Problem

In 1915, Einstein computed the precession of the perihelion of Mercury [1] as the first application of his new theory of general relativity, finding that advance in angle of the perihelion with respect to the Sun (assuming to be spherical and not rotating) to be,

$$\Delta \theta = \frac{6\pi GM}{ac^2(1 - \epsilon^2)} = \frac{24\pi^3a^2}{T^2c^2(1 - \epsilon^2)},$$

per revolution, where $G$ is Newton’s gravitational constant, $M$ is the (rest) mass of the Sun, the orbit has semimajor axis $a$ and eccentricity $\epsilon$, $c$ is the speed of light in vacuum, and $T^2 = 4\pi^2a^3/GM$ relates the period $T$ of the orbit to $a$ via Kepler’s third law. The result (1) is accurate to order $v^2/c^2$, where $v$ is the velocity of Mercury with respect to the Sun, and the mass $m$ of Mercury has been neglected in comparison to mass $M$ of the Sun.\(^1\)

\(^1\)In 1967, Dicke [2, 3] noted that the precession of the perihelion of Mercury could be explained by the possible oblateness of the Sun. Evidence for this is marginal, as reviewed in [4].

\(^2\)The famous data on the precession of the perihelion of Mercury were compiled by Le Verrier (1859) [5, 6]. For a review as of 1903, see secs. 23-24 of [7].

Among the many attempts around 1890 to explain the precession of the perihelion of Mercury, Lévy [8] proposed that gravity is deducible from a scalar potential,

$$V = \frac{GM}{r}\left(1 - \frac{\dot{r}^2}{v_g^2}\right),$$

where $r$ is the present distance from the source to the observer, $\dot{r}$ is the speed of the source, and $v_g$ is the speed of gravity. Apparently Lévy was inspired by Weber’s electrodynamics, and hoped that $v_g = c$ would explain the data; however it did not quite.

In 1898, Gerber [9, 10] gave a model of gravity based on the scalar potential,

$$V = \frac{GM}{r(1 - \dot{r}/v_g)^2} \approx \frac{GM}{r}\left(1 + \frac{2\dot{r}}{v_g} + \frac{3\dot{r}^2}{v_g^2}\right),$$

This potential is an approximate form of an approximate retarded potential, as discussed in [11]. From this potential he computed the precession of the perihelion of Mercury per revolution, finding,

$$\Delta \theta = \frac{24\pi^3a^2}{T^2v_g^2(1 - \epsilon^2)}.$$  

Based on the data, Gerber inferred that $v_g = c$ to good accuracy.

In retrospect, this is less surprising in that eq. (4) is identical to the result (1) of Einstein [1], computed via his theory of general relativity with $v_g = c$. This “coincidence” led to accusations that Einstein plagiarized Gerber’s result [12, 13], which reverberate to this day. See [14] for extensive comments on various approximations to gravity in the solar system.
Einstein’s result (1) agrees well with data.\(^3\)

The question arises as to what is the prediction from special relativity. The literature on this is rather erratic.

It was stated by Goldstein in Exercise 6, Chap. 13 of [18] (1950)\(^4\) that the result from special relativity is 1/6 that of general relativity, our eq. (1).

In 1984, Phipps [19] claimed that the result of special relativity is the same as that of general relativity.

In 1986, Peters [20] noted that Phipps made a computational error, and claimed the correct result of Phipps’ model is 1/2 that of general relativity. Peters then endorsed the claim of Goldstein [18] as the “standard” view of special relativity.

In 1987, Biswas [21] claimed that the result of (his interpretation of) special relativity is the same as that of general relativity.

In 1988, Frisch [22] discussed “post-Newtonian” approximations, claiming that use of “relativistic momentum” but Newtonian gravity gives the result of Goldstein [18], 1/6 of the observed precession of the perihelion of mercury, while including the gravitation due to gravitational field energy doubles the result, to 1/3 of the observed precession of the perihelion of mercury.

In 1989, Peters [23] argued that Biswas’ calculation was in error.

In 2008, Biswas [24] published another version of his 1998 paper [21], again claiming that his model, based on special relativity, explains the full precession of the perihelion of Mercury.

In 2015, Wayne [25] claimed that special relativity can explain the precession of the perihelion of Mercury.

In 2016, Lemmon and Mondragon [26] argued that special relativity predicts 1/3 of the rate of the precession of the perihelion according to general relativity.

In 2020, Corda [27] claimed that special relativity can explain the precession of the perihelion of Mercury, invoking time dilation.

In 2022, D’Abramo [28] claimed that Corda [27] was wrong.

What is going on here?

## 2 Solution

### 2.1 Goldstein

Goldstein [18] deduced the form of a single-particle Lagrangian in special relativity as,

$$\mathcal{L} = -mc^2\sqrt{1 - v^2/c^2} - V,$$

where \(m\) is the rest mass of the particle, whose velocity is \(v\), and \(V\) is the potential energy of the particle. The low-velocity limit of eq. (5) is the usual nonrelativistic form \(\mathcal{L} = T - V\) with kinetic energy \(T = mv^2/2\).

\(^3\)An illustration of how an \textit{ad hoc} modification to Newton’s “orbit equation” can reproduce the result of general relativity for the precession of the perihelion is given, for example, in Sec. 8.9, p. 312 of [17].

\(^4\)Exercise 26, Chap. 7 of the 3\textsuperscript{rd} ed. (2002)
Goldstein argued that for a particle of electric charge \( q \) in an electromagnetic field, one can use \( V = q\phi - \mathbf{v} \cdot \mathbf{A} \) (in SI units), where \( (\phi, \mathbf{A}) \) are the electromagnetic potentials of the field in some gauge. However, Goldstein did not mention what he considered the gravitational potential to be in special relativity. One infers that he assumed it to be just the nonrelativistic form \( V = -\frac{GMm}{r} \) for (spherical) masses \( M \) and \( m \) distance \( r \) apart.

### 2.2 Phipps

Phipps [19] considered the Lagrangian (5), but supposed the gravitational potential in special relativity, for (spherical) mass \( m \) with velocity \( \mathbf{v} \) with respect to (large, spherical) mass \( M \) is,

\[
V = -\frac{GMm}{r\sqrt{1 - v^2/c^2}} = -\frac{\gamma GMm}{r}, \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.
\]

That is, he supposed that the “relativistic mass” \( \gamma m \) is also the “gravitational mass” of \( m \).

### 2.3 Peters

This section follows the analysis by Peters in [20]. He used the Lagrangian (5) and Phipps’ form (6) of the gravitational potential energy, and expanded these in powers of \( 1/c \), writing,

\[
\mathcal{L} \approx -mc^2 + \frac{mv^2}{2} + \frac{mv^4}{8c^2} + \frac{GMm}{r} \left(1 + \frac{\alpha v^2}{2c^2}\right),
\]

where \( \alpha = 1 \) for the potential (6), while \( \alpha = 0 \) for the nonrelativistic potential used by Goldstein [18].

Rather than proceeding to deduce Lagrange’s equations of motion, Peters considered the Hamiltonian (anticipating that \( dH/d\theta \) will be an “orbit equation”),

\[
H = \sum_{i=1,3} p_i v_i - \mathcal{L}, \quad p_i = \frac{\partial \mathcal{L}}{\partial \dot{v}_i} \approx mv_i \left(1 + \frac{v^2}{2c^2} + \frac{\alpha GM}{rc^2}\right),
\]

\[
H \approx mv^2 \left(1 + \frac{v^2}{2c^2} + \frac{\alpha GM}{rc^2}\right) + mc^2 - \frac{mv^2}{2} - \frac{mv^4}{8c^2} - \frac{GMm}{r} \left(1 + \frac{\alpha v^2}{2c^2}\right)
\]

\[
= mc^2 + \frac{mv^2}{2} + \frac{3mv^4}{8c^2} - \frac{GMm}{r} \left(1 - \frac{\alpha v^2}{2c^2}\right).
\]

The Hamiltonian (9) is independent of time, and so is a constant of the motion, but it is not the total energy \( E = T + V \), from which it differs by the sign of the term in \( \alpha \). For \( \alpha = 0 \), Goldstein’s assumption, the Hamiltonian is the energy.

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5 In 1911, Einstein [29] considered that ”gravitational” mass equals (relativistic) inertial mass, although this notion does not appear in the later theory of general relativity, where only rest mass is emphasized.

6 In sec. III of [19], Phipps considered the motion of mass \( m \) along a radius with respect to (fixed) mass \( M \), and speculated that this might involve some kind of “antigravity” effect. He seemed unaware of the literature on this topic, which is reviewed in [30].
This has the implication that the total energy is not conserved in Phipps’ model, which suggests that Phipps’ model is not physically plausible, and perhaps should not be considered further.\footnote{Instead of using the approximate Lagrangian (14), we can consider form (5) with the potential energy (6). Then, instead of eqs. (8)-(9) we have,}

\[\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{\gamma GMm}{r}, \quad H = \sum_{i=1}^{3} p_i v_i - \mathcal{L}, \quad p_i = \frac{\partial \mathcal{L}}{\partial v_i} = \gamma m v_i + \frac{\gamma^2 G M m v_i}{r c^2}, \quad H = \gamma m v^2 + \frac{\gamma^2 G M m v^2}{r c^2} + \frac{m c^2}{\gamma} - \frac{\gamma G M m}{r} = \gamma m c^2 + (\gamma^2 - \gamma - 1) \frac{G M m}{r}. \]

\footnote{If we ignore gravity, setting $G = 0$, we just have a free particle of mass $m$, with total energy $\gamma m c^2$, which equals the Hamiltonian of eq. (12) in this case.}

\footnote{For Goldstein’s assumption that Newtonian gravity should hold in special relativity, we set $\gamma = 1$ in the terms involving $G$, and the Hamiltonian (12) is the total energy $\gamma m c^2 + V$ (for the Newtonian potential $V = -G M m/r$).}

\footnote{But, for Phipps’ model, total energy is not conserved, while the Hamiltonian of (12) is a conserved quantity.}

\footnote{Conservation of energy is a complicated issue in general relativity. See, for example, [31]. But, if we ignore the nonlinear effect of the curvature of spacetime by the moving object on that object itself, and ignore the gravitational radiation of the moving object, one can consider that energy in conserved in the general-relativistic description of the orbit of the moving object about the much larger “fixed” mass $M$.}

\subsection{Continuation of Peters’ Analysis}

For completeness, I transcribe the rest of Peters analysis in [20].

The (planar) orbit can be described by coordinates $r$ and $\theta$, in which the velocity can be expressed as,

\[v^2 = r^2 + r^2 \dot{\theta}^2. \quad (13)\]

The Lagrangian (7) does not depend on $\theta$, so there is a conserved canonical momentum,

\[L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \approx m r^2 \dot{\theta} \left( 1 + \frac{v^2}{2 c^2} + \frac{\alpha G M}{r c^2} \right), \quad (14)\]

which we identify as the orbital angular momentum. With this, we can express the angular velocity as,

\[\dot{\theta} \approx \frac{L}{m r^2} \left( 1 - \frac{v^2}{2 c^2} - \frac{\alpha G M}{r c^2} \right). \quad (15)\]

Next, Peters followed a method of Newton to replace the radius $r$ by its reciprocal $u = 1/r$. For this, we note that,

\[v^2 = r^2 + r^2 \dot{\theta}^2 = \left( \frac{d}{dt} \frac{1}{u} \right)^2 + \frac{\dot{\theta}^2}{u^2} = \left( \frac{d \theta}{dt} \frac{d}{d \theta} \frac{1}{u} \right)^2 + \frac{\dot{\theta}^2}{u^2} = \left( -\dot{\theta} \frac{u'}{u^2} \right)^2 + \frac{\dot{\theta}^2}{u^2} = \frac{\dot{\theta}^2 u'^2}{u^4} + \frac{\dot{\theta}^2}{u^2} = (u'^2 + u^2) \frac{\dot{\theta}^2}{u^4} = (u'^2 + u^2) \frac{\dot{\theta}^2}{u^4}. \quad (16)\]
where \( u' = du/d\theta \). From eq. (15), we have,

\[
r^4\dot{\theta}^2 \approx \frac{L^2}{m^2} \left( 1 - \frac{v^2}{c^2} - \frac{2\alpha GM}{rc^2} \right). \tag{17}
\]

Using this in eq. (16), we find,

\[
v^2 \approx \frac{L^2}{m^2}(u'^2 + u^2) \left( 1 - \frac{v^2}{c^2} - \frac{2\alpha GM}{rc^2} \right) + \left( 4u^2 - \frac{u}{a} + \frac{1}{a^2} \right) - GMmu \left( 1 + \frac{\alpha v^2}{2c^2} \right) = \frac{2GMm}{r} - \frac{GM}{a} = GM\left( 2u - \frac{1}{a} \right). \tag{18}
\]

Peters noted that in the first approximation, \( v^2 \approx (L^2/m^2)(u'^2 + u^2) \), to rewrite eq. (18) as,

\[
v^2 \approx \frac{L^2}{m^2}(u'^2 + u^2) - \frac{m^2v^4}{8c^2} - \frac{2\alpha GMv^2}{rc^2}. \tag{19}
\]

With this, the Hamiltonian (9) becomes,

\[
H \approx mc^2 + \frac{L^2}{2m}(u'^2 + u^2) - \frac{mv^4}{8c^2} - GMmu \left( 1 + \frac{\alpha v^2}{2c^2} \right). \tag{20}
\]

For a Newtonian orbit, the total energy \( E \) is related to the semimajor axis \( a \) by,

\[
E = \frac{mv^2}{2} - \frac{GMm}{r} = -\frac{GMm}{2a}, \quad v^2 = \frac{2GM}{r} - \frac{GM}{a} = GM\left( 2u - \frac{1}{a} \right). \tag{21}
\]

We follow Peters in using the Newtonian relation (21) for \( v^2 \) in the terms of order \( 1/c^2 \) of the special-relativistic Hamiltonian (20) to find,

\[
H \approx mc^2 + \frac{L^2}{2m}(u'^2 + u^2) - \frac{G^2M^2m}{8c^2} \left( 4u^2 - \frac{u}{a} + \frac{1}{a^2} \right) - GMmu \left( 1 + \frac{\alpha GM}{2c^2} \left( 2u - \frac{1}{a} \right) \right) = mc^2 + \frac{L^2}{2m}(u'^2 + u^2) - GMmu - \frac{G^2M^2m}{2c^2} \left( (1 + 2\alpha)u^2 - \frac{(1 + \alpha)u}{a} + \frac{1}{4a^2} \right). \tag{22}
\]

Then, we can find an “orbit equation” by taking the derivative of the (constant) Hamiltonian with respect to \( \theta \),

\[
H' = \frac{dH}{d\theta} \approx \frac{L^2}{m}u''(u' + u) - GMmu' - \frac{G^2M^2m}{2c^2} \left( 2(1 + 2\alpha)uu' - \frac{(1 + \alpha)u'}{a} \right) = 0, \tag{23}
\]

\[
u'' + u \left( 1 - \frac{(1 + 2\alpha)G^2M^2m^2}{c^2L^2} \right) \approx \frac{GMm^2}{L^2} - \frac{G^2M^2m^2(1 + \alpha)}{2c^2L^2} = \text{const.} \tag{24}
\]

To order \( 1/c^2 \), it suffices to use the value \( L^2 = GMm^2a(1 - \epsilon^2) \) of the Newtonian orbit,\(^9\) so the special-relativistic orbit equation is,

\[
u'' + (1 - k)u = \text{const}, \quad \text{where} \quad k = \frac{(1 + 2\alpha)GM}{ac^2(1 - \epsilon^2)} \ll 1. \tag{25}
\]

\(^9\)See, for example, eq. (3.63) of [18].
This equation has solutions of the form,
\[ u(\theta) = u_0 + u_1 \cos(\sqrt{1-k}\, \theta), \tag{26} \]
whose period \( T \) in angle \( \theta \) is, for small \( k \),
\[ T = \frac{2\pi}{\sqrt{1-k}} \approx 2\pi \left( 1 + \frac{k}{2} \right) = 2\pi + \pi k. \tag{27} \]
That is, the perihelion of the orbit advances/precesses by angle,
\[ \Delta \theta = \pi k = (1 + 2\alpha) \frac{\pi GM}{ac^2(1-\epsilon^2)}, \tag{28} \]
per revolution. For \( \alpha = 0 \) (Newtonian gravity) this is 1/6 of the result (1) of general relativity, as claimed in [18]. For the model that the relativistic mass \( \gamma m \) is also the gravitational mass, as in eq. (5), \( \alpha = 1 \) and the precession of the perihelion is 1/2 that of general relativity.

The result (1) of general relativity would be predicted by a version of Phipps’ model with \( \alpha = 5/2 \), but this model would not conserve energy, and so cannot be considered a reasonable explanation.

### 2.4 Biswas

Biswas’ model [21] seems to be essentially the same as that of Phipps, but with an additional dimensionless parameter, \( \kappa \) of his eq. (19b), which he sets to 2 to obtain agreement with the observed precession of the perihelion of Mercury.

### 2.5 Frisch

Frisch’s second “post-Newtonian approximation” [22] included an effect of “time dilation” vs. distance \( r \) from the mass \( M \) (in the rest frame of that mass), which is a kind of “curved time” that goes beyond special relativity, the nominal theme of the present note. He claimed this led to 1/3 of the observed precession of the perihelion of Mercury.

I consider that “post-Newtonian approximations” start from knowledge of general relativity and “work backwards” to formalism that resembles Newtonian mechanics, (without metric tensors and notions of curved spacetime). In contrast, this note concerns additions to special relativity to include gravity that is related to a single scalar potential for a 2-body system, perhaps via the “weak equivalence principle, that inertial mass is the same as “gravitational mass”.

### 2.6 Wayne

Wayne [25] seemed to argue that the “relativistic mass” of a moving object with rest mass \( m \) should be \( m(1 + v^2/c^2) \) at order \( 1/c^2 \) rather then the usual approximation \( m(1 + v^2/2c^2) \), and then claimed that “special relativity” predicts the observed precession of the perihelion of Mercury.
2.7 Lemmon and Mondragon

Lemmon and Mondragon [26] first reviewed that model of Goldstein [18], using “relativistic momentum” but Newtonian gravity, confirming that this model predicts 1/6 of the observed precession of the perihelion of Mercury. In their Sec. IV, they appeared to consider the model of Phipps [19], which assumes that the “gravitational mass” of moving object $m$ is its “relativistic mass $\gamma m$. But they proceeded in a slightly different manner than the analysis of Peters [20] reviewed in our sec. 2.3 above, arriving at the potential energy given in their eq. (37). From this, they inferred that the precession of the perihelion of Mercury is 1/3 that predicted by general relativity (which happens to agree with the “post-Newtonian” result of Frisch, which is a different model).

I am skeptical of the approach of Lemmon and Mondragon, and consider that the analysis of Peters, given above, is the more correct version of Phipps’ model.

2.8 Corda

The paper of Corda [27] invokes effects that are more in the spirit of a “post-Newtonian approximation” to general relativity (as in the paper of Frisch [22]), than an extension of special relativity.

Thanks to Derek Abbott and Germano D’Abramo for e-discussions of the problem.

References

http://kirkmcd.princeton.edu/examples/GR/einstein_skpaw_831_15.pdf


http://kirkmcd.princeton.edu/examples/GR/dicke_science_184_419_74.pdf


See p. 25 for a summary of the data.

http://kirkmcd.princeton.edu/examples/GR/leverrier_aoip_5_1_59.pdf
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Several subsequent discussion are appended to this file.

http://kirkmcd.princeton.edu/examples/GR/einstein_BT_200827.pdf
See the appended notes in English.


http://kirkmcd.princeton.edu/examples/GR/biswas_aip_56_1032_88.pdf
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