tedious to carry out. At any rate, when one talks about a high-Q circuit, one means a very sharply tuned device with a narrow frequency response.