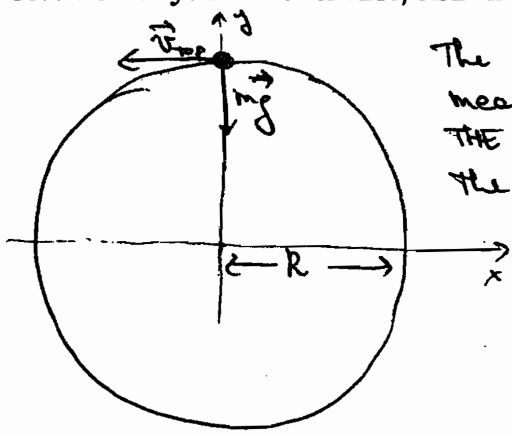
Problem 1. Flying in Circles (20 pts)

An airplane flies in a loop (a circular path in a vertical plane) of radius $R = 200$ m. The pilot's head always points toward the center of the loop. The speed of the airplane is not constant; it goes most slowly at the top of the loop and most quickly at the bottom. You may express your analytical responses below in terms of variables given and g.

a) (10 pts) At the top of the loop, the pilot feels weightless. What is his speed at the top? Give an analytical answer first, and then a numerical one.



The pilot feels weightless: that means that NO FORCE IS EXERTED BY THE PLANE ON THE PILOT.
The total force ~~acts~~ on the pilot is:

$$\vec{F}_{TOT} = m\vec{g} = \frac{m v_{Top}^2}{R}(-\hat{j})$$

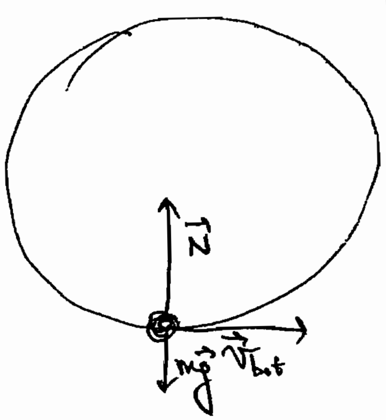
$$\Rightarrow \frac{v_{Top}^2}{R} = g$$

$$\Rightarrow \boxed{v_{Top} = \sqrt{gR}}$$

Numerically: $v_{Top} = \sqrt{(9.8\frac{m}{s^2})(200\,m)} = \boxed{44.3\ \frac{m}{s}}$

b) (10 pts) At the bottom of the loop, the speed of the plane is 270 km/hr $\approx$ 75 m/s. What is the apparent weight of the pilot at the bottom if his true weight is 800.0 N?

The apparent weight is equal to the force $\vec{N}$ exerted by the plane on the pilot in the upward direction:



The 2nd Newton's law reads:

$$\vec{N} + m\vec{g} = m\frac{v_{bot}^2}{R}\hat{j}$$

$$\Rightarrow N - mg = \frac{m v_{bot}^2}{R}$$

$$\Rightarrow N = \{\text{APPARENT WEIGHT}\} = mg\left(1 + \frac{v_{bot}^2}{gR}\right) =$$

$$= (800\,N)\cdot\left(1 + \frac{(75\,m/s)^2}{(9.8\frac{m}{s^2})(200\,m)}\right) =$$

$$= \boxed{3096\ N}$$

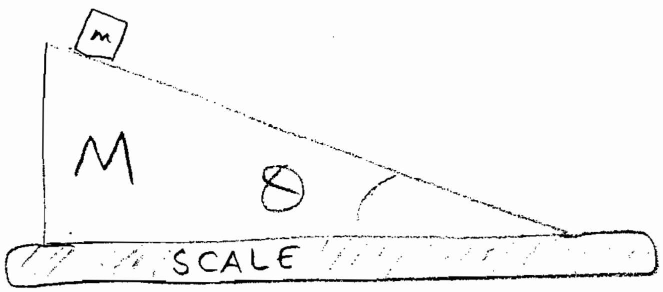
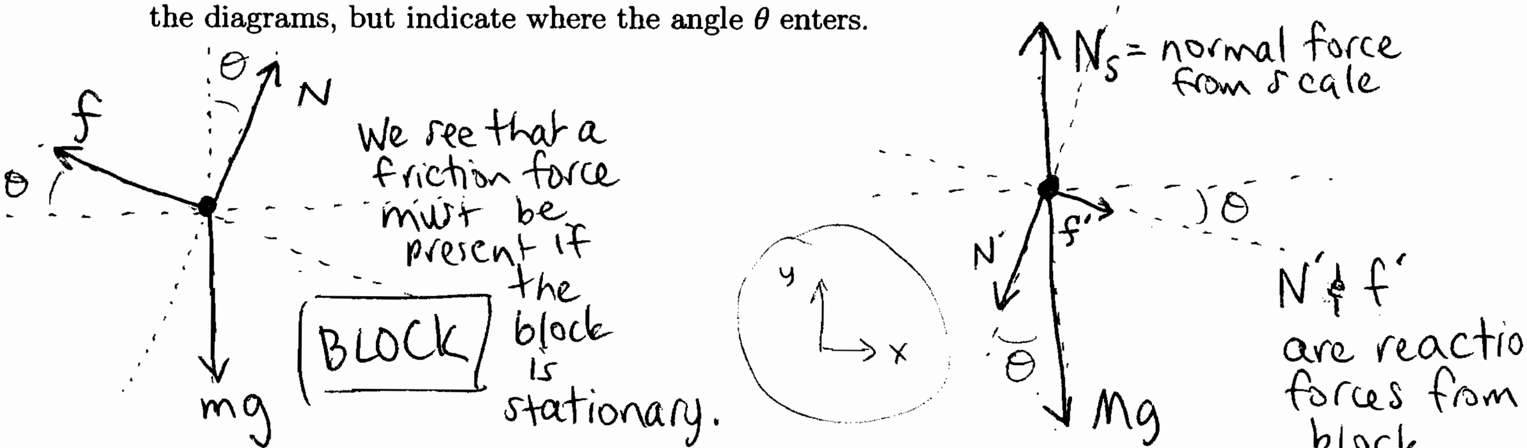


Figure 1: The block is atop the wedge; the wedge is atop the spring scale.

Problem 2. Block, Wedge and Scale (40 pts)

A wedge of mass M and opening angle θ (see Figure [?]) rests on a spring scale. A small block of mass m is at rest near the top of the wedge, as shown in Figure [?]. The wedge is at rest with respect to the scale at all times. You may express answers below in terms of variables given and g.

a) (10 pts) Draw free body diagrams for the wedge and the block. Clearly label all the forces for full credit. You do not need to resolve the forces into components for the diagrams, but indicate where the angle θ enters.



b) (6 pts) What is the total force from the block on the wedge? Give both magnitude and direction.

FIRST APPLY NEWTONS LAWS TO THE BLOCK:

$$\sum \vec{F} = ma = 0 = \vec{N} + \vec{f} + m\vec{g} \Rightarrow m\vec{g} = -(\vec{N} + \vec{f})$$

Let $\vec{g}$ be a vector of magnitude g, directed downward.

SINCE $\vec{N}'$ and $\vec{f}'$ are reaction forces we have

$$\vec{N}' = -\vec{N} \quad \text{ad} \quad \vec{f}' = -\vec{f} \quad \text{so}$$

$$\sum \vec{F}_{\text{from block on wedge}} = \vec{N}' + \vec{f}' = -(\vec{N} + \vec{f}) = m\vec{g}$$

$\boxed{\text{Magnitude} = mg \quad \text{direction} = \text{down}}$

c) (6 pts) Apply Newton's laws and solve for the normal force from the spring scale on the wedge. What does the spring scale read? (It is calibrated in Newtons.) You must show all your work for full credit.

From the diagram in (a)

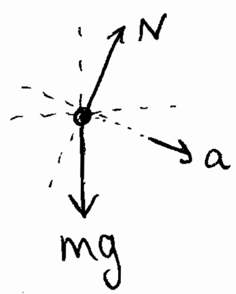
$$\sum \vec{F}_{\substack{\text{ON} \\ \text{WEDGE}}} = M\vec{a} = 0 = \vec{N}_s + \vec{N}' + \vec{f}' + M\vec{g} = 0$$

$$\Rightarrow \vec{N}_s + m\vec{g} + M\vec{g} = 0$$

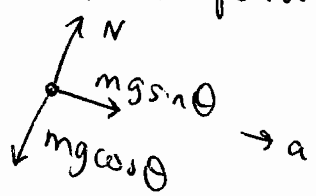
$$\Rightarrow \boxed{N_s = (M+m)g \text{, upward}}$$

d) (6 pts) The block is replaced with a teflon block, also of mass m. The friction between the teflon block and the wedge is negligible, so the teflon block accelerates after it is released. Find the magnitude of its acceleration before it hits bottom.

BLOCK



Resolve forces parallel and perpendicular to plane.

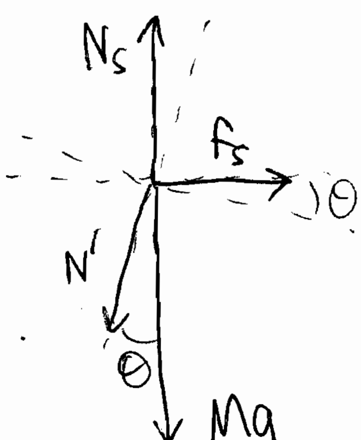


$$\sum F_{\text{parallel}} = ma = mg\sin\theta$$

$$\rightarrow \boxed{a = g\sin\theta}$$

Note $\sum F_{\text{perpendicular}} = 0 = N - mg\cos\theta \rightarrow N = mg\cos\theta$

e) (8 pts) What does the spring scale read as the teflon block accelerates down the wedge? Draw a new freebody diagram for the wedge without friction from the block.



We see that if the wedge is to remain stationary, the scale must exert a static friction force on it which we call $\vec{f}_s$.

The scale reads $|\vec{N}_s| = Mg + N'\cos\theta$

$$= \boxed{Mg + mg\cos^2\theta}$$

f) (4 pts) What do you expect the spring scale to read in the limit $\theta = 90°$? Does your answer to (e) conform to your prediction?

Mg because the block will be in free fall.

$$Mg + mg(\overset{\to 0}{\cancel{\cos^2 \pi/2}}) = Mg$$

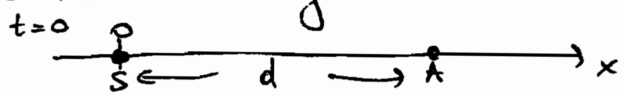
Yes the answer conforms to the prediction

Problem 3. Jerking to Avoid a Collision (40 pts)

The space vessel Serenity is traveling at constant speed v_{0s} when its pilot notices an abandoned ship at rest a distance d in front of it. The Serenity pilot does not change direction, but instantly begins slowing his ship so that his acceleration is $a = -jt$ where j is a constant and t is the time measured from that instant. He chooses j so that the Serenity just avoids colliding with the abandoned ship.

a) (16 points) Take the direction of motion as the x-axis. Using v_{0s}, d and j as needed, write analytical expressions for the velocity and position of the Serenity as functions of time: $v_s(t)$ and $x_s(t)$.

We choose a RF with origin on the initial position of the Serenity :



Data : $a_S(t) = -jt$, $v_S(0) = v_{0S}$, $x_S(0) = 0$

$$\Rightarrow \quad v_S(t) = v_S(0) + \int_0^t dt'\, a_S(t') = v_{0S} - \int_0^t dt'\, jt' = \boxed{v_{0S} - \frac{1}{2}jt^2}$$

$$x_S(t) = x_S(0) + \int_0^t dt'\, v_S(t') = 0 + \int_0^t dt' \left[v_{0S} - \frac{1}{2}j(t')^2\right]$$

$$= \boxed{v_{0S}t - \frac{1}{6}jt^3}$$

b) (16 points) What is the value of j for which a collision is just barely avoided? Let us call this value the critical jerk. Express your answer in terms of v_{0s} and d as needed.

If the collision is barely avoided, that means that there is a time T such that :

$$x_S(T) = d \quad , \quad v_S(T) = 0$$

$$\Rightarrow \quad v_{0S}T - \frac{1}{6}jT^3 = d \quad , \quad v_{0S} - \frac{1}{2}jT^2 = 0$$

From the 2nd eqn we get : $T^2 = \frac{2v_{0S}}{j}$

$$\Rightarrow T = \sqrt{\frac{2v_{0S}}{j}}$$

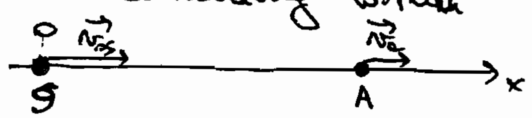
From the 1st eqn we get $v_{0S}\sqrt{\frac{2v_{0S}}{j}} - \frac{1}{\cancel{6}\,3}\cancel{j}\,\frac{\cancel{2}v_{0S}}{\cancel{j}}\sqrt{\frac{2v_{0S}}{j}} = d$

$$\Rightarrow \quad \frac{2}{3}v_{0S}\sqrt{\frac{2v_{0S}}{j}} = d \quad \Rightarrow \quad \frac{8v_{0S}^3}{9j} = d^2 \quad \Rightarrow \quad \boxed{j = \frac{8v_{0S}^3}{9d^2}}$$

c) (8 points) If instead the abandoned ship were drifting along the x-axis with constant speed v_a when the Serenity approached with speed v_{0s}, what would be the value of the critical jerk? HINT: you may work the problem in the inertial frame in which the abandoned ship is at rest. What is v'_{0s}, the initial speed of the Serenity in that frame?

We choose a RF' comoving with the abandoned ship and coinciding with the RF in part (a) at $t=0$:

$t=0$:



Fixed RF

$t=0$:

Comoving RF'

In the comoving RF' the velocity of Serenity at time $t=0$ is : $\vec{v}'_{0s} = \vec{v}_{0s} - \vec{v}_a$

$$v'_{0s} = v_{0s} - v_a$$

and its acceleration at all times is the same as in the original RF : $a'_s(t) = a_s(t) = -jt$

However in the comoving RF' by construction the abandoned ship is at rest : $v'_A = 0$ and its position is constantly equal to $x'_A = d$.

In the comoving frame the problem looks exactly the same as the problem in parts (a) and (b), with a new initial velocity equal to $v'_{0s} = v_{0s} - v_a$.

Then the new critical jerk will be given by the result in part (b) with v_{0s} replaced by $v'_{0s} = v_{0s} - v_a$:

$$\boxed{j' = \frac{8\,(v_{0s} - v_a)^3}{9\,d^2}}$$