

A Pendulum Problem

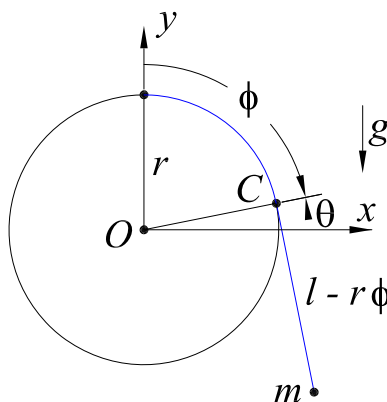
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1 Problem

Discuss the motion of a not-so-simple pendulum consisting of a point mass m attached to a massless string of fixed length $l > \pi r/2$ whose other end is attached to the top of a fixed, right-circular cylinder of radius r whose (z) axis is horizontal, as sketched below. The pendulum is released from rest in a configuration where the string lies in the vertical plane $z = 0$. You may assume that the portion of the string not in contact with the cylinder is always straight.



This problem appeared on an exam at UC Berkeley in 2010 [1].

2 Solution

This system has an equilibrium for $\phi = \pi/2$, where ϕ is the angle of the lowest contact point C to the vertical axis y (with the x -axis horizontal), in which case the portion of the string below point C is vertical. We anticipate that small oscillations about this equilibrium are those of a simple pendulum of length $l = \pi r/2$.

If the more general motion is to have the string straight at all times, it must be that the initial, lowest point of contact C of the string with the cylinder has $0 < \phi_0 < \pi/2$.

2.1 Lagrange's Method

We first use Lagrange's method, noting that the system can be characterized by the single coordinate ϕ .

The lowest point of contact C has (x, y, z) coordinates, with respect to the origin O on the axis of the cylinder,

$$x_C = r \sin \phi, \quad y_C = r \cos \phi, \quad z_C = 0. \quad (1)$$

The portion of the string not in contact with the cylinder has length $l - r\phi$, so mass m is offset from the point of contact by $\Delta = (\Delta x, \Delta y, \Delta z)$,

$$\Delta x = (l - r\phi) \cos \phi, \quad \Delta y = -(l - r\phi) \sin \phi, \quad \Delta z = 0. \quad (2)$$

Hence, mass m has position $\mathbf{x}$ with (x, y, z) coordinates

$$x = r \sin \phi + (l - r\phi) \cos \phi, \quad y = r \cos \phi - (l - r\phi) \sin \phi, \quad z = 0. \quad (3)$$

The velocity $\mathbf{v} = d\mathbf{x}/dt = \dot{\mathbf{x}}$ of the mass is

$$v_x = \dot{x} = r\dot{\phi} \cos \phi - (l - r\phi)\dot{\phi} \sin \phi - r\dot{\phi} \cos \phi = -(l - r\phi)\dot{\phi} \sin \phi \quad (4)$$

$$v_y = \dot{y} = -r\dot{\phi} \sin \phi - (l - r\phi)\dot{\phi} \cos \phi + r\dot{\phi} \sin \phi = -(l - r\phi)\dot{\phi} \cos \phi, \quad (5)$$

$$v_z = 0, \quad (6)$$

and so

$$v^2 = v_x^2 + v_y^2 + v_z^2 = (l - r\phi)^2 \dot{\phi}^2. \quad (7)$$

Mass m has kinetic energy T ,

$$T = \frac{mv^2}{2} = \frac{m(l - r\phi)^2 \dot{\phi}^2}{2}, \quad (8)$$

and potential energy V relative to the axis of the cylinder,

$$V = mgy = mg[r \cos \phi - (l - r\phi) \sin \phi]. \quad (9)$$

The equations of motion follows from the Lagrangian $\mathcal{L} = T - V$ as

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} &= \frac{d}{dt} m(l - r\phi)^2 \dot{\phi} = m(l - r\phi)^2 \ddot{\phi} - 2mr(l - r\phi) \dot{\phi}^2 \\ &= \frac{\partial \mathcal{L}}{\partial \phi} = -mr(l - r\phi) \dot{\phi}^2 + mg(l - r\phi) \cos \phi. \end{aligned} \quad (10)$$

We can rewrite this as

$$(l - r\phi) \ddot{\phi} - r\dot{\phi}^2 = g \cos \phi. \quad (11)$$

The equilibrium condition is $\phi = \pi/2$, such that the portion of the string not in contact with the cylinder simply hangs vertically.

To consider small oscillations about this equilibrium, it is convenient to change variables to angle $\theta = \pi/2 - \phi$, which is the angle to the horizontal x -axis of the radius vector to point C . Then, $\ddot{\phi} = -\ddot{\theta}$, $\dot{\phi}^2 = \dot{\theta}^2$, $\cos \phi = \sin \theta$, and

$$l - r\phi = l - r(\pi/2 - \theta) = l - \pi r/2 + r\theta. \quad (12)$$

The equation of motion (11) becomes

$$(l - \pi r/2 + r\theta) \ddot{\theta} + r\dot{\theta}^2 = -g \sin \theta. \quad (13)$$

Small oscillations have small θ and $\dot{\theta}$, in which case the equation of motion simplifies to

$$(l - \pi r/2) \ddot{\theta} \approx -g\theta, \quad (14)$$

which is the equation of motion for a simple pendulum of length $l - \pi r/2$, as anticipated above.¹

¹For $\theta \approx \theta_0 \sin(\omega t)$ with $\omega = \sqrt{g/(l - \pi r/2)}$, $\dot{\theta} \approx \theta_0 \omega \cos(\omega t)$, which is small for small θ_0 .

2.2 Newtonian Analyses

2.2.1 Analysis in the Lab Frame

A torque analysis about the fixed point O would involve the torque on mass m due to the tension $\mathbf{T}$ in the string as well as that due to gravity. But the tension is unknown, so this analysis cannot usefully proceed.²

Instead, we consider analyses based on the contact point C , about which the tension $\mathbf{T}$ exerts no torque. However, point C is accelerating with respect to the lab frame.

2.2.2 Use of a Nonrotating Frame

For possible simplicity, we next consider a nonrotating, but translating frame with origin at point C , which we call the $'$ frame. In this frame, $\hat{\mathbf{x}}' = \hat{\mathbf{x}}$, $\hat{\mathbf{y}}' = \hat{\mathbf{y}}$, $\hat{\mathbf{z}}' = \hat{\mathbf{z}}$, and $t' = t$.

We can make a Newtonian analysis of this system by considering the torque about the (accelerating) contact point C , but now we must include the torque due to the “fictitious force” $\mathbf{F}' = -m\mathbf{a}_C$ that acts on the (center of) mass m , where $\mathbf{a}_C$ is the acceleration of point C in the inertial lab frame.³

From eq. (1), we see that the acceleration $\mathbf{a}_C$ can be written as

$$\mathbf{a}_C = r\ddot{\phi}\hat{\mathbf{x}}_{\parallel} - r\dot{\phi}^2\hat{\mathbf{x}}_{\perp}, \quad (15)$$

where $\hat{\mathbf{x}}_{\parallel}$ is along the direction of the portion on the string not on the cylinder, and $\hat{\mathbf{x}}_{\perp}$ is along the radius vector OC .

The “fictitious force” $\mathbf{F}' = -m\mathbf{a}_C$ that acts on mass m , has lever arm $\Delta' = (l - r\phi)\hat{\mathbf{x}}_{\parallel}$, so the torque $\tau'_{C,\mathbf{F}'}$ in the $'$ frame about point C on mass m is

$$\tau'_{C,\mathbf{F}'} = \Delta' \times \mathbf{F}' = \Delta' \times (-m\mathbf{a}_C) = \Delta'(-ma_{C,\perp})\hat{\mathbf{z}}' = mr(l - r\phi)\dot{\phi}^2\hat{\mathbf{z}}'. \quad (16)$$

The torque $\tau'_{C,g}$ in the $'$ frame due to gravity about point C is

$$\tau'_{C,g} = -mg(l - r\phi)\cos\phi\hat{\mathbf{z}}'. \quad (17)$$

The angular momentum $\mathbf{L}'_C$ in the $'$ frame of mass m about point C is

$$\mathbf{L}'_C = -m(l - r\phi)^2\dot{\phi}\hat{\mathbf{z}}'. \quad (18)$$

The torque equation of motion, $d\mathbf{L}'_C/dt' = d\mathbf{L}'_C/dt = \tau'_{C,\text{total}} = \tau'_{C,\mathbf{F}'} + \tau'_{C,g}$, has only a $\hat{\mathbf{z}}'$ component,

$$-m(l - r\phi)^2\ddot{\phi} + 2r(l - r\phi)\dot{\phi}^2 = mr(l - r\phi)\dot{\phi}^2 - mg(l - r\phi)\cos\phi, \quad (19)$$

which reduces to eq. (11) as found above by Lagrange’s method.

²We could use $m\mathbf{a} = m\dot{\mathbf{v}} = \mathbf{T} - mg\hat{\mathbf{y}}$ to write $\mathbf{T} = m\dot{\mathbf{v}} + mg\hat{\mathbf{y}}$. Then, with the angular momentum of mass m about point O as $\mathbf{L}_O = \mathbf{x} \times m\mathbf{v}$, the torque equation would be $d\mathbf{L}_O/dt = \mathbf{x} \times m\dot{\mathbf{v}} = \tau_O = \mathbf{x} \times (\mathbf{T} - mg\hat{\mathbf{y}}) = \mathbf{x} \times m\dot{\mathbf{v}}$, which is a tautology.

³See, for example, eq. (15) of [2], eq. (39.7) of [3] or sec. 7.1 of [4].

2.2.3 Use of a Rotating Frame

We can also use a rotating and translating frame (the " frame), again with its origin at point C .

We take the x'' axis to be along the portion of the string not on the cylinder, and the y'' axis to be along the radius vector through point C . Also, $\hat{\mathbf{z}}'' = \hat{\mathbf{z}}$ and $t'' = t$. Then, mass m is at position $\Delta'' = (l - r\phi) \hat{\mathbf{x}}''$, with velocity $\mathbf{v}'' = d\Delta''/dt'' = d\Delta''/dt = -r\dot{\phi} \hat{\mathbf{x}}''$.

The acceleration $\mathbf{a}_C$ of point C with respect to the lab frame can be expressed in terms of the unit vectors of the " frame as $\mathbf{a}_C = r\ddot{\phi} \hat{\mathbf{x}}'' - r\dot{\phi}^2 \hat{\mathbf{y}}''$.⁴

The angular velocity Ω of the " frame with respect to the nonrotating lab frame is

$$\Omega = -\dot{\phi} \hat{\mathbf{z}} = -\dot{\phi} \hat{\mathbf{z}}''. \quad (20)$$

The total force on mass m in the " frame now includes four "fictitious forces",⁵ in addition to the force of gravity $-mg \hat{\mathbf{y}}$,

$$\begin{aligned} \mathbf{F}_{\text{total}}'' &= -mg \hat{\mathbf{y}} - m\mathbf{a}_C + m\Delta'' \times \dot{\Omega} + 2m\mathbf{v}'' \times \Omega + m\Omega \times (\Delta'' \times \Omega) \\ &= mg \sin \phi \hat{\mathbf{x}}'' - mg \cos \phi \hat{\mathbf{y}}'' - mr\ddot{\phi} \hat{\mathbf{x}}'' + mr\dot{\phi}^2 \hat{\mathbf{y}}'' + m(l - r\phi)\ddot{\phi} \hat{\mathbf{y}}'' - 2mr\dot{\phi}^2 \hat{\mathbf{y}}'' + m(l - r\phi)\dot{\phi}^2 \hat{\mathbf{x}}'', \end{aligned} \quad (21)$$

where the fourth term in the first line of eq. (21) is the Coriolis force and the fifth term is the centrifugal force.

In the " frame there is no angular momentum $\mathbf{L}_C''$ of mass m with respect to point C , so

$$d\mathbf{L}_C''/dt'' = 0 = \boldsymbol{\tau}_C'' = \Delta'' \times \mathbf{F}_{\text{total}}'' = (l - r\phi) \hat{\mathbf{x}}'' \times \mathbf{F}_{\text{total}}''. \quad (22)$$

This has only a z'' -component,⁶ which is, after dividing out the common factors of $m(l - r\phi)$,

$$0 = -g \cos \phi - r\dot{\phi}^2 + (l - r\phi)\ddot{\phi}, \quad (23)$$

i.e., the same as eq. (11).

References

- [1] K. Grosvenor, *U.C. Berkeley Physics Preliminary Exam Review Problems* (Aug. 28, 2011). http://kirkmc.d.princeton.edu/examples/mechanics/grosvenor_11.pdf
- [2] K.T. McDonald, *Comments on Torque Analyses* (April 28, 2019). <http://kirkmc.d.princeton.edu/examples/torque.pdf>
- [3] L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed. (Pergamon, 1976). See §39. http://kirkmc.d.princeton.edu/examples/mechanics/landau_mechanics_76.pdf
- [4] V. Barger and M. Olsson, *Classical Mechanics A Modern Perspective*, 2nd ed. (McGraw-Hill), 1995). http://kirkmc.d.princeton.edu/examples/mechanics/barger_95.pdf

⁴Note that $\hat{\mathbf{x}}'' = \hat{\mathbf{x}}_{\parallel}$ and $\hat{\mathbf{y}}'' = \hat{\mathbf{x}}_{\perp}$ of sec. 2.2.2.

⁵Again, see, for example, eq. (39.7) of [3].

⁶Note that the centrifugal force produces no torque about point C .