Accelerated Pendulum

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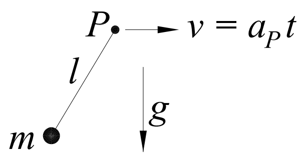
1 Problem

What is the angular frequency of small oscillations of a simple pendulum whose support point P has constant horizontal acceleration a_P while in a uniform gravitation field of strength g?



2 Solution

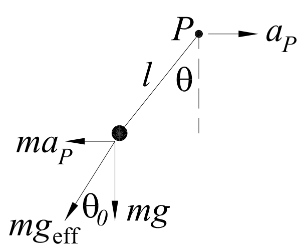
For zero acceleration a_P, the angular frequency of small oscillations of a simple pendulum of length l is $\omega = \sqrt{g/l}$, independent of its mass m.

It is well known that this problem is quickly solved using an accelerated (but not rotating) frame whose origin is the support point P. After reviewing this approach, we confirm the solution via Lagrange's method, and then consider the use of an accelerated and rotating frame.

2.1 Accelerated but Not Rotating Frame

We recall that in an accelerated, but not rotating frame, an object of mass m experiences the forces on it in an inertial frame plus a "fictitious"/coordinate force $-m\mathbf{a}$ where $\mathbf{a}$ is the acceleration of the coordinate axes of the frame. See, for example, eq. (39.7) of [1] or pp. 168-172 of [2].

A force diagram is shown below for a simple pendulum of length l and mass m at the surface of the Earth, whose local acceleration due to gravity is g. The support point P has constant horizontal acceleration a_P (and velocity $v_P = a_P t$) to the right.



The total force on the bob of the pendulum can be written as $m\mathbf{g}_{\rm eff}$ in terms of an effective gravitational acceleration $\mathbf{g}_{\rm eff}$ where

$$g_{\rm eff} = \sqrt{g^2 + a_P^2} \tag{1}$$

and $\mathbf{g}_{\rm eff}$ makes angle θ_0 to the vertical related by

$$\cos\theta_0 = \frac{g}{g_{\rm eff}}\,, \qquad \sin\theta_0 = \frac{a_P}{g_{\rm eff}}\,. \tag{2}$$

Then, the angular frequency of small oscillations of the pendulum about angle θ_0 is

$$\omega = \sqrt{\frac{g_{\rm eff}}{l}}\,. \tag{3}$$

2.2 Lagrange's Method

To confirm the result (3) via Lagrange's method, we consider the single coordinate θ of the pendulum to the vertical.

The velocity of the bob (in the inertial lab frame) is

$$\mathbf{v} = \mathbf{v}_P + l\dot\theta\,\hat{\boldsymbol\theta}, \tag{4}$$

so the kinetic energy of the system is

$$T = \frac{mv^2}{2} = \frac{m}{2}\left(v_P^2 + l^2\dot\theta^2 + 2\mathbf{v}_P\cdot l\dot\theta\,\hat{\boldsymbol\theta}\right) = \frac{mv^2}{2} = \frac{m}{2}\left(v_P^2 + l^2\dot\theta^2 - 2v_P\,l\dot\theta\cos\theta\right), \tag{5}$$

and the potential energy (relative to the support point) is

$$V = -mgl\cos\theta. \tag{6}$$

With $L = T - V$, Lagrange's equation of motion for the system is

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = \frac{d}{dt}\left(ml^2\dot\theta - mv_P\,l\cos\theta\right) &= ml^2\ddot\theta - ma_P\,l\cos\theta + mv_P\,l\dot\theta\sin\theta \\ &= \frac{\partial L}{\partial\theta} = mv_P\,l\dot\theta\sin\theta - mgl\sin\theta, \qquad (7)\\ \ddot\theta &= \frac{a_P\cos\theta}{l} - \frac{g\sin\theta}{l}\,. \qquad (8)\end{aligned}$$

We intuit that there exists an equilibrium angle θ_0, and introduce the variable $\phi = \theta - \theta_0$. Then for small ϕ,

$$\cos\theta = \cos(\phi + \theta_0) \approx \cos\theta_0 - \phi\sin\theta_0, \qquad \sin\theta = \sin(\phi+\theta_0) \approx \phi\cos\theta_0 + \sin\theta_0, \tag{9}$$

and the equation of motion (8) can be written as

$$\ddot\phi \approx \frac{a_P\cos\theta_0}{l} - \frac{a_P\,\phi\sin\theta_0}{l} - \frac{g\phi\cos\theta_0}{l} - \frac{g\sin\theta_0}{l}\,. \tag{10}$$

The constant terms in eq. (10) sum to zero if we take

$$\tan\theta_0 = \frac{a_P}{g}\,, \qquad \cos\theta_0 = \frac{g}{\sqrt{g^2 + a_P^2}} = \frac{g}{g_{\rm eff}} \qquad \sin\theta_0 = \frac{a_P}{\sqrt{g^2+a_P^2}} = \frac{a_P}{g_{\rm eff}}\,, \tag{11}$$

with $g_{\rm eff} = \sqrt{g^2 + a_P^2}$ as in eqs. (1)-(2). Then, the equation of motion (10) simplifies to

$$\ddot\phi \approx -\frac{\phi}{l}\frac{a_P^2 + g^2}{g_{\rm eff}} = -\frac{g_{\rm eff}}{l}\phi, \tag{12}$$

which corresponds to simple harmonic motion with angular frequency $\omega \approx \sqrt{g_{\rm eff}/l}$, as found in eq. (3).

2.3 Accelerated and Rotating Frames

In principle, we could also analyze this problem in a frame whose origin has arbitrary acceleration $\mathbf{a}$, and whose axes have arbitrary angular velocity $\mathbf{\Omega}$ with respect to the inertial lab frame. This would would require consideration of the additional "fictitious" forces $m\mathbf{r} \times \dot{\mathbf{\Omega}} + 2m\mathbf{v} \times \mathbf{\Omega} + m\mathbf{\Omega} \times (\mathbf{r} \times \mathbf{\Omega})$ act on the center of bob of the pendulum (of mass m, at position $\mathbf{r} = \mathbf{l}$, with velocity $\mathbf{v}$, in this frame.

A particular case is that $\mathbf{a} = \mathbf{a}_P$ and $\mathbf{\Omega} = \dot{\theta}\,\hat{\mathbf{z}}$ where the z-axis is into the page of the figures above. In this frame the bob of the pendulum is at rest, so the Coriolis force $2m\mathbf{v} \times \mathbf{\Omega}$ would be zero, while the centrifugal force $m\mathbf{\Omega} \times (\mathbf{r} \times \mathbf{\Omega})$ is along the string of the pendulum and exerts no torque about point P.

In this ($'$) frame there is no rotation of the pendulum so the torque equation (about point P would be

$$\boldsymbol{\tau}' = 0 = \mathbf{l} \times m\mathbf{g}_{\rm eff} + \mathbf{l} \times m(\mathbf{l} \times \dot{\mathbf{\Omega}}) = \mathbf{l} \times m\mathbf{g}_{\rm eff} + ml^2\ddot{\theta}\,\hat{\mathbf{z}}, \tag{13}$$

considering the "fictitious" torque due to the "fictitious" force $m\mathbf{l} \times \dot{\mathbf{\Omega}}$, and taking $\hat{\mathbf{z}}$ to be out of the page. This is the same as the torque equation in the accelerated, but nonrotating frame considered in Sec. 2.1 above,

$$\boldsymbol{\tau} = \mathbf{l} \times m\mathbf{g}_{\rm eff} = \frac{d\mathbf{L}}{dt} = -ml^2\ddot{\theta}\,\hat{\mathbf{z}}. \tag{14}$$

However, it appears that additional "fictitious" torques also must be considered unless the rotating axes are those of the rest frame of the pendulum.

It suffices not to consider such accelerated/rotating frames in problems like this.[1]

References

[1] L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed. (Pergamon, 1976),
http://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics.pdf

[2] K.T. McDonald, *Accelerated Coordinate Systems* (1980),
http://kirkmcd.princeton.edu/examples/ph205l16.pdf

[3] K.T. McDonald, *Torque Analyses of a Sliding Ladder* (May 6, 2007),
http://kirkmcd.princeton.edu/examples/ladder.pdf

[4] K.T. McDonald, *Falling Chimney* (Oct. 1, 1980),
http://kirkmcd.princeton.edu/examples/chimeny.pdf

[1] See, for example, the sliding ladder [3] and the falling chimney [4].