Pitching Pennies into a Magnet

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(Jan. 15, 1997; updated October 3, 2013)

1 Problem

If one pitches a penny into a large magnet, eddy currents are induced in the penny, and their interaction with the magnetic field results in a repulsive force, according to Lenz' law. Estimate the minimum velocity needed for a penny to enter a long, 1-T solenoid magnet whose diameter is 0.1 m.

You may suppose that the penny moves so that its axis always coincides with that of the magnet, and that gravity may be ignored. The speed of the penny is low enough that the magnetic field caused by the eddy currents may be neglected compared to that of the solenoid. Equivalently, you may assume that the magnetic diffusion time is small.

2 Solution

The penny has radius a and thickness Δz. For the motion as stated in the problem, the eddy current will flow in concentric rings about the center of the disk. Therefore, we first examine a ring of radius r and radial extent Δr.

The magnetic flux through the ring at position z is,

$$\Phi \approx \pi r^2 B_z(0, z), \tag{1}$$

whose time rate of change is,

$$\dot{\Phi} = \pi r^2 \dot{B}_z = \pi r^2 B'_z v, \tag{2}$$

where $\dot{}$ indicates differentiation with respect to time, $'$ is differentiation with respect to z, B_z stands for $B_z(0, z)$, and v is the velocity of the center of mass of the ring.

The penny has electrical conductivity σ. Its resistance to currents around the ring is,

$$R = \frac{2\pi r}{\sigma \Delta r \Delta z}, \tag{3}$$

so the (absolute value of the) induced current is,

$$I = \frac{\mathcal{E}}{R} = \frac{\dot{\Phi}}{R} = \frac{\sigma r B'_z v \Delta r \Delta z}{2}, \tag{4}$$

(in MKSA units).

The azimuthal eddy current interacts with the radial component of the magnetic field to produce the axial retarding force. Close to the magnetic axis, we estimate the radial field in term of the axial field according to,

$$B_r(r, z) \approx r \frac{\partial B_r(0, z)}{\partial r} = -\frac{r}{2} \frac{\partial B_z(0, z)}{\partial z} \equiv -\frac{r B'_z}{2}, \tag{5}$$

as can be deduced from the Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, noting that on the magnetic axis $\partial B_r/\partial r = \partial B_x/\partial x = \partial B_y/\partial y$. Then, the retarding force on the ring is,

$$\Delta F_z = 2\pi r B_r I = -\pi\sigma r^2 B_r B_z' v \Delta r \Delta z \approx -\frac{\pi\sigma r^3 (B_z')^2 v \Delta r \Delta z}{2}. \tag{6}$$

Alternatively, we note that the kinetic energy lost by the penny appears as Joule heating. Hence, for the ring analyzed above,

$$v\Delta F_z = \frac{dU}{dt} = -I^2 R = -\frac{\pi\sigma r^3 (B_z')^2 v^2 \Delta r \Delta z}{2}, \tag{7}$$

using eqs. (3) and (4), which agains leads to eq. (6).

The equation of motion of the ring is,

$$dF_z = -\frac{\pi\sigma r^3 (B_z')^2 v \Delta r \Delta z}{2} = m\dot{v} = 2\pi\rho r \Delta r \Delta z \, v' v, \tag{8}$$

where ρ is the mass density of the metal. We integrate this equation with respect to radius to find,

$$-\frac{\pi\sigma a^4 (B_z')^2 v \Delta z}{8} = \pi\rho a^2 \Delta z \, v' v, \tag{9}$$

After dividing out the common factor $\pi a^2 \Delta z \, v$, we find,

$$v' = -\frac{\sigma a^2 (B_z')^2}{8\rho}. \tag{10}$$

Since the righthand side is independent of v, the change in speed of the penny is independent of its initial speed, so the penny can be stopped (but the sign of v cannot be reversed).

For an estimate, we note that the peak gradient of the axial field of a solenoid of diameter D is about B_0/D, and the gradient is significant over a region $\Delta z \approx D$. Hence, on entering a solenoid the jet velocity is reduced by,

$$\Delta v \approx \frac{\sigma a^2 B_0^2}{8\rho D}. \tag{11}$$

The penny must have initial velocity $v_0 > \Delta v$ to reach the center of the magnet.

Another estimate can be made by approximating the solenoid as semi-infinite. Then, following the useful result of prob. 5.3 of [2], the axial field of a coil at $z > 0$ is,

$$B_z(z) = \frac{B_0}{2}\left(1 + \frac{z}{\sqrt{z^2 + (D/2)^2}}\right), \qquad \textit{for which} \qquad B_z'(z) = \frac{B_0 (D/2)^2}{2[z^2 + (D/2)^2]^{3/2}}, \tag{12}$$

and hence, the change in the speed of the penny before it enters the magnet (at $z = 0$) is,

$$\Delta v = \frac{\sigma a^2}{8\rho}\int_{-\infty}^{0} (B_z')^2 \, dz = \frac{\sigma a^2 B_0^2 (D/2)^4}{32\rho}\int_{-\infty}^{0} \frac{dz}{[z^2 + (D/2)^2]^3} = \frac{3\pi\sigma a^2 B_0^2}{256\rho D} \approx \frac{\sigma a^2 B_0^2}{27\rho D}. \tag{13}$$

Note that the penny cannot reach $z = +\infty$ unless $v_0(z = -\infty) > 2\Delta v$.

A copper penny has $a \approx 1$ cm $= 10^{-2}$ m, density ≈ 10 g/cm^3 $= 10^4$ kg/m^4, electrical resistivity $\approx 10^{-6}$ Ω-cm, and therefore conductivity $\approx 10^8$ MKSA units. The minimum initial velocity to enter (and pass through) a 1-T magnet with diameter $D = 0.1$ m is then,

$$v_{\min} \approx \frac{10^8 \cdot (10^{-2})^2 \cdot 1^2}{10 \cdot 10^4 \cdot 0.1} \approx 1 \text{ m/s}. \tag{14}$$

Acknowledgment

The author thanks Ahmed Hassanein, Bob Palmer and Bob Weggel for discussions of this problem. The case of a sphere rather than a disk has been presented in [1].

References

[1] J. Walker and W.H. Wells, *Drag Force on a Conducting Spherical Drop in a Nonuniform Magnetic Field*, ORNL/TN-6976 (Sept. 1979),
http://physucs.princeton.edu/~mcdonald/examples/fluids/walker_ornl_tm6976_79.pdf

[2] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999),
http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf