4 The Behavior of a Simple Pendulum and a Precision Measurement of g

Introduction

The simple pendulum (Fig. 1) is one of the oldest known precision devices. Lore has it that Galileo was the first to note how the uniformity of the period of oscillation of a pendulum could be the basis of a clock.

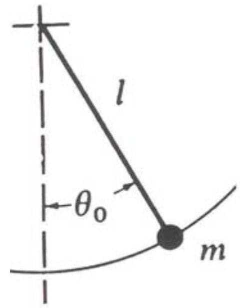

Figure 1: A simple pendulum consisting of a compact mass m suspended from a string of length l (measured from the pivot point to the center of mass). The pendulum is launched at angle θ_0 to the vertical.

The period T of the pendulum is the time to complete a swing: the mass returns to the same position and is moving again in its original direction after one period. Note that the pendulum passes through each point along its path twice per period (except for the two extreme points at large angles).

If you have not encountered an analysis of the simple pendulum in the course yet, you may wish to read the Appendix to this Lab which discusses how a relation for the period of the pendulum can be deduced without $F = ma$.

An advanced analysis of the simple pendulum tells us that the period T is

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} + \frac{9}{64}\sin^4\frac{\theta_0}{2} + ...\right) \tag{1}$$

For 'small enough' launch angles θ_0 the terms $\sin^2(\theta_0/2)$, *etc.*, are very small and we have

$$T \approx 2\pi\sqrt{\frac{l}{g}}, \tag{2}$$

independent of the (small) launch angle.

In this Lab you will investigate the validity of expressions (1) and (2), and then use the latter to deduce g according to

$$g = 4\pi^2\frac{l}{T^2}. \tag{3}$$

An error analysis of eq. (3) relates the expected error σ_g on g to those on your measurements of length and time:

$$\frac{\sigma_g}{g} = \sqrt{\left(\frac{\sigma_l}{l}\right)^2 + 4\left(\frac{\sigma_T}{T}\right)^2}. \tag{4}$$

For a pendulum of length $l \approx 1$ m the measurement error σ_l will be about 1 mm when using a standard meter stick; hence we expect $\sigma_l/l \approx 0.001$. The period of such a pendulum should be about 2 sec according to eq. (2), while the measurement error with the computer electronic timer should be less than 0.001 sec. By measuring not one but ten periods, we expect $\sigma_t/t \approx 0.0003$. Hence if you are careful your error on g should be only

$$\frac{\sigma_g}{g} \approx 0.001. \tag{5}$$

That is, expect to measure values of g between 9.79 and 9.81 m/s^2.

The measurements in this Lab proceed in three steps as follows.

4.1 Variation of Period with Angle

In this part you will explore the dependence of the period of a simple pendulum on the launch angle, for a fixed length. According to eq. (1) there is a small variation of period with launch angle, but there exists a minimum period for a given length l that we will call $T_{\min}$:

$$T_{\min} = 2\pi\sqrt{\frac{l}{g}}. \tag{6}$$

If we take the ratio of the observed period at launch angle θ_0 to that observed at very small angles, we predict

$$\frac{T(\theta)}{T_{\min}} = 1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} + +\frac{9}{64}\sin^4\frac{\theta_0}{2} + ... \tag{7}$$

Thus if you plot $T(\theta)/T_{\min}$ *vs.* $\sin^2(\theta_0/2)$ for 'small' angles you should find a straight line with intercept 1 and slope 1/4; at larger angles the plot should turn upwards.

Use a pendulum consisting of a steel ball and string of length 1 m, measured from the pivot point to the center of the ball. The center of the ball should intercept the light path from LED to photodiode on the photogate when the pendulum is at rest. The center of the protractor should be well aligned with the pivot point of the pendulum, and the string should be aligned with $0°$ *on the protractor when the pendulum is at rest.*

First measure the duration of one period for a launch angle of $5°$*, which should be about 2 sec. Use the photogate for this, with the computer running the timer program* pt *of directory* c:\timer. *Use* Miscellaneous Timer Modes *from the main menu, then* Pendulum Timer, *then* Normal Time Display. *In this timing mode the computer records the times between the first and third blockages of the LED beam, which is just what is needed to measure a complete period of pendulum motion. (Do you see why this is so?)*

To gain precision, all further data should be be taken for at least 10 periods. After the 10th period press Enter/Return *to stop data collection, then select* Display Table of Data *to view the data plus a statistical summary. The standard deviation,* SD, *gives the best measure* *thus far in the Ph101 Lab of the accuracy of the computer timer. When satisfied with the quality of the data,* Print Table of Data *for your lab book.*

Now make a measurement of 10 periods at each of the launch angles 10, 20, 30, 40, 50°.

If the string lengthens during the measurements because the clamp is not tight enough, the data are invalid.

You or your partner should watch the angle of the pendulum during the measurement to see if it decreases significantly compared to the launch value. If so, record the value at the end of the 10th period, and use the average of the initial and final angles in later analysis. (Or, repeat the measurement at a launch angle larger than nominal by 1/2 the amount the angle decreased.)

Exit program pt *and start* Windows *to use the* StatMost *program to do the calculations and produce the graph. Start a new spreadsheet and enter the angles* θ_0 *in column* A *and the periods* T *in column* B. *Use* Data, *then* Transform, *then* Simple Math *to create new columns containing* $T(\theta_0)/T(5°)$ *and* $\sin^2(\theta_0/2)$. *Since* StatMost *uses radian measure for angles, if you entered the angles in degrees, include a division by 1 rad = 57.296° in your math expression before taking the sine. That is, type in a* Formula: SSQ=(SIN(A/2/57.296))^2 *if the angles are in column* A. *Plot a graph and a polynomial fit using* Analyses, *then* Polynomial Regression *of* Order: 2.

The defaults for numerical axis labels in StatMost *are poor for the vertical axis,* $T(\theta_0)/T(5°)$, *which varies over only a small range. You might wish to improve the appearance of your plot by double clicking on the left vertical axis to bring up the window* 2D Axis Properties. *Some useful settings might be* Minimum = 0.99, Maximum = 1.05, MajorNum = 6, MinorNum = 0, *and* Precision = 2. *(There are several other graph-modifications windows that can be activated by double clicking on various portions of the graph.)*

Compare the value of the order-1 fit coefficient to the prediction of 1/4 in eq. (7). Do you have agreement within the calculated error? Do you see any evidence for the order-2 coefficient of 9/64?

In part 4.3 of this Lab you should try to measure g to 1 part in 1000. If you wish to avoid making a correction of this size or larger for the effect of launch angle on period, what is the largest launch angle you can use (refer to eq. (7))?

4.2 Variation of Period with Mass

[Analyze the data from part 4.1 before proceeding to part 4.2.]

In parts 4.2 and 4.3 you will explore the validity of eq. (2). A prominent feature is that the period should not depend on the mass of the pendulum ball, so long as the length from the pivot point to the center of the ball in constant.

Confirm this be measuring the duration of 10 periods of a pendulum with a steel, an aluminum and a wooden ball. You can use a data point from part 4.1 for the steel ball if you wish.

Do your periods for the three masses agree to within the timing error expected from your repetitions of the 5° case in part 4.1?

Note that eq. (2) implies that the period changes with length according to

$$\frac{dT}{T} = \frac{1}{2}\frac{dl}{l}. \tag{8}$$

Hence to achieve good consistency the lengths of the three pendula must be equal to within 1 part in 1000.

4.3 Variation of Period with Length

[Analyze part 4.2 before proceeding to part 4.3]

In this part you examine the validity of eq. (2), which can be rewritten as

$$l = g\frac{T^2}{4\pi^2}. \tag{9}$$

Using a steel ball, measure the duration of 10 periods for pendula of lengths 0.5, 0.75, 1.0, 1.25 and 1.5 m. Choose a launch angle which will not affect the period to greater than 1 part in 1000, based on your results from part 4.1. Measure the distance l carefully (i.e., don't use the end of a meter stick in your measurement; don't just guess where the center of the ball is...).

Use program pt *to collect the data and then* StatMost *to analyze the mean period as a function of length. Enter the lengths and times in the columns of a new spreadsheet. Then select* Data*, then* Transform*, then* Simple Math *with* Formula: TSQ_4PISQ=(B/6.283)^2 *to create a new column containing $T^2/4\pi^2$, assuming the periods are in column* B*. Select* Analyses*, then* Polynomial Regression *of* Order: 2 *to produce a graph of l vs. $T^2/4\pi^2$.*

Is your graph of l *vs.* $T^2/4\pi^2$ consistent with a straight line, in confirmation of eq. (9)? The regression coefficient of order 1 is your value of g. Is it consistent with the nominal value within the reported error?

As a check, calculate a separate value of g from the data at each length l using $g = 4\pi^2 l/T^2 = \mathsf{C}/\mathsf{A}$, assuming l is in column A and $T^2/4\pi^2$ is in column C of your spreadsheet.

If your value for g is outside the range 9.75-9.85 m/s^2, please check for numerical errors. If necessary, repeat the measurement for the longest l and analyze just this measurement to extract another value of g.

4.4 Appendix: Dimensional Analysis of the Simple Pendulum

The period T of the simple pendulum might depend on the mass m, the length l, the acceleration of gravity g, and the launch angle θ_0. With 'dimensional analysis' we can deduce the functional form of T on m, l and g, the parameters that have dimensions (*i.e.*, units). This technique cannot deduce the dependence on angle θ_0 since an angle is dimensionless.

The hypothesis of dimensional analysis is that the period T depends on products of powers of m, l and g:

$$T = Cm^\alpha l^\beta g^\gamma, \tag{10}$$

where C is a dimensionless number, possibly a function of θ_0. For this equation to be true, the dimensions (units) of both sides must be the same. In mechanics we deal with three distinct units: *mass*, *length* and *time*. The required equality of units in an equation permits up to three parameters to be determined in that equation.

For the simple pendulum, we note that the dimensions of the period T is just *time*, and those of acceleration g are $length/(time)^2$. So in terms of dimensions, eq. (10) becomes

$$[time] = [mass]^{\alpha}[length]^{\beta}\left[\frac{length}{(time)^2}\right]^{\gamma}. \tag{11}$$

For this to be true the dimensions of both sides must be the same, which means that the exponents of each unit must the be same on both sides. To emphasize the exponents, the lefthand side of eq. (11) can be rewritten as

$$[time] = [time]^1[length]^0[mass]^0. \tag{12}$$

Hence we deduce the three equations:

$$time \;:\; 1 = -2\gamma, \tag{13}$$
$$length \;:\; 0 = \beta + \gamma, \tag{14}$$
$$mass \;:\; 0 = \alpha. \tag{15}$$

These equations readily imply that

$$\alpha = 0, \qquad \beta = 1/2 \qquad \text{and} \qquad \gamma = -1/2. \tag{16}$$

Recalling eq. (10) we see that the period should not depend on the mass, and depends on length and g according to

$$T = C\sqrt{\frac{l}{g}}. \tag{17}$$

While dimensional analysis cannot determine the value of the number C, it correctly predicts the form of the dependence of the period on quantities that have units, without resort to $F = ma$.