Perfectly Conducting Cylinder in an External, Static Magnetic Field

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(October 29, 2011; updated July 26, 2019)

1 Problem

Discuss the magnetic field outside, and the currents on the surface, of a perfectly conducting cylinder that is in an external, static magnetic field which, in the absence of the cylinder, is uniform and perpendicular to the axis of the cylinder.

Consider also the case that the cylinder has uniform velocity $v \ll c$ perpendicular to its axis and to the direction of the external magnetic field, where c is the speed of light in vacuum. Discuss the flow of energy in a dynamo based on a perfectly conducting cylinder that slides on a U-shaped track in a transverse magnetic field.

2 Solution

This problem is similar to that of a perfectly conducting cylinder in a uniform, static electric field. Outside the cylinder, (and inside the sources of the external, static electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$), we have that $\boldsymbol{\nabla} \times \mathbf{E} = 0 = \boldsymbol{\nabla} \times \mathbf{B}$, so the fields outside the cylinder (or radius a) can be derived from a scalar potential Φ,

$$\mathbf{B}(r > a) = -\boldsymbol{\nabla}\Phi_B, \qquad \mathbf{E}(r > a) = -\boldsymbol{\nabla}\Phi_E, \tag{1}$$

in a cylindrical coordinate system (r, θ, z) with the z-axis being that of the cylinder. Of course, $\boldsymbol{\nabla} \times \mathbf{E} = 0$ everywhere for a static electric field, while at the surface of the perfectly conducting cylinder $\boldsymbol{\nabla} \times \mathbf{B}$ is nonzero (and proportional to the surface current $\mathbf{K}$ on the cylinder). Inside the perfectly conducting cylinder there can be no (static) currents, so $\boldsymbol{\nabla} \times \mathbf{B} = 0$ there, and the magnetic scalar potential Φ_B can have any constant value for $r < a$. The electric scalar potential Φ_E also has a constant value for $r < a$, which is equal to the value of the potential at the surface $r = a$.

In addition, $\boldsymbol{\nabla} \times \mathbf{B} = 0$ everywhere, while $\boldsymbol{\nabla} \times \mathbf{E} = 0$ except on the surface $r = a$ which can support a nonzero surface charge density σ. Thus, for both $r > a$ and $r < a$, the scalar potentials obey Laplace's equation $\nabla^2\Phi = 0$, whose general solution in cylindrical coordinates has the form (see, for example, sec. 2.11 of [11]),

$$\Phi = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta + \alpha_n) + \sum_{n=1}^{\infty} \frac{b_n}{r^n} \cos(n\theta + \beta_n). \tag{2}$$

At the surface of the perfectly conducting cylinder the magnetic field can have no radial component, $B_r(r = a) = 0$, and the electric field can have no tangential component, $E_z(r = a) = 0 = E_\theta(r = a)$; otherwise there would be infinite surface currents on the cylinder.

We suppose that the external electric and magnetic fields $\mathbf{E}_0$ and $\mathbf{B}_0$ are in the $+x$-direction, so they can be derived from the scalar potentials,

$$\Phi_{0,E} = -E_0\, x = -E_0\, r\cos\theta, \qquad \Phi_{0,B} = -B_0\, x = -B_0\, r\cos\theta. \tag{3}$$

Then, the total potentials for $r > a$ must have the forms,

$$\Phi_E = -E_0\, r\cos\theta + \frac{E_1\cos\theta}{r}\,, \qquad \Phi_B = -B_0\, r\cos\theta + \frac{B_1\cos\theta}{r}\,, \tag{4}$$

referring to eq. (2). The field components for $r > a$ are,

$$E_r(B_r) = -\frac{\partial\Phi_{E(B)}}{\partial r} = E_0(B_0)\,\cos\theta + \frac{E_1(B_1)}{r^2}\cos\theta\,, \tag{5}$$

$$E_\theta(B_\theta) = -\frac{1}{r}\frac{\partial\Phi_{E(B)}}{\partial\theta} = -E_0(B_0)\,\sin\theta + \frac{E_1(B_1)}{r^2}\sin\theta\,. \tag{6}$$

The tangential electric field, E_θ, and the perpendicular magnetic field, B_r, vanish at the surface $r = a$ of a perfect conductor, so we learn that,

$$E_1 = a^2 E_0, \qquad \text{and} \qquad B_1 = -a^2 B_0. \tag{7}$$

Hence, the fields for $r > a$ are,

$$E_r = E_0\,\cos\theta\left(1 + \frac{a^2}{r^2}\right), \tag{8}$$

$$E_\theta = -E_0\,\sin\theta\left(1 - \frac{a^2}{r^2}\right), \tag{9}$$

$$B_r = B_0\,\cos\theta\left(1 - \frac{a^2}{r^2}\right), \tag{10}$$

$$B_\theta = -B_0\,\sin\theta\left(1 + \frac{a^2}{r^2}\right). \tag{11}$$

Since the electric and magnetic fields are zero inside the perfectly conducting cylinder, the surface charge density σ at $r = a$ is (in Gaussian units),

$$\sigma = \frac{E_r(r = a)}{4\pi} = \frac{E_0}{2\pi}\cos\theta, \tag{12}$$

and the surface current $\mathbf{K} = (c/4\pi)\,\hat{\mathbf{r}} \times \mathbf{B}(r = a)$ is,[1]

$$K_z = \frac{c}{4\pi}B_\theta(r = a) = -\frac{c}{2\pi}B_0\,\sin\theta. \tag{13}$$

Both the total charge and current densities on the surface $r = a$ are zero. The surface current is in the $-z$ direction for $y > 0$, and in the $+z$ direction for $y < 0$. The magnetic field due to these currents is in the $-x$ direction for $|y| < a$, as needed to cancel the external magnetic field $B_0\,\hat{\mathbf{x}}$ at the surface of the perfectly conducting cylinder, where the perpendicular component (B_x at $y = 0$) of the magnetic field must be zero.

[1]The surface charges and currents experience outward radial forces proportional to E_r^2 and B_θ^2, respectively. A nonclassical force is required to keep the charges and currents from leaving the surface. Here, we simply assume that charges and currents can exist on the surface of a perfect conductor.

2.1 Rotating Cylinder

This section added July 23, 2019, following a suggestion by J. Castro Paredes.

If the perfectly conducting cylinder rotates about its axis, the surface charge and current distributions (12)-(13) remain the same in the lab frame, while the cylinder rotates with respect to them. No energy is dissipated during this motion, which in principle could persist forever.

2.1.1 Poor Conductor

It may be of interest to consider also the case of a poorly conducting cylinder, such that the external magnetic field completely penetrates into the cylinder.[2] In this section we suppose that the external electric field $\mathbf{E}_0$ is zero, and we neglect the effect of magnetic fields induced by currents in the poorly conducting cylinder, such that $\mathbf{B} \approx \mathbf{B}_0$. Then, if the cylinder is not rotating, there are no charge or current distributions associated with the cylinder, and the electric field remains zero.

If the cylinder rotates with angular velocity ω about its (z) axis, an axial eddy-current density $J_z(r, \phi)$ is induced (plus radial currents at the ends of the long/infinite cylinder) by the changing magnetic flux through loops that rotate with the cylinder. In particular, we consider a "loop" consisting of the axis of the cylinder plus the line parallel (r_0, ϕ_0, z) to the z-axis, which is completed by radial line segments at $z = \pm\infty$. If we define the normal to this loop to be in the $+\hat{\boldsymbol{\phi}}$ direction, then the magnetic flux Φ per unit length in z through this loop is,

$$\Phi(r_0, \phi_0) \approx -B_0 r_0 \sin\phi_0. \tag{14}$$

As the loop rotates about the z-axis with angular velocity $\omega = d\phi_0/dt$, an electric field is induced along the line (r_0, ϕ_0, z) with strength,

$$E_z(r_0, \phi_0) = -\frac{\mathcal{EMF}}{\text{unit length}} = \frac{1}{c}\frac{d\Phi(r_0, \phi_0)}{dt} \approx \frac{B_0 r_0 \omega \cos\phi_0}{c}. \tag{15}$$

If the small, but nonzero, resistivity of the cylinder is ϱ, then the current density along the line (r_0, ϕ_0, z) is,

$$J_z(r_0, \phi_0) = \frac{E_z(r_0, \phi_0)}{\varrho} \approx \frac{B_0 r_0 \omega \cos\phi_0}{c\varrho}. \tag{16}$$

Note that the total current crossing any plane of constant z is zero.

The total power P per unit length dissipated by Joule heating is,

$$P = \tau\omega = \int_0^a r_0\, dr_0 \int_0^{2\pi} d\phi_0\, J_z^2 \varrho \approx \frac{\pi a^4 B_0^2 \omega^2}{4c^2 \varrho} = \frac{a^2 B_0^2 \omega^2}{4c^2 R}, \tag{17}$$

where $\tau \approx a^2 B_0^2 \omega / 4c^2 R$ is the torque per unit length required to rotate the cylinder, and $R = \varrho/\pi a^2$ is the resistance per unit length of the cylinder to axial currents. Thus, power PL would be required to keep a cylinder of length L rotating with angular velocity ω in the presence of static magnetic field $\mathbf{B}_0$ perpendicular to its axis.

[2] The case of an infinite cylinder made of a good conductor with relative permeability μ is considered in probs. 7.30-38 of [1], and in sec. 5.7 of [2]. Much of the technical literature on this theme emphasizes hollow cylinders of finite length; see, for example [3]-[6]. Rotating spheres and ellipsoids are discussed in [7]-[9]. For a general review, see [10]. It requires considerable effort to relate these analyses involving Bessel functions of imaginary argument to the simple results of this section.

3 The Cylinder Moves in the y-Direction at Constant Speed

The preceding analysis assumed that the cylinder was at rest. Suppose instead that the cylinder has speed $v \ll c$ in the y-direction, where c is the speed of light in vacuum.

In the rest frame of the cylinder, in which quantities are denoted with the superscript *, the external fields are,

$$\mathbf{E}^\star \approx \mathbf{E}_0 + \frac{\mathbf{v}}{c} \times \mathbf{B}_0, \qquad \mathbf{B}^\star \approx \mathbf{B}_0 - \frac{\mathbf{v}}{c} \times \mathbf{E}_0, \tag{18}$$

for $v \ll c$. When the external field in the lab frame is either purely electric or purely magnetic, we have that in the rest frame (whose velocity in the lab frame is $\mathbf{v} = v\,\hat{\mathbf{y}}$),

$$\mathbf{E}_0^\star \approx E_0\,\hat{\mathbf{x}}, \qquad \mathbf{B}_0^\star \approx \frac{v}{c} E_0\,\hat{\mathbf{z}}, \qquad \text{or} \qquad \mathbf{E}_0^\star \approx -\frac{v}{c} B_0\,\hat{\mathbf{z}}, \qquad \mathbf{B}_0^\star \approx B_0\,\hat{\mathbf{x}}. \tag{19}$$

Then, the results of sec. 2 apply in the rest frame of the cylinder for the effects of the x-components of the external fields, if we work in cylindrical coordinates (r', θ', z) with origin at the instantaneous axis of the moving cylinder. Then, $\hat{\mathbf{y}} = \sin\theta'\,\hat{\mathbf{r}}' + \cos\theta'\,\hat{\boldsymbol{\theta}}'$, *etc.*

The transformations of the fields from the rest frame to the lab frame are (for $v \ll c$),

$$\mathbf{E} \approx \mathbf{E}^\star - \frac{\mathbf{v}}{c} \times \mathbf{B}^\star, \qquad \mathbf{B} \approx \mathbf{B}^\star + \frac{\mathbf{v}}{c} \times \mathbf{E}^\star. \tag{20}$$

For an external field $E_0\,\hat{\mathbf{x}}$ in the lab frame, the magnetic field in the rest frame must vanish inside the cylinder. To cancel the magnetic field $\mathbf{B}_0^\star = vE_0\,\hat{\mathbf{z}}/c$ the cylinder must have an axial surface current,

$$K_{\theta'}^\star = -\frac{v}{4\pi} E_0, \tag{21}$$

which creates axial field $B_z^\star = -vE_0/c$ inside the cylinder. In the idealization of an infinite cylinder, this surface current creates vanishing small magnetic field outside the cylinder, where the magnetic field remains $\mathbf{B}_0^\star$. Hence, the fields in the lab frame for $r' \geq a$ are,

$$E_{r'} = E_r^\star = E_0 \cos\theta' \left(1 + \frac{a^2}{r'^2}\right), \tag{22}$$

$$E_{\theta'} = E_\theta^\star = -E_0 \sin\theta' \left(1 - \frac{a^2}{r'^2}\right), \tag{23}$$

$$B_z = \frac{v}{c} E_0 - \frac{v}{c}(\cos\theta' E_r^\star - \sin\theta' E_\theta^\star) = -\frac{v}{c} E_0 \frac{a^2}{r'^2} \cos 2\theta'. \tag{24}$$

The surface charge and current densities obey the transformations (for $v \ll c$),

$$\sigma \approx \sigma^\star + \frac{\mathbf{v} \cdot \mathbf{K}^\star}{c^2} = \frac{E_0}{2\pi} \cos\theta' \left(1 - \frac{v^2}{2c^2}\right) \approx \frac{E_0}{2\pi} \cos\theta', \tag{25}$$

$$\mathbf{K} \approx \mathbf{K}^\star + \sigma^\star \mathbf{v}_{\theta'} = \frac{vE_0}{4\pi} \cos 2\theta'\,\hat{\boldsymbol{\theta}}', \tag{26}$$

recalling eq. (12). The lab-frame surface current (26) is the source of the external, lab-frame magnetic field (24), and obeys the condition that $\mathbf{B}(r' = a^+) = (4\pi/c)\mathbf{K} \times \hat{\mathbf{r}}'$, noting that $\mathbf{B}$ vanishes inside the cylinder.

For the case of only an external magnetic field $B_0\,\hat{\mathbf{x}}$ in the lab frame, the electric field inside the cylinder must vanish in its rest frame. To cancel the field $\mathbf{E}_0^\star \approx -vB_0\,\hat{\mathbf{z}}/c$ the cylinder takes on a surface charge density $\sigma^\star = kz$, which leads to an electric scalar potential $V^\star$ given by,

$$V^\star(r', 0, z) = \int_0^{2\pi} a\,d\theta' \int_{-\infty}^{\infty} dz' \frac{kz'}{\sqrt{(z'-z)^2 + a^2 + r'^2 + 2ar'\cos\theta'}} = 2\pi akz. \tag{27}$$

The corresponding electric field is $E_z^\star = -2\pi ak$, so this cancels the axial electric field $-vB_0/c$ everywhere in the rest frame of the cylinder for $k = -vB_0/2\pi ac$. Hence, the fields in the lab frame are,

$$B_{r'} = B_r^\star = B_0\,\cos\theta' \left(1 - \frac{a^2}{r'^2}\right), \tag{28}$$

$$B_{\theta'} = B_\theta^\star = -B_0\,\sin\theta' \left(1 + \frac{a^2}{r'^2}\right), \tag{29}$$

$$E_z = \frac{v}{c}(\cos\theta' B_r^\star - \sin\theta' B_\theta^\star) = \frac{v}{c}B_0 \left(1 - \frac{a^2\cos 2\theta'}{r'^2}\right), \tag{30}$$

for $r' \geq a$. The surface charge and current densities in the lab frame are given by (for $v \ll c$),

$$\sigma \approx \sigma^\star + \frac{\mathbf{v}\cdot\mathbf{K}^\star}{c^2} = -\frac{vB_0 z}{2\pi ac}, \tag{31}$$

$$\mathbf{K} \approx \mathbf{K}^\star + \sigma^\star \mathbf{v}_{\theta'} \approx -\frac{cB_0}{2\pi}\sin\theta'\,\hat{\mathbf{z}}, \tag{32}$$

recalling eq. (13).

It is noteworthy that the lab-frame electric field at the surface of the conducting cylinder,

$$E_z(r' = a) = \frac{v}{c}B_0(1 - \cos 2\theta') = \frac{2v}{c}B_0\sin^2\theta', \tag{33}$$

is tangential to the surface of the cylinder. Thus, we see that **the condition that the electric field be perpendicular to the surface of a perfectly conducting cylinder holds only in the rest frame of the cylinder**.[3] Note also that the electric field (33) is everywhere positive, while the surface current (32) is positive only for $\sin\theta' < 0$. That is, **in a moving perfect conductor the signs of the surface current and surface electric field are not correlated**.

[3]The condition that the magnetic field have no component normal to the surface of a perfect conductor holds in any frame, in that the electric field in the rest frame is normal to the surface, so its transform $\propto \mathbf{v} \times \mathbf{E}_\perp$ to a magnetic field in a frame with velocity $\mathbf{v}$ also has no normal component. Hence, lines of the magnetic field can never cross a perfect conductor, whether at rest or in motion, since they would need to be normal to the surface at some point if they were to cross. An implication for the present example is that the total magnetic flux in y "above" and "below" the perfectly conducting cylinder remains constant as the cylinder moves. Only for an external field of infinite extent in y can that field remain constant during the motion of the cylinder, as tacitly assumed here.

3.1 Stored Electromagnetic Energy

The volume density u of electromagnetic energy stored in the fields is given by,

$$u = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} = \frac{(\mathbf{E}_0 + \mathbf{E}_a)^2 + (\mathbf{B}_0 + \mathbf{B}_a)^2}{8\pi} \equiv \frac{\mathbf{E}_0^2 + \mathbf{B}_0^2}{8\pi} + u_a, \tag{34}$$

where $\mathbf{E}_a$, $\mathbf{B}_a$ and u_a are the changes to the fields and stored energy when a perfectly conducting cylinder of radius a is present (and moving with velocity $v\,\hat{\mathbf{y}}$); these terms all include factors of a or v.

For the case of an external electric field, we have from eqs. (22)-(24) that (to order v/c),

$$u_a = \frac{E_0^2}{8\pi}\left(\frac{2a^2}{r'^2}\cos 2\theta' + \frac{a^4}{r'^4}\,.\right) \tag{35}$$

Similarly, for the case of an external magnetic field, we have from eqs. (28)-(30) that (to order v/c),

$$u_a = \frac{B_0^2}{8\pi}\left(-\frac{2a^2}{r'^2}\cos 2\theta' + \frac{a^4}{r'^4}\,.\right) \tag{36}$$

For nonzero v these energy densities are time dependent, so there must be a flow of energy in the vicinity of the cylinder in these cases.

3.2 Energy Flow

When the cylinder is in motion, we can expect it to be associated with a flow of electromagnetic energy. To discuss this, we consider the Poynting vector $\mathbf{S}$,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}\left[(E_{\theta'}B_z - E_z B_{\theta'})\,\hat{\mathbf{r}}' + (E_z B_{r'} - E_{r'}B_z)\,\hat{\boldsymbol{\theta}}' + (E_{r'}B_{\theta'} - E'_\theta z B_{r'})\,\hat{\mathbf{z}}\right], \tag{37}$$

which implements conservation of energy according to,

$$\boldsymbol{\nabla}\cdot\mathbf{S} = -\frac{\partial u}{\partial t}\,, \tag{38}$$

for $r' > a$.

For the case of an external electric field, we have from eqs. (22)-(24) that,

$$\mathbf{S} \;=\; -\frac{vE_0^2}{4\pi}\frac{a^2}{r'^2}\cos 2\theta'\left[\sin\theta'\left(1 - \frac{a^2}{r'^2}\right)\hat{\mathbf{r}}' + \cos\theta'\left(1 + \frac{a^2}{r'^2}\right)\hat{\boldsymbol{\theta}}'\right]. \tag{39}$$

In the limit $a \to 0$ there is no flow of energy in the lab frame, where there is only a constant electric field plus a moving charged wire that (perhaps surprisingly) produces no $\mathbf{E}$ or $\mathbf{B}$ fields.

Similarly, for the case of an external magnetic field, we have from eqs. (28)-(30) that,

$$\mathbf{S} \;=\; \frac{vB_0^2}{4\pi}\left(1 - \frac{a^2}{r'^2}\cos 2\theta'\right)\left[\sin\theta'\left(1 + \frac{a^2}{r'^2}\right)\hat{\mathbf{r}}' - \cos\theta'\left(1 - \frac{a^2}{r'^2}\right)\hat{\boldsymbol{\theta}}'\right]. \tag{40}$$

In the limit $a \to 0$ the energy flow is nonzero, $\mathbf{S} = -vE_0^2\, \hat{\mathbf{y}}/4\pi$, since according to eq. (30) the moving, perfectly conducting wire is associated with a uniform electric field $E_z = -vB_0/c$ in the lab frame.

Note that $S_{r'}(r' = a)$ is nonzero for the case of a moving cylinder in a magnetic field, while averaging to zero over azimuth θ'. A perfect conductor at rest cannot have any Poynting flux perpendicular to its surface, but we see that this is not necessarily so for a perfect conductor in motion. This is an example of the relativity of steady energy flow, which is explored further in sec. 4 below.

The flow of energy described by eqs. (39) and (40) falls off at least as fast as $1/r'^2$ (in cylindrical coordinates), and does not correspond to "radiation to infinity". The energy flow is significant only in the "near zone", whose extent is a few times the radius a of the cylinder. Within this "near zone" the stored electromagnetic energy is rearranged as the cylinder moves, and the Poynting vector $\mathbf{S}$ describes the details of this rearrangement.

3.3 Forces on the Cylinder

As discussed elsewhere [12], the least ambiguous method to calculate force $\mathbf{F}$ on a rigid body in an electromagnetic field is via the Maxwell stress tensor T_{ij}, which relates to the force F_i on a surface element $d\text{Area}_j$ according to,

$$F_i = \int \sum_k T_{ij}\; d\text{Area}_j, \tag{41}$$

where,

$$T_{ij} = \frac{E_i D_j + B_i H_j}{4\pi} - \delta_{ij} \frac{\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}}{8\pi}\,. \tag{42}$$

For the case of an external electric field, we have from eqs. (22)-(24) that at the surface $r' = a$ of the moving cylinder,

$$\begin{pmatrix} T_{r'r'} & T_{r'\theta'} & T_{r'z} \\ T_{\theta'r'} & T_{\theta'\theta'} & T_{\theta'z} \\ T_{zr'} & T_{z\theta'} & T_{zz} \end{pmatrix} = \frac{E_0^2 \cos^2\theta'}{2\pi} \begin{pmatrix} 1 - \frac{v^2}{c^2}\cos^2\theta' & 0 & 0 \\ 0 & -1 - \frac{v^2}{c^2}\cos^2\theta' & 0 \\ 0 & 0 & -1 + \frac{v^2}{c^2}\cos^2\theta' \end{pmatrix}, \tag{43}$$

which is independent of the speed v of the cylinder at order v/c.

The area element is $d\text{Area} = a\, d\theta'\, dz\, \hat{\mathbf{r}}'$, so the only nonzero force element on the surface is (to order v/c),[4]

$$dF_{r'} = T_{r'r'}\, d\text{Area}_{r'} = \frac{aE_0^2\cos^2\theta'}{2\pi}\, d\theta'\, dz = \frac{\sigma E_{r'}}{2}\, a\, d\theta'\, dz. \tag{44}$$

The x- and y-force elements are related by,

$$dF_x = dF_{r'}\cos\theta' - dF_{\theta'}\sin\theta' = \frac{aE_0^2\cos^3\theta'}{2\pi}\, d\theta'\, dz = \frac{aE_0^2(3\cos\theta' + \cos 3\theta')}{8\pi}\, d\theta'\, dz\,, \tag{45}$$

$$dF_y = dF_{r'}\sin\theta' + dF_{\theta'}\cos\theta' = \frac{aE_0^2\sin\theta'\cos^2\theta'}{2\pi}\, d\theta'\, dz = \frac{aE_0^2(\sin\theta' + \sin 3\theta')}{8\pi}\, d\theta'\, dz\,. \tag{46}$$

[4]Note that the radial force (44) is the product of the surface charge density and 1/2 the surface field $E_{r'}$.

Integrating over θ', we find that $F_x = F_y = 0$.

For the case of an external magnetic field, we have from eqs. (28)-(30) that at the surface $r' = a$ of the moving cylinder,

$$\begin{pmatrix} T_{r'r'} & T_{r'\theta'} & T_{r'z} \\ T_{\theta'r'} & T_{\theta'\theta'} & T_{\theta'z} \\ T_{zr'} & T_{z\theta'} & T_{zz} \end{pmatrix} = \frac{B_0^2 \sin^2\theta'}{2\pi} \begin{pmatrix} -1 - \frac{v^2}{c^2}\sin^2\theta' & 0 & 0 \\ 0 & 1 - \frac{v^2}{c^2}\sin^2\theta' & 0 \\ 0 & 0 & -1 + \frac{v^2}{c^2}\sin^2\theta' \end{pmatrix}, \tag{47}$$

which is independent of the speed v of the cylinder at order v/c.

The area element is $d\text{Area} = a\, d\theta'\, dz\, \hat{\mathbf{r}}'$, so the only nonzero force element on the surface is (to order v/c)[5]

$$dF_{r'} = T_{r'r'}\, d\text{Area}_{r'} = -\frac{aB_0^2 \sin^2\theta'}{2\pi}\, d\theta'\, dz = -\frac{K_z B_{\theta'}}{2c}\, a\, d\theta'\, dz. \tag{48}$$

The x- and y-force elements are related by,

$$dF_x = dF_{r'}\cos\theta' - dF_{\theta'}\sin\theta' = \frac{aB_0^2\cos\theta'\sin^2\theta'}{2\pi}\, d\theta'\, dz = \frac{aB_0^2(\cos\theta' - \cos 3\theta')}{8\pi}\, d\theta'\, dz\,, \tag{49}$$

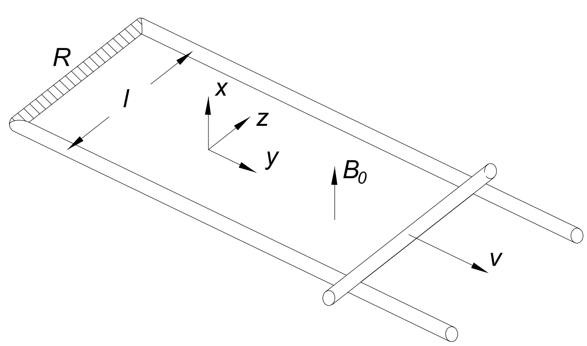
$$dF_y = dF_{r'}\sin\theta' + dF_{\theta'}\cos\theta' = \frac{aB_0^2\sin^3\theta'}{2\pi}\, d\theta'\, dz = \frac{aB_0^2(3\sin\theta' - \sin 3\theta')}{8\pi}\, d\theta'\, dz\,. \tag{50}$$

Integrating over θ', we find that $F_x = F_y = 0$.

Thus, no external force is required for uniform motion of a perfectly conducting cylinder through an external, static electric or magnetic field that is perpendicular to the axis of the cylinder.

4 Use of a Perfectly Conducting Cylinder in a Dynamo

A conceptually simple dynamo consists of a perfectly conducting cylinder that slides with velocity $v\,\hat{\mathbf{y}}$ along a U-shaped track whose cross piece (of length l) is a resistive load R, subject to external magnetic field $B_0\,\hat{\mathbf{x}}$. as shown in the sketch below. For an interesting example of such a dynamo, see [13].



[5]Note that the force (48) is the product of the surface current density and 1/2 the surface field $B_{\theta'}$.

The magnetic flux through this partially moving circuit increases linearly with time, $\Phi_M = B_0\, lvt$, so application of Faraday's law[6] indicates that there should be an $\mathcal{EMF} = -B_0\, lv/c$ around the circuit, which drives a clockwise current $I = B_0\, lv/cR$.[7]

The methods used in the preceding sections for a moving, perfectly conducting cylinder applies to this configuration, to a good approximation, if the length of the cross piece/load, and its distance from the moving conductor, are large compared to the radius a of the conductor. Then, the effect on the sliding bar of the magnetic field of the currents in the U can be neglected.

The z-component of the electric field along the surface of the sliding, perfectly conducting cylinder varies with azimuth as $\sin^2\theta'$ according to eq. (33), which vanishes at $\theta' = \pi$ where the sliding cylinder contacts the U-shaped conductor. Hence, we might infer that there is no $\mathcal{EMF} = \int E_z dl$ between the long conductors of the U, and no current flows in the circuit. However, this analysis does not agree with our expectation from Faraday's law.

Instead, we suppose that the field relevant for a calculation of the potential difference $\Delta V = -\langle E_z\rangle\, l$ is the average electric field, $\langle E_z\rangle = vB_0/c$, along the perfectly conducting cylinder. Then, $\Delta V = -vB_0\, l/c$, and the current in the resistor is in the $+z$ direction with magnitude,

$$I = \frac{vB_0\, l}{cR}\,. \tag{51}$$

This current flows throughout the circuit, and adds a surface current $\Delta K_z = -I f(\theta')/\pi a$ to the moving cylinder (in the $-z$ direction) for some even function $f(\theta')$ that obeys $\int_0^{2\pi} f(\theta')\, d\theta' = \pi$. As a result, there is an additional azimuthal magnetic field outside this cylinder, such that the nonzero components of the electric and magnetic fields at the surface of the cylinder are now, recalling eqs. (29)-(30) and (32), and that $\mathbf{B}(r' = a^+) = (4\pi/c)\mathbf{K}\times\hat{\mathbf{r}}'$,

$$\begin{aligned} B_{\theta'}(r' = a^+) &= -2B_0\, \sin\theta' - \frac{4If}{ac}\,, && (52) \\ E_z(r' = a^+) &= -\frac{vB_{\theta'}(r' = a^+)\sin\theta'}{c}\,, && (53) \end{aligned}$$

as the total surface current density is,

$$K_z = -\frac{cB_0\, \sin\theta'}{2\pi} - \frac{If}{\pi a} = \frac{c}{4\pi}B_{\theta'}(r' = a^+). \tag{54}$$

The electric field does work on the surface current at the rate,

$$P_E = l\int_0^{2\pi} a\, d\theta'\, K_z E_z = -\frac{alv}{4\pi}\int_0^{2\pi} d\theta'\, B_{\theta'}^2(r' = a^+)\sin\theta' = -\frac{4lvB_0 I}{\pi c}\int_0^{2\pi} d\theta'\, f\sin^2\theta', \tag{55}$$

noting that f is an even function of θ'.

---

[6]See [14] for discussion of the use of Faraday's law for circuits with moving parts.

[7]If $R = 0$ such that the circuit consists entirely of perfect conductors, the magnetic field cannot enter the circuit as it would need a component perpendicular to the surface of the sliding bar to do so. See also footnote 2. Hence, there would be no $\mathcal{EMF}$ and no net current flowing in the circuit in this case.

The Poynting vector at the surface has only the radial component,

$$
\begin{aligned}
S_{r'}(r'=a^+) &= -\frac{c}{4\pi}E_z B_{\theta'} = \frac{v}{4\pi}B_{\theta'}^2(r'=a^+)\sin\theta' \\
&= \frac{vB_0^2\sin^3\theta'}{\pi} + \frac{4vB_0 I f\sin^2\theta'}{\pi a c} + \frac{4vI^2 f^2\sin\theta'}{\pi a^2 c^2}\,. \qquad (56)
\end{aligned}
$$

The total power flowing off the moving cylinder is, noting that $f(\theta')$ is an even function,

$$
\begin{aligned}
P &= l\int_0^{2\pi} S_{r'}(r'=a^+)\,a\,d\theta' = \frac{4lvB_0 I}{\pi c}\int_0^{2\pi} f\sin^2\theta'\,d\theta' = -P_E \\
&= I^2R\begin{cases} 2 & (f=1/2), \\ 1 & (f=\cos^2\theta'), \end{cases} \qquad (57)
\end{aligned}
$$

recalling eq. (51). With $f=\cos^2\theta'$ we have that $P=I^2R$, as seems reasonable (while with $f=1/2$, for azimuthally symmetric ΔK_z as naïvely expected, we obtain $P=2I^2R$.)

Because of the surface current on the cylinder, it experiences a Lorentz force,

$$
\begin{aligned}
\mathbf{F} &= l\int_0^{2\pi} a\,d\theta'\,\frac{K_z\,\hat{\mathbf{z}}}{c}\times\frac{\mathbf{B}_{\theta'}}{2} = \frac{al}{8\pi}\int_0^{2\pi} d\theta'\left(2B_0\sin\theta' + \frac{4If}{ac}\right)^2 \hat{\mathbf{z}}\times(-\sin\theta'\,\hat{\mathbf{x}} + \cos\theta'\,\hat{\mathbf{y}}) \\
&= -\frac{al}{8\pi}\int_0^{2\pi} d\theta'\left(2B_0\sin\theta' + \frac{4If}{ac}\right)^2(\sin\theta'\,\hat{\mathbf{y}} + \cos\theta'\,\hat{\mathbf{x}}) = -\frac{2lB_0 I}{\pi c}\int_0^{2\pi} d\theta'\,f\sin^2\theta'\,\hat{\mathbf{y}} \\
&= -\frac{P}{2v}\,\hat{\mathbf{y}}, \qquad (58)
\end{aligned}
$$

noting that the average magnetic field on the surface current is 1/2 the magnetic field just outside the surface. To keep the cylinder in steady motion, some mechanical agent must provide an opposing force, which delivers energy into the system at rate,[8]

$$
P_{\rm mech} = -F_y v = \frac{P}{2} = I^2R\begin{cases} 1 & (f=1/2), \\ 1/2 & (f=\cos^2\theta'). \end{cases} \qquad (59)
$$

We expect that in the lab frame the mechanical force on the moving, perfectly conducting cylinder provides the power I^2R dissipated by the load resistor, which suggests that $f = 1/2$, *i.e.*, the additional current density ΔK_z is azimuthally symmetric. *But, then the electromagnetic power P flowing off the cylinder is double the power dissipated in the load resistor, as if the system emits "radiation to infinity".*

Thus, the moving cylinder acts as a transducer of mechanical power to electromagnetic power, according to the above analysis in the lab frame.[9]

[8] If we used the full field $\mathbf{B}_{\theta'}$ in the Lorentz-force calculation (59), rather than half this value, then use of $f=\cos^2\theta'$ would lead to $P_{\rm mech}=I^2R$, consistent with eq. (57). However, using the Maxwell stress tensor to calculate the force leads to $T_{r'r'} = -B_{\theta'}^2/8\pi$ (neglecting the term in $E_z^2 \propto B_{\theta'}^2 v^2/c^2$), which confirms the use of $\mathbf{B}_{\theta'}/2$ in eq. (58).

[9] The currents in this problem are steady, but the magnetic flux through the circuit varies with time, so we expect an $\mathcal{EMF}$ in the (moving) circuit according to a broad interpretations of Faraday's law. The resulting power dissipated in the resistor agreeably flows in the form of electromagnetic energy out from the surface of the sliding bar. But this analysis gives no microscopic picture of how that energy flow arises.

The above analysis is almost, but not quite, satisfactory. Taking $f = 1/2$, corresponding to azimuthally symmetric current I in the sliding bar, we have that the mechanical power equals the power I^2R dissipated in the load resistor. I would have expected that the power transferred from the conduction electrons to the electromagnetic fields would also be I^2R, and that the power leaving the surface of the sliding bar would be I^2R. But instead these are both $2I^2R$ – which doesn't make sense to me.

4.1 Analysis in the Rest Frame of the Cylinder

In the rest frame of the perfectly conducting cylinder it can have no electric field component along it surface, and in this frame the Poynting vector has no component perpendicular to the surface. Hence, in this frame we cannot say that the cylinder is the source/transducer of the power dissipated in the resistor.

Instead, we note that in the rest frame of the cylinder the magnet that provides the external field $\mathbf{B}_0 = B_0\,\hat{\mathbf{x}}$ has velocity $-v\,\hat{\mathbf{y}}$. This magnet exerts total force $\mathbf{F}' = \mathbf{F} = F_y\,\hat{\mathbf{y}}$ on the cylinder, so there is a reaction force $-\mathbf{F}'$ on the magnet. A mechanical force $-(-\mathbf{F}')$ is required in the lab frame to keep the magnet at rest there, and in the rest frame of the cylinder this force does work on the moving magnet at rate $-v\,\hat{\mathbf{y}} \cdot \mathbf{F}' = -vF_y = I^2R$. Thus, in the rest frame of the cylinder, we are led to say that the magnet acts as the transducer of mechanical power to electromagnetic power. This is an example of the relativity of steady energy flow (also discussed in [16]).

Acknowledgment

The author thanks Alan Boswell and Bruce Cragin for insightful e-discussions of this topic.

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