6 The Physics of Springs

Introduction

A spring is a system that tends to return to its equilibrium position when displaced from that position. According to this definition, the spring concept has much wider application than simply to the metal spirals commonly called springs.

The tendency of a spring to return to equilibrium is described by a restoring force that depends on the size of the departure from equilibrium. The larger the departure, the larger the restoring force. For departures that are not too large, the restoring force F of any spring is linear in the departure Δx from equilibrium:

$$F = -k\Delta x, \tag{1}$$

where k is the so-called spring constant. The minus sign in eq. (1) indicates that the restoring force opposes the departure from equilibrium.

In this laboratory you will explore three situations involving springs:

1. Simple harmonic motion of a single spring/mass system.
2. The more complicated motion of a system of two masses and three springs.
3. A rotating spring/mass system in which the spring supplies the centripetal force.

A remarkable feature of springs is that they will vibrate at a particular frequency if displaced from equilibrium and let go. For a spring with constant k as defined by eq. (1) that moves a mass m the frequency f of vibration is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}, \qquad \text{and the corresponding period is} \qquad T = 2\pi\sqrt{\frac{m}{k}}. \tag{2}$$

While a derivation of eq. (2) follows from considerations of $F = ma$ it is useful to note that the form of the result can be obtained from dimensional analysis, as shown in the Appendix.

6.1 Simple Harmonic Motion

In part 1 of this lab you will verify to form of eq. (2) and measure the constant k of a spring.

The apparatus consists of an air track to reduce friction, a spring, a glider and a hanging weight attached to the glider via a piece of magnetic tape (shiny side down). Weigh the glider before proceeding.

Measurements of the period of the motion of the glider can be made with the computer timer, program pt *of directory* c:\timer *in* Motion Timer *mode. First place the photogate in line with one end of the glider at its equilibrium position, as can be verified using the* Photogate Status Check. *As always,* Print *and* Save *your good data files.*

6.1.1 Independence of Period and Amplitude

The period T of springy motion is claimed to be independent of the amplitude (so long as the amplitude is small enough that the force is linear in the displacement). Verify this by recording the period for displacements of 2, 4 and 6 cm of the glider from equilibrium.

Use the 100-g hanger with no additional weights for this.

6.1.2 Dependence of the Period on Mass

Measure the period of the glider when additional weights of 10, 20, 30 and 40 g are added to the hanger. Use a standard displacement of 4 cm for this part.

Also, record the equilibrium position for each weight (including no extra weight). Use Polynomial Regression *of* Order: 1 *of* StatMost *to plot a graph of the weight (not mass) on the vertical axis vs. the displacement on the horizontal axis for the 5 weights (including 0 g extra). The fitted slope of the graph is the spring constant k.*

If the graph is not a straight line it is possible the weights are too large for the spring. If so the measurements should be repeated with a smaller set of weights.

Analyze your data as follows to study the dependence of the period on mass. For the case with no extra weight, the mass of the system is

$$m_0 = m_{glider} + m_{hanger}, \tag{3}$$

and the corresponding period should be

$$T_0 = 2\pi\sqrt{\frac{m_0}{k}}. \tag{4}$$

How well does your measurement of T_0 agree with eq. (4) using your measured values of the masses and the value of k deduced above?

When the additional weights Δm were added to the hanger the period increased only slightly, because $\Delta m/m_0$ is small. Equation (2) predicts

$$T = 2\pi\sqrt{\frac{m_0 + \Delta m}{k}} = 2\pi\sqrt{\frac{m_0}{k}}\sqrt{1 + \frac{\Delta m}{m_0}} \approx T_0\left(1 + \frac{1}{2}\frac{\Delta m}{m_0}\right), \tag{5}$$

using the Taylor series expansion for $\sqrt{1 + \Delta m/m_0}$.

Use StatMost *to plot a graph of T/T_0 on the vertical axis vs. $\Delta m/m_0$ on the horizontal axis, and to determine the slope of a straight-line fit to your data. Be sure to include the data point at (0,1) in the fit.* How well does you result agree with the prediction of eq. (5) that the slope be 0.5? The significance of your measured slope is that the period varies with the mass according to

$$T \propto m^{\text{slope}}. \tag{6}$$

6.2 A System with 2 Masses and 3 Springs

[Analyze part 1 before proceeding to part 2.]

Reconfigure the apparatus of part 1 by removing the tape and hanging weights and adding a second glider and two more springs, as sketched in Fig. 1. The three springs should be identical, and the two gliders should have the same mass.

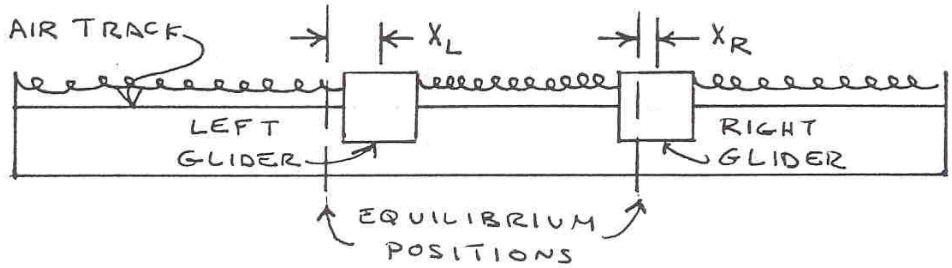

Figure 1: The apparatus for the study of coupled oscillations.

After the gliders come to rest at their equilibrium positions displace one while holding the other fixed. Then let both gliders go. The resulting motion is somewhat complicated! The displaced glider has the largest oscillation at first, but soon the second glider seems to take over the motion, then the first glider again, and so on. Furthermore, there does not appear to be a unique frequency to the oscillations.

6.2.1 Normal Modes

The motion of systems with multiple springs is clearly much more complex than that of a single spring. In this part you will not analyze the full complexity, but rather study those special cases, called normal modes, that always exist for multiple spring systems. A normal mode is an oscillation in which all movement takes place at a single frequency. The general motion of a multiple spring system can be thought of as a combination of normal modes, as will be explored in part 6.2.2.

The present apparatus includes 3 springs and 2 gliders. If you focus your attention on the gliders, there are only 2 things to watch. This implies that the system has exactly 2 normal modes.

One normal mode occurs when the gliders move always keeping the same distance between them. *Displace both gliders by 4 cm to the right of their equilibrium positions and let them go. The middle spring should keep a constant length and both gliders should oscillate with a common frequency. Measure the period of this oscillation using* Motion Timer *with the photogate aligned on the equilibrium position of one of the gliders.*

The other normal mode occurs when the gliders move in opposite directions with equal amplitudes. *Displace one glider 4 cm to the left of its equilibrium position and the other by 4 cm to the right of its equilibrium position and then let them go. Measure the period of oscillation of one of the gliders. The period of the second normal mode should be shorter than that of the first.*

Compare you measured periods with those deduced from the following analysis. During the first normal mode the force never changes in the middle spring. The changing force on, say, the left glider is only due to the change in length of the leftmost spring. If k is the spring constant of the 3 springs, then the force on the left glider is just

$$F_L = -k\Delta x_L, \tag{7}$$

where Δx_L is the displacement of the left glider. Hence if m is the mass of the glider we expect

$$T_1 = 2\pi\sqrt{\frac{m}{k}} \tag{8}$$

as the period of the first normal mode. Does your measurement agree with this equation, using the spring constant k measured in part 1?

During the second normal mode the left glider experiences changing forces from both the left and middle springs. After a displacement Δx_L the leftmost spring exerts force $-k\Delta x_L$ on the left glider. Meanwhile the middle spring exerts force $-k(2\Delta x_L)$ on the left glider since the middle spring is compressed (stretched) twice as much as the leftmost spring is stretched (compressed). Note that the direction of the force on the leftmost glider is the same for both springs. Hence we can write

$$F_L = -3k\Delta x_L, \tag{9}$$

and so we predict

$$T_2 = 2\pi\sqrt{\frac{m}{3k}} = \frac{T_1}{\sqrt{3}} \tag{10}$$

for the period of the second normal mode. How well do these predictions agree with your measurements?

6.2.2 The Beat Frequency and Transference of Oscillation

In this part we return to the complex motion observed at the beginning of part 6.2. The claim is that this motion can be thought of as a sum of the two normal modes you have just studied.

We will now label the positions of the left and right glider as x_L and x_R, where the values of these coordinates are zero at equilibrium. Then the motion of the first normal mode can be written

$$x_L = A_1 \cos\frac{2\pi t}{T_1}, \qquad x_R = A_1 \cos\frac{2\pi t}{T_1}, \tag{11}$$

while that of the second normal mode is

$$x_L = A_2 \cos\frac{2\pi t}{T_2}, \qquad x_R = -A_2 \cos\frac{2\pi t}{T_2}. \tag{12}$$

The general case is

$$\begin{aligned} x_L &= A_1 \cos\tfrac{2\pi t}{T_1} + A_2 \cos\tfrac{2\pi t}{T_2}, \\ x_R &= A_1 \cos\tfrac{2\pi t}{T_1} - A_2 \cos\tfrac{2\pi t}{T_2}. \end{aligned} \tag{13}$$

Consider again the case that at $t = 0$ you displace the left glider by amount A, but hold the right glider at rest, and then let both of them go. This case is described by eq. (13) with $A_1 = A_2 = A/2$, and hence

$$\begin{aligned} x_L = &\ \tfrac{A}{2}(\cos \tfrac{2\pi t}{T_1} + \cos \tfrac{2\pi\sqrt{3}t}{T_1}), \\ x_R = &\ \tfrac{A}{2}(\cos \tfrac{2\pi t}{T_1} - \cos \tfrac{2\pi\sqrt{3}t}{T_1}), \end{aligned} \tag{14}$$

recalling that $T_2 = T_1/\sqrt{3}$. Using the 'well-known' trig identities for combining cosines we find

$$\begin{aligned} x_L = &\ A\cos \tfrac{\pi(\sqrt{3}+1)t}{T_1} \cos \tfrac{\pi(\sqrt{3}-1)t}{T_1}, \\ x_R = &\ A\sin \tfrac{\pi(\sqrt{3}+1)t}{T_1} \sin \tfrac{\pi(\sqrt{3}-1)t}{T_1}. \end{aligned} \tag{15}$$

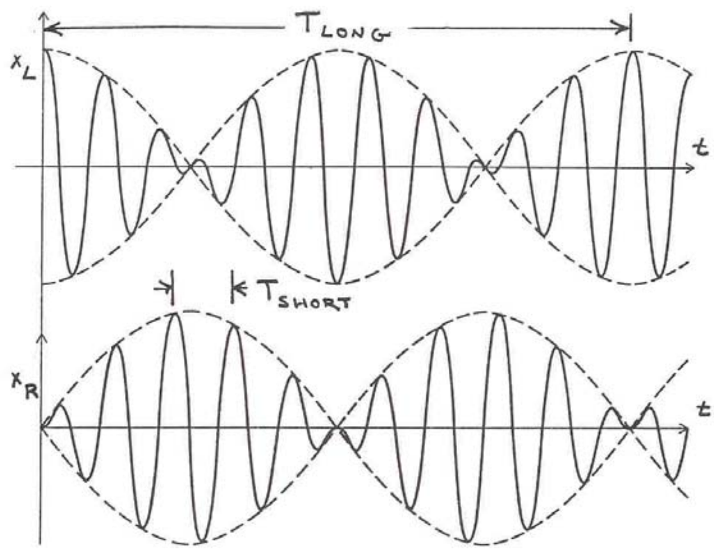

Figure 2: Graph of the motion of two gliders coupled by three springs for the conditions of eq. (15).

This analysis tells us that the motion involves the combination of two new periods

$$T_{short} = \frac{2T_1}{\sqrt{3}+1} = (\sqrt{3}-1)T_1, \qquad \text{and} \qquad T_{long} = \frac{2T_1}{\sqrt{3}-1} = (\sqrt{3}+1)T_1. \tag{16}$$

The low frequency corresponding to the long period is called the 'beat frequency'.

A way of thinking about the motion described by eq. (15) is that the gliders oscillate with period T_{short} subject to a slow change in amplitude with period T_{long}. First only the left glider has a large amplitude, but after time $T_{long}/4$ its amplitude in fact becomes very small. Meanwhile, the right glider started with almost no motion, but by time $T_{long}/4$ its amplitude has become large. This situation keeps reversing every additional time $T_{long}/4$, and is called 'transference of oscillation'.

Make measurements of T_{short} and T_{long} by arranging the initial conditions of the gliders as described above. With care you can use the Motion Timer *to determine T_{short}; because of the complexity of the motion only about one recorded interval in 5 will correspond to T_{short}. You will need to use a stopwatch (or* Keyboard Timing Modes *of program* pt*) to measure T_{long}: measure the time interval over which the motion of the left glider becomes very small and then large again 10 times; this will be $5T_{long}$.*

How well do your observations agree with the theory sketched above?

6.3 Centripetal Force

In this part you will confirm that circular motion of a mass m at radius r and tangential velocity v requires an inward force of

$$F = \frac{mv^2}{r} = m\omega^2 r, \tag{17}$$

where ω is the angular velocity.

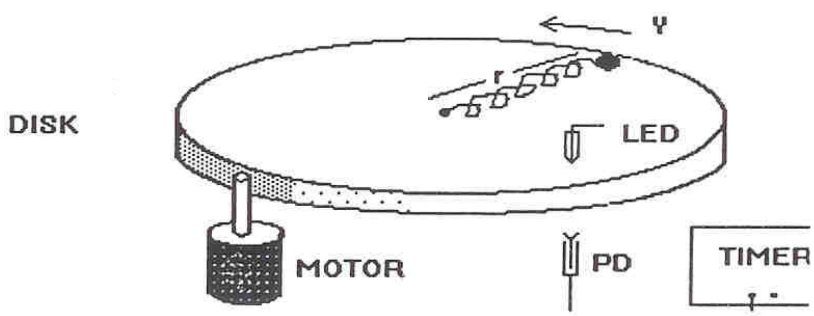

Figure 3: The apparatus to measure centripetal force.

The apparatus is sketch in Fig. 3. Measure the mass m and diameter d of a steel ball and attach it to a spring whose other end is fixed on a turntable. Set up a photogate just beyond the edge of the turntable. Use the electric motor to spin the turntable, increasing the control voltage until the spring stretched so the ball extends slightly beyond the edge of the turntable. CAUTION: if the turntable rotates too rapidly the ball will move off the turntable!

Use the One Gate *option of* Gate Timing Modes *of program* pt *to position the photogate so that the ball interrupts the gate for the briefest possible interval. Then switch to* Motion Timer *to measure the period T of revolution of the turntable.*

Turn off the electric motor. Measure the distance D from the center of the turntable to the light beam (the Photogate Status Check *may help here.) Attach a piece of string to the outer edge of the ball, over the pulley, and connect the other end of the string to a hanger. Add weights to the hanger until the outer edge of the ball is again distance D from the center of the turntable. Record the mass M required for this.*

The centripetal force on the ball when the turntable was rotating is thus determined to be

$$F = Mg. \tag{18}$$

The center of the ball was at radius $r = D - d/2$. The angular velocity of the ball was $\omega = 2\pi/T$. Hence

$$m\omega^2 r = 4\pi^2 \frac{m(D - d/2)}{T^2}. \tag{19}$$

Evaluate expressions (18) and (19) and comment on how well you have confirmed eq. (17).

6.4 Appendix: Dimensional Analysis

The period T of a spring/mass system can in general depend on the spring constant k, on the mass m and on the amplitude A of the displacement from equilibrium. We 'guess' that the functional form of this dependence can be written are products of powers:

$$T \propto k^\alpha m^\beta A^\gamma. \tag{20}$$

The constants α, β and γ are to be determined by the requirement that both sides of expression (20) must have the same dimensions.

According to the basic definition of the spring force (1) the dimensions of the spring constant k are $force/distance = mass/(time)^2$. The dimension of the amplitude A is, of course, *length*. Thus the required equality of dimensions in expression (20) can be written

$$[time] = [time]^1[mass]^0[length]^0 = \left[\frac{mass}{(time)^2}\right]^\alpha [mass]^\beta [length]^\gamma. \tag{21}$$

Equating the powers of *time*, *mass* and *length* we find

$$\begin{aligned} 1 &= -2\alpha, \\ 0 &= \alpha + \beta, \\ 0 &= \gamma, \end{aligned} \tag{22}$$

which immediately tells us that

$$\begin{aligned} \alpha &= -\tfrac{1}{2}, \\ \beta &= \tfrac{1}{2}, \\ \gamma &= 0, \end{aligned} \tag{23}$$

and hence

$$T \propto \sqrt{\frac{m}{k}}. \tag{24}$$

Our dimensional analysis tells us the period T varies as $\sqrt{m/k}$ and not $\sqrt{k/m}$ (which has dimensions $1/time$), and that the period does not depend on the amplitude. Of course, dimensional analysis cannot reveal the dimensionless coefficient 2π of eq. (2).