THE DIFFRACTION THEORY OF ABERRATIONS

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ABSTRACT. A historical and critical survey is given of investigations concerned with image formation in optical instruments in the presence of aberrations. The development of the subject is traced from the early researches of Airy on an aberration-free image. The advances made in recent years are discussed in greater detail; these mainly concern the effects of small aberrations, tolerance criteria, and the asymptotic treatment of diffraction problems. A section is also included on investigations into the effects of waves of non-uniform amplitude. The detailed light distribution in typical images is illustrated by isophote-diagrams and photographs.

§ 1. INTRODUCTION

In a perfect optical system the light waves which proceed from each element of the object emerge in the image space as convergent waves spherical in form. Such an instrument represents an idealization which cannot be realized in practice; in the image space of an actual instrument, the wave surfaces will, as a rule, be of a more complicated form. The deviations from the ideal spherical form of these surfaces may range in practice from a fraction of a wavelength in a well corrected telescope or microscope objective, to several dozen wavelengths in instruments required for less precise work. In many cases the resulting image bears little resemblance either to the Airy diffraction pattern or to the confusion figure predicted by geometrical optics.

The diffraction theory of aberrations is concerned with the study of images formed by actual optical instruments. Recent years have seen important advances in this branch of optics and many valuable results have been obtained. In particular, in the domain of very small aberrations (amounting only to a fraction of a wavelength) greatly simplified series expansions for the light distribution in diffraction patterns have been given, supplemented by a detailed study of a number of typical cases. The effects of aberrations of intermediate size (about one to ten wavelengths) have been studied with the help of a specially constructed mechanical integrator. Attempts have also been made to examine the effects of large aberrations. Further, a new and systematic investigation of the maximum amount of aberrations which may be tolerated in optical instruments has been carried out. The possibility has also been examined of improving the quality of

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the image by deliberately introducing non-uniformity of amplitude and sometimes, in addition, varying the phase of the disturbance over the exit pupil.

As no article on the diffraction theory of aberrations has been published in the preceding volumes of these Reports we have included in the present account a historical review of the subject. The period before 1940 is, however, treated only briefly; additional information can be found in the articles of von Laue (1928), König (1929), Martin (1946) and in a thesis by Nijboer (1942). A survey of experimental investigations on diffraction images was recently given in a thesis by Nienhuis (1948). We begin with a short preliminary section on the geometrical treatment of aberrations; this serves as an introduction to the classification of image errors used in the main discussion.

§2. GEOMETRICAL TREATMENT OF ABERRATIONS

The geometrical theory of aberrations is usually developed by methods due to Hamilton based on Fermat's principle of stationary path. This principle asserts that if light is propagated from a point \( P' \) to a point \( P \) in a medium of which the refractive index at a typical point is denoted by \( n \), it will travel along a path for which the optical length

\[
\int_{P'}^{P} \mu \, ds
\]

is stationary with respect to a small variation of that path. The integral (2.1) when taken along a natural ray may be regarded as a function of six variables, namely the rectangular Cartesian coordinates of \( P' \) and \( P \) with respect to some fixed reference system. In consequence of Fermat's principle, this function, sometimes called the characteristic function of the medium and denoted by \( V(x', y', z'; x, y, z) \) satisfies the relations

\[
\begin{align*}
\frac{\partial V}{\partial x'} &= -\mu' l', & \frac{\partial V}{\partial y'} &= -\mu' m', & \frac{\partial V}{\partial z'} &= -\mu' n', \\
\frac{\partial V}{\partial x} &= \mu l, & \frac{\partial V}{\partial y} &= \mu m, & \frac{\partial V}{\partial z} &= \mu n.
\end{align*}
\]

Here \((l', m', n')\) and \((l, m, n)\) denote the direction cosines of the rays at \( P' \) and at \( P \) respectively, while \( \mu' \) and \( \mu \) are the refractive indices at these points.

Consider an optical instrument with a point source at \( P'_0 \) \((x'_0, y'_0, z'_0)\) emitting monochromatic light. From (2.2) it follows that the equations

\[
V(x'_0, y'_0, z'_0; x, y, z) = \text{constant}
\]

represent the orthogonal trajectories of the rays, i.e. the wave fronts of geometrical optics. Except in the ideal case when (2.3) represents concentric spherical surfaces in the image space of the system, the rays will intersect the image plane at different points. Geometrical optics identifies the light intensity at a typical element of the image plane with the density of these ray intersections. Apart from the details of structure, this gives an approximation to the actual light distribution when the deformations of the wave fronts are of a considerable amount.

To investigate the appearance of the image it is convenient to introduce an aberration function \( \Phi \) which measures the deviation of the wave fronts from the spherical form. We consider a system with cylindrical symmetry and take a reference sphere centred on the Gaussian image \( P_0 \) of \( P'_0 \), passing through the
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centre of $O$ of the exit pupil. Further we denote by $A$ a typical point on this sphere and by $B$ the intersection of $P_0A$ with the wave which passes through $O$ (Figure 1).

The aberration function, defined by the equation

$$\Phi = AB,$$

may be expressed as function of the three rotational invariants $\sigma^2$, $r^2$ and $\sigma r \cos (\chi - \theta)$, where $(r, \theta)$, and $(\sigma, \chi)$ denote polar coordinates in the exit pupil and of the image point $P_0$ respectively. It may as a rule be expanded in the form

$$\Phi(\sigma, r, \phi) = \sum_{i, j, k} a_{ijk} \sigma^{2i+k} r^j \cos^{k} \phi,$$

where $\phi = \chi - \theta$, $i, j, k$ are non-negative integers, $j \geq k$, $j - k$ is even ($\neq 0$), $2i + j + k \geq 4$, and the $a$'s are constants.

Let $P$ be the actual point of intersection with the image plane of a typical ray proceeding from the exit pupil. From $\Phi$ it is possible to determine with the help of Fermat’s principle the ray aberration displacements $P_0P$. Those terms in (2.5) for which $2i + j + k = N$ are called the wave aberrations of the $N$th order

$$(N \text{ is always even})$$

and give rise to ray aberrations of order $N - 1$. The five aberrations of the lowest order ($N = 4$) are as a rule dominant in the paraxial region. The corresponding ray aberrations are known respectively as third order (sometimes primary or Seidel) spherical aberrations, coma, astigmatism, field curvature and distortion and have been discussed in the literature in much detail. Their physical meaning is too well known to need repeating here. For detailed discussions of geometrical aberrations we refer the reader to papers by Schwarzschild (1905), by Steward (1926) and by Nijboer (1943).

§ 3. THE DIFFRACTION THEORY OF ABERRATIONS

3.1. A Historical Review

We mentioned that the geometrical theory of aberrations identifies the intensity in the image plane with the ray density. This approximation, which in many cases gives an adequate picture of the light distribution in the image, gradually loses its validity as the aberrations become smaller. In the limiting case of perfectly spherical waves, for example, issuing from a circular opening,

* A different expansion recently proposed by Nijboer (1942, 1943) will be discussed later. The present notation is substantially due to him.

† A certain confusion exists in the literature about the terminology. Some earlier authors call first order ray aberrations those which we have denoted as ray aberrations of the third order. The classification used here has the advantage that it can be generalized to cover errors of focusing which are represented by first order terms. Such errors are often associated with the chromatic aberration.
geometrical optics predicts in the focal plane an infinite intensity at focus and zero intensity elsewhere; in reality the image consists of a bright central patch surrounded by rings—the familiar Airy pattern. It is therefore clear that in certain cases more refined investigations are needed. A direct application of electromagnetic theory presents considerable mathematical difficulties; instead one often uses approximations based on Kirchhoff’s formula

\[ U(P) = \frac{1}{4\pi} \int \left\{ \frac{U}{\partial n} \left( \frac{e^{-ikr}}{r} \right) - \frac{e^{-ikr}}{r} \frac{\partial U}{\partial n} \right\} dS. \]  

\[ \ldots \ldots (3.1) \]

This expresses the wave disturbance at any point P inside a closed region D as a surface integral involving the disturbance U and its gradient \( \partial U/\partial n \) over the boundary. Here r represents the distance of a typical surface element from P, \( \partial/\partial n \) denotes differentiation along the inward normal to D, \( k = 2\pi/\lambda \) and \( \lambda \) is the wavelength. In the investigations which we shall discuss, the scalar disturbance U is identified with a component of the electric or the magnetic vector or of a Hertz vector. From it the intensity I is derived by means of the approximate relation

\[ I = \text{constant} |U|^2. \]  

\[ \ldots \ldots (3.2) \]

When applied to problems of light distribution due to the passage of aberrant waves through the exit pupil of an optical instrument, Kirchhoff’s formula with certain approximations reduces to Huyghens’ principle of geometrical optics, taking into account however the mutual interference of the secondary wavelets proceeding from the opening. Using Huyghens’ principle in this form, Airy (1835) obtained the solution for the ideal case mentioned above. It is usually written as

\[ I(v) = \left( \frac{2J_1(v)}{v} \right)^2, \]  

\[ \ldots \ldots (3.3) \]

where in suitable units \( I(v) \) is the intensity in the pattern at distance v from its centre and \( J_1 \) is a Bessel function of the first kind.*

In two important papers Lommel (1885, 1886) extended Airy’s classical result by deriving expressions for the intensity distribution due to spherical waves in planes other than the geometrical focal plane. For diffraction at a circular opening his solution takes the form†

\[ I(u, v) = \left( \frac{2}{u} \right)^2 \left[ U_1^2(u, v) + U_2^2(u, v) \right], \]  

\[ \ldots \ldots (3.4) \]

where the U’s are two of the functions

\[ U_n(u, v) = \sum_{s=0}^{\infty} (-1)^s \left( \frac{u}{v} \right)^{n+2s} J_{n+2s}(v), \]  

\[ \ldots \ldots (3.5) \]

\( \alpha \) is a parameter which specifies the position of the receiving plane and \( \psi \) has the same meaning as before. When \( |u/v| > 1 \) the convergence of (3.5) is slow and for this case Lommel expressed the solution in terms of the functions

\[ V_n(u, v) = \sum_{s=0}^{\infty} (-1)^s \left( \frac{v}{u} \right)^{n+2s} J_{n+2s}(\psi), \]  

\[ \ldots \ldots (3.6) \]

* The derivation of (3.3) involved the assumption that the angular semi-aperture of the opening is small and that the disturbance over the converging wave is of uniform amplitude. In practice these assumptions may be far from valid but it has been shown by Hopkins (1943, 1944) that with a suitable interpretation of the argument \( \psi \), Airy’s formula is accurate to within a few per cent for systems with angular semi-aperture up to about 30°.
† The case of a circular opening was treated in an almost identical manner by Struve (1886). Formulæ more convenient for calculating the intensity far from focus were given by Schwarzschild (1897).
The functions $U_n$ and $V_n$ which Lommel introduced and which now bear his name play an important part in a number of related problems. Lommel evaluated numerically the solution for a large number of particular cases. Full use of his results was however not made until forty years later when Berek (1926) obtained with their help a graphical representation of the three-dimensional light distribution near focus.* Recently, Zernike and Nijboer (1949) gave a more accurate and detailed diagram (see Figure 2) based on a different expansion of the diffraction integral.

The first investigations concerning diffraction images in the presence of monochromatic aberrations appear to be due to Rayleigh and Strehl. Rayleigh (1879) studied images formed by cylindrical waves affected by a certain unsymmetrical aberration which are diffracted at a rectangular aperture, and also examined the effects of third order spherical aberration. In the latter case, however, he confined his investigations to determining the intensity at the centre of the pattern only. He formulated an important tolerance criterion, known in an extended form as Rayleigh's limit. This criterion asserts that the quality of an instrument is not sensibly affected by the presence of certain commonly occurring types of aberration if they are such that the waves in the image space do not deviate by more than a quarter of a wavelength from suitably chosen spherical surfaces. This criterion has since become widely used in formulating conditions concerning the maximum amount of aberrations which may be tolerated in the practical design of optical instruments.

Strehl, in his book Théorie des Fernrohres (1894) and in numerous papers mostly published between 1893 and 1930 in the Zeitschrift für Instrumentkunde and in Zentralzeitung für Optik und Mechanik, studied the effect of third order aberrations but he confined his researches mainly to investigating the variation of intensity along the principal ray. To him we owe the important

* Berek's figure is rather inaccurate. Later versions of it show the outline of the geometrical shadow incorrectly as, for example, in the following books: Handbuch der Physik, 1929, 21, 885 and Picht, J., 1931, Optische Abbildung (Braunschweig : Vieweg), p. 71.
concept now known as Strehl definition (Definitionshelligkeit), viz. the ratio of the maximum intensity in a particular receiving plane of an instrument to the intensity at the centre of the Airy disc in a perfect system of the same aperture and focal length. This quantity supplies in many cases a good measure of the quality of the system.

Conrady (1919), Buxton (1921, 1923) and Martin (1922) investigated the effects of various aberrations with the help of numerical integrations. They computed a number of particular cases and obtained valuable information concerning Rayleigh's criterion. Some work of these authors was also concerned with balancing spherical aberrations of different order against each other so as to obtain high intensity at the centre of the pattern; this question was later studied in a more general manner by Richter (1925).

In important papers Steward (1925) and Picht (1925, 1926) derived series expansions for the intensity distribution in typical diffraction images. In Steward's treatment the diffraction integral, derived by an immediate application of Huyghens' principle, gives the intensity \( I \) at a point \( P \) of the pattern in the form

\[
I = \left| \int \int e^{ik(V + d)} dS \right|^2,
\]

\[ \ldots \ldots \quad (3.8) \]

where \( V \) is a characteristic function and \( d \) is the distance from \( P \) to a typical point of the exit pupil over which the integration is carried out. Steward considered at first the effect of spherical aberration in an arbitrary receiving plane parallel to the Gaussian image plane, the exit pupil being assumed circular. (3.8) then reduces to

\[
I(u, v) = \left| \int_0^1 \exp \left( \frac{1}{2} i ut + i \sum \lambda A_s \right) J_0(v \sqrt{t}) \, dt \right|^2.
\]

\[ \ldots \ldots \quad (3.9) \]

The problem treated earlier by Lommel corresponds to the case \( A_s = 0 \) for all \( s \). By an argument similar to Lommel's, Steward showed that (3.9) can be expressed in terms of so-called generalized Lommel functions, these again being series involving Bessel functions.

When other aberrations are present (3.8) takes a more complicated form, since simplifying symmetry conditions no longer exist. Steward restricted the rest of his analysis to third order aberrations. His paper contains two diagrams showing isophotes (lines of equal intensity) in diffraction images, one affected by a certain amount of third order coma, the other by third order astigmatism. Unfortunately these diagrams are rather incomplete and contain some errors [see Nijboer (1947, p. 619) and Kingslake (1948, p. 152)]. Steward also studied the effects of different forms of aperture on the resolution. In particular he examined the influence of an annular aperture when third order spherical aberration is present under out-of-focus conditions.

Picht (1925, 1926) took as the starting point of his researches a well-known result due to Debye that the effect of spherical waves emerging from an optical instrument may be represented by a superposition of plane waves with different phases, amplitudes and directions of propagation. By generalizing Debye's formulae, Picht showed that the intensity distribution associated with the passage of aberrant waves through an exit pupil may be written in the form

\[
I(x, y, z) = \left| \int \int \psi(m, n) \exp \left\{ -i \frac{k}{2} [(x - \xi)l + (y - \gamma)m + (z - \zeta)n] \right\} \frac{dm \, dn}{l} \right|^2
\]

where \( \xi = \xi(m, n), \quad \gamma = \gamma(m, n), \quad \zeta = \zeta(m, n) \)

\[ \ldots \ldots \quad (3.10) \]
are parametric equations of a typical wave, \((l, m, n)\) are direction cosines of its normals and \(\psi\) is an amplitude factor. In the presence of third order aberrations, Picht obtained the development of \((3.10)\) into series of Bessel functions. From these series he computed, for a small amount of spherical aberration, the three-dimensional intensity distribution in the neighbourhood of the paraxial focus. He also studied a number of images affected by third order astigmatism in systems with a rectangular exit pupil.

Born (1932, 1938) studied the influence of very small aberrations of the third order. He derived an expression for the intensity, applicable whether one or more observations are present, in terms of the functions

\[ K_0(v) = \left( \frac{J_1(v)}{v} \right)^2, \quad K_1(v) = \frac{2J_1(v)J_2(v)}{v}, \quad K_2(v) = \frac{2J_1(v)J_2(v)}{v^2}. \]

Born's formula applies however to distributions in the Gaussian image plane only.

No further substantial advances were made up to about 1940. The following years witnessed a renewed interest in this subject and led to a number of important developments which we shall now discuss.

3.2. Advances since 1940

3.21 Images in the presence of small aberrations.

Some of the researches so far described led to results which in many cases permitted the calculation of the light distribution in the diffraction images. These solutions, however, were not entirely satisfactory as they were either too restricted or involved very heavy computations.

In an important thesis* Nijboer (1942) took up this problem afresh and obtained a simpler and more satisfactory solution for cases where the wave deformation is small, only a fraction of a wavelength. Nijboer considered two series expansions of the aberration function \(\Phi\), viz.

\[ \Phi(\sigma, r, \phi) = \sum_{l, m, n} b_{lmn} \sigma^{2l+m} r^n \cos m\phi \]  

\[ \Phi(\sigma, r, \phi) = \sum_{l, m, n} b_{lmn} \sigma^{2l+m} R_n^m(r) \cos m\phi, \]

where \(l, m, n\) are non-negative integers, \(n \geq m, n - m\) is even, \(R_n^m(r)\) are certain polynomials which we discuss later, the other symbols having same meaning as in § 2. A closer examination shows that the first expansion is particularly suitable for a geometrical treatment of aberrations, the second for a diffraction treatment. We recall that the traditional expansion with the same variables would contain terms \(\cos^{m+1}\phi\) in place of \(\cos m\phi\). The occurrence of the Fourier terms is essential in Nijboer's theory.

Some advantages of the new expansions can be illustrated by considering first the geometrical aberrations. It can be shown that in the presence of a single aberration

\[ b_{lmn}^m \sigma^{2l+m} r^n \cos m\phi \]

the rays from the point \((r, \phi)\) of the exit pupil intersect the plane through the Gaussian image point perpendicular to the principal ray in a point \((x, y)\) given by

\[ x + iy = \frac{1}{2} b_{lmn}^m \sigma^{2l+m} r^{n-1} \frac{R}{a} \left[ (n+m)e^{i(m-1)\phi} + (n-m)e^{i(m+1)\phi} \right], \]

* An account of this work will also be found in Nijboer (1943, 1947). As mentioned by Nijboer the method was to some extent suggested by earlier unpublished researches of Zernike.
where \( a \) and \( R \) denote the radius of the exit pupil and of the reference sphere respectively. From (3.14) it follows that the main features of the aberration figure (given by \( r = \text{constant} \)), such as symmetry, depend on \( m \) while details depend both on \( m \) and \( n \). It will be shown further that \( m \) and \( n \) play a similar role in the appearance of the diffraction pattern for an aberration specified by the term
\[
b_{nm} \sigma^{2l+m} R_n^m(r) \cos m \phi.
\]........(3.15)

This suggests a new classification of image errors in which the value of \( m \) determines the general type. By analogy with Seidel aberrations, the terms with \( m = 0, 1 \) and \( 2 \) may be called spherical aberration, coma and astigmatism respectively. Curvature and distortion now appear as degenerate cases of spherical aberration and of coma. The terms with \( m \geq 3 \) do not occur in Seidel theory. This classification considerably simplifies the discussion of the higher order effects.

In Nijboer's diffraction treatment the expansion of the aberration function is of the form (3.12) where
\[
R_n^m(r) \cos m \phi
\]........(3.16)
are so-called circle polynomials introduced by Zernike (1934) in his diffraction theory of phase contrast. These polynomials are orthogonal within the unit circle and the \( R_n^m(r) \) are given by
\[
R_n^m(r) = (-1)^{(n-m)/2} \binom{n+m}{m} r^m F \left[ \frac{1}{2}(n+m+2), -\frac{1}{2}(n+m), -\frac{1}{2}(n-m), m+1, r^2 \right]
\]........(3.17)
where \( F \) is a hypergeometric function. The expansion in terms of circle polynomials is particularly advantageous for the balancing of very small aberrations against each other so as to obtain maximum Strehl definition. We observe that in (3.15) a number of terms of the customary form \( \sigma^{2l+m} r^l \cos \phi \) are combined; with the help of the orthogonality relations for the circle polynomials it may be shown that they have been combined in such a way as to give maximum intensity in the centre of coordinates for a given value of the coefficient of
\[
\sigma^{2l+m} r^m \cos^m \phi.
\]

This may be illustrated by recalling that Richter (1925) showed (in a different notation) that in the presence of a small amount of spherical aberration of the form
\[
\Phi = A_6 r^6 + A_4 r^4 + A_2 r^2 + A_0,
\]........(3.18)
the Strehl definition is a maximum for a given value of \( A_6 \) if
\[
A_4 / A_6 = -3/2, \quad A_2 / A_6 = 3/5.
\]........(3.19)

In Nijboer's diffraction treatment the fifth order spherical aberration is given by the term
\[
b_{060} R_6^0(r) = b_{060} (20r^6 - 30r^4 + 12r^2 - 1),
\]........(3.20)
and Richter's condition is automatically satisfied. However, we must insist that this balancing of aberrations as achieved in Nijboer's theory holds for very small aberrations only; in some cases, as Figure 8 indicates, it no longer holds when the maximum wave deformation is only about \( 0.6 \lambda \).
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With the usual approximations the Huyghens–Kirchhoff integral for the disturbance at a point $(q, \psi)$ in a receiving plane perpendicular to the principal ray and specified by a parameter $p$, can be written in the form

$$U(p, q, \psi) = \frac{1}{\pi} \int_0^{2\pi} \exp \{ipr^2 + iqr \cos (\phi - \psi) - ik\Phi(\sigma, r, \phi)\} r \, dr \, d\phi. \quad \ldots \ldots (3.21)$$

In the presence of a single aberration Nijboer writes

$$\Phi(\sigma, r, \phi) = \frac{1}{k} \beta_{\text{nm}} R_n^m(r) \cos m\phi,$$

where

$$\beta_{\text{nm}} = k\alpha^{2l+m} b_{\text{nm}}, \quad \ldots \ldots (3.22)$$

and expands the term $\exp ik\Phi$ in powers of $\beta$. Neglecting fifth and higher powers of $\beta$ in (3.21) and integrating with respect to $\phi$ Nijboer obtains

$$U(p, q, \psi) = \int_0^{2\pi} e^{iprJ_0(qr)} r \, dr - 2i\beta_{\text{nm}} i^m \cos m\psi \int_0^{2\pi} e^{ipr^*} R_n^m(r) J_m(qr) r \, dr$$

$$+ \frac{(i\beta_{\text{nm}})^3}{2!} \left\{ \int_0^{2\pi} e^{ipr^*} R_n^m(r)^2 J_0(qr) r \, dr + i^3m \cos 2m\psi \right\}$$

$$\times \int_0^{2\pi} e^{ipr^*} R_n^m(r)^3 J_m(qr) r \, dr \right\} - \frac{(i\beta_{\text{nm}})^3}{2 \cdot 3 !} \left\{ 3i^m \cos m\psi \right\}$$

$$+ \frac{(i\beta_{\text{nm}})^4}{2^4 \cdot 4 !} \left\{ 3 \int_0^{2\pi} e^{ipr^*} R_n^m(r)^4 J_0(qr) r \, dr + 4i^3m \cos 2m\psi \right\}$$

$$\times \int_0^{2\pi} e^{ipr^*} R_n^m(r)^3 J_m(qr) r \, dr + i^4m \cos 4m\psi$$

$$\times \int_0^{2\pi} e^{ipr^*} R_n^m(r)^4 J_m(qr) r \, dr \right\}. \quad \ldots \ldots (3.23)$$

From this expression a number of symmetry properties of the diffraction pattern may immediately be deduced. It is seen for example that the $p$-axis is an $m$-fold axis of symmetry. Moreover if $m$ is odd (e.g. as in the case of coma) the intensity distribution is symmetrical about the plane $p = 0$. If $m$ is even (but $m \neq 0$) the intensity at any point is equal to that in the point resulting from a reflection in the plane $p = 0$ and the additional rotation about the $p$-axis through an angle $\pi/m$. For some special cases this rule has previously been stated by Strehl, by Picht and by Steward.

Nijboer outlined a method based on various properties of the circle polynomials which makes possible the development of the integrals in (3.23) into series of Bessel functions. The series terminate in the important case $p = 0$. From these expansions the isophotes near focus may be determined for a small arbitrary single aberration. Some of the diagrams obtained in this way by Nijboer (1942, 1947) and Zernike and Nijboer (1949) are reproduced in Figures 3, 4, 5, 7. Additional cases of slightly larger aberrations were studied by Nienhuis and Nijboer (1949). This later investigation was confined to third order effects and contains new formulae and diagrams (Figures 6 and 8) for the

* Unless otherwise stated, all figures refer to systems with a circular exit pupil and $\Phi_{\text{max}}$ denotes the maximum deviation of the wave from the reference sphere, shown in Figure 1. The Figures are not all reproduced on the same scale.
Figure 3. Isophotes as in Figure 2, in presence of third order spherical aberration of amount $\beta_{140} = \frac{1}{2} (\Phi_{\text{max}} = 0.48\lambda)$. The thick line indicates the geometrical caustic. Strehl definition 0.95. After Zernike and Nijboer (1949).

Figure 4. Isophotes as in Figure 2, in presence of spherical aberration of amount $\beta_{000} = \frac{1}{2} (1.6\lambda)$ of fifth order spherical aberration balanced against $2.4\lambda$ of third order spherical aberration. The thick line indicates the geometrical caustic. Strehl definition 0.965. After Zernike and Nijboer (1949).

Figure 5. Isophotes in the Gaussian image plane in presence of third order coma of amount $\beta_{211} = 1$ ($\Phi_{\text{max}} = 0.48\lambda$). Strehl definition 0.879. The dotted curve represents the line of zero intensity. The boundary of the geometrical confusion figure is also indicated. The numbers represent intensities corresponding to a value of 1,000 in the centre of the ideal Airy pattern. After Nijboer (1947).

Figure 6. Isophotes in the Gaussian image plane in presence of third order coma of amount $\beta_{211} = 3$ ($\Phi_{\text{max}} = 1.4\lambda$). Strehl definition 0.306. The dotted curves represent the line of zero intensity. The boundary of the geometrical confusion figure is also indicated. After Nienhuis and Nijboer (1949).
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Figure 8. Isophotes in the central plane in presence of third order astigmatism of amount $\beta_{22} = 4(\phi_{\text{max}} = 0.64\lambda)$. Strehl definition 0.840. The dotted circle indicates the boundary of the geometrical confusion figure. *After Nijboer (1947).*

light distribution in certain receiving planes. It was supplemented by photographs (see Plates *) of actual images showing a good agreement with the theoretical predictions. In this work the authors rediscover the formula

$$I = \left[ \frac{2}{\beta} \sum_{s=0}^{\infty} J_{2s+1}(\beta) \right]^2,$$

originally given in a slightly different notation by Picht (1931, p. 202) for the Strehl definition in the central plane $p = 0$ when third order astigmatism is present.
Some of the isophote diagrams involved the computation of intensity at as many as 600 points. This heavy labour is amply rewarded by the results which give the first real insight into the complex structure of important types of diffraction image.

A few cases of small aberrations were also studied by Maréchal and by Kingslake by methods which we shall discuss later.

3.22. Tolerance conditions.

Wang Ta-Hang (1941) discussed the location of the best axial focus in the presence of spherical aberration and deduced tolerance conditions for the maximum permissible amounts of third, fifth and seventh order spherical aberration for systems where the lower order terms are under control. Most of Wang Ta-Hang's results were previously found by Conrady, Buxton, Martin and Richter and also follow as special cases from Nijboer's researches.

Optimum correction of small aberrations and tolerance conditions were studied in an elegant and comprehensive manner by Maréchal (1944a, b, c, 1947a). These investigations are based on a relation between the intensity at points near the maximum of the diffraction pattern and the mean square deviation of the wave front from a certain reference sphere.

![Figure 18.](image)

Let $\Sigma$ be a wave surface in the exit pupil of an instrument and let $S$ be a sphere in its neighbourhood, centred on $C$ and of radius $R$. Further let $A$ be a point on $S$ and let $AC$ intersect $\Sigma$ in $B$ (Figure 18). Finally let

$$AB = \Delta, \quad BC = l, \quad d\sigma = d\sigma/\int_{\Sigma} d\sigma,$$

$$\ldots \ldots (3.24)$$

where $d\sigma$ is an element of $\Sigma$. (In general $\Delta \neq \Phi$ since $C$ is not necessarily the Gauss image point and $S$ does not necessarily pass through the centre of the exit pupil.)

For a given wave front and for fixed $C$, the mean square deviation

$$E = \int \int_{\Sigma} \Delta^2 d\omega,$$

$$\ldots \ldots (3.25)$$

of the wave front from the sphere is a function of the radius $R$. This function has a minimum

$$E = E_0 = \int \int_{\Sigma} l^2 d\omega - \left( \int \int_{\Sigma} l d\omega \right)^2,$$

$$\ldots \ldots (3.26)$$

when

$$R = R_0 = \int \int_{\Sigma} l d\omega.$$
Maréchal showed, that when $|\Delta|<\lambda/4$ (a condition twice less severe than that imposed by Rayleigh's criterion), the intensity $I$ (taken as unity at the centre of the Airy pattern) at C satisfies the relation

$$I \sim \left[1 - \frac{2\pi^2}{\lambda^2} E_0\right]^2.$$ ...... (3.27)

Earlier authors found that in presence of certain aberrations of amount $\frac{1}{4}\lambda$, the central intensity takes on values between about 0.73 and 0.87, the exact amount depending on the type of aberration. With the help of (3.27) Maréchal was able to formulate a new tolerance criterion which unlike that of Rayleigh corresponds to a fixed lower limit for the central intensity.

A system is generally regarded as well corrected if the loss in intensity at the centre of the pattern does not exceed 20%, i.e. when $I \geq 0.8$. For such a case (3.27) implies that $E_0 < \lambda^2/180$. Maréchal therefore concluded that for a well corrected system the root-mean-square deviation of the wave from the 'mean focal square' (given by $R = R_0$) is less than $\lambda^2/180$. In accordance with this criterion, Maréchal then deduced the following tolerance conditions for defocusing and for aberrations of the third order:

**Defocusing:** $\lambda/2x^2$.

**Spherical aberration:** $4\lambda/x^2$.

**Coma:** $|g| \leq 0.63 \frac{\lambda}{\sigma x}$.

**Astigmatism:** $|t-s| \leq \lambda/x^2/2$.

**Astigmatism and curvature:** $(t+s)^2 + 2(t-s)^2 \leq \lambda^2/x^4$.

**Combined aberrations:**

$$\left(\frac{L}{L_{\text{max}}}\right)^2 + \left(\frac{\delta g}{\delta g_{\text{max}}}\right)^2 + \left(\frac{t-s}{(t-s)_{\text{max}}}\right)^2 + \left(\frac{t+s}{(t+s)_{\text{max}}}\right)^2 \leq 1.$$  

Here $x$ denotes the maximum angular semi-aperture in the image space, $L$ denotes the longitudinal spherical aberration, $g$ the sine condition ratio, and $t$ and $s$ are the abscissae of the tangential and the sagittal focal lines. Further, $L_{\text{max}}$ denotes the maximum amount of the longitudinal spherical aberration which may be tolerated when no other aberration is present, $\delta g_{\text{max}}$, $(t-s)_{\text{max}}$ and $(t+s)_{\text{max}}$ having analogous meanings. For fifth order spherical aberration and fifth order coma Maréchal found that the tolerances may be increased by 50% provided these aberrations are suitably 'balanced'. A tolerance condition for third order coma substantially in agreement with Maréchal's was recently deduced by Kingslake (1948).

Toraldo di Francia (1946) derived an expression for the lower limit of intensity at the centre of the reference sphere in terms of the maximum wave aberration. In particular, this expression shows that when the aberration is less than $\lambda/4$ the intensity at the centre does not fall below 0.5.

Françon (1944, 1947a, b, 1948a) studied both theoretically and experimentally the efficiency of visual optical instruments suffering from spherical aberration. (Aberrations may be tolerated when the efficiency, defined as the ratio of the limiting resolvable separation of the unaided eye to that of the instrument–eye combination, is nearly 1.) He found that for a small pupil (0.8–0.9 mm. in diameter), the eye may be regarded as a perfect instrument and in this case a

* Väisälä (1922) derived a weaker condition $I \sim 1 - 4\pi^2 E_0/\lambda^2$. 
high degree of correction is called for. For pupils of larger size poorer correction may be tolerated. Françon showed that for small pupils and objects of very low contrast (\(\sim 0.3\)), aberrations of the order of \(\lambda/16\) may be significant. Rayleigh's and Maréchal's criteria apply to a perfect eye, and contrast 1. Françon's investigations give, for the case of spherical aberration, an extension of Rayleigh's criterion for pupils up to 4 mm. and for a range of contrasts.

3.23. Asymptotic behaviour of the diffraction integral.

The effects of aberrations which are large compared with the wavelength may be investigated by examining the asymptotic behaviour of the diffraction integral for large values of \(k = 2\pi/\lambda\). Van Kampen (1949) studied this problem by using a two-dimensional analogue of the principle of stationary phase (cf. van der Corput (1948), p. 206).

The Huyghens-Kirchhoff diffraction integral which describes the complex displacement at a point \(P\) in the image space of an instrument may be written in the form

\[
U(P) = \int \int_D k g(x, y) e^{ikf(x, y)}\, dx\, dy, \quad \ldots \ldots (3.28)
\]

where \(D\) is a domain bounded by a finite number of analytic curves \(C\). Van Kampen found that, on account of the rapidity of fluctuation of the exponential term in (3.28), the value of \(U(P)\) depends substantially on the behaviour of the integrand near a limited number of 'critical points' which may be of three kinds: (1) internal points of \(D\) at which \(f\) is stationary, i.e. where \(\partial f/\partial x = \partial f/\partial y = 0\); (2) boundary points at which \(f\) is stationary, i.e. where \(\partial f/\partial s = 0\), \(ds\) being element of \(C\); (3) corner points or boundary points where two analytic curves join.

In the neighbourhood of a critical point of the first kind, the integrand of (3.28) can be expanded in the form

\[
k g e^{ikf} = k \exp \left(ika_{00} \exp ik(a_{20}x^2 + a_{11}xy + a_{02}y^2 + \ldots)\right) \left(b_{00} + b_{10}x + \ldots\right). \ldots (3.29)
\]

To find the corresponding contribution to the integral van Kampen neglects in the exponent third and higher powers of \(x\) and \(y\) and integrates term by term over the ranges \(-\infty < x < \infty\), \(-\infty < y < \infty\). The resulting expansion is a power series in \(k^{-1}\) beginning with a constant term. For the contribution of a critical point of the second kind, van Kampen obtains in an analogous manner a series with \(k^{-1/2}\), \(k^{-3/2}\), \(k^{-5/2}\) and for the contribution of corner points, a power series in \(k^{-1}\), in both these cases the constant term being absent. The total effect is then obtained by adding the contribution from each of the critical points.

The principal term of the final expansion is independent of the wavelength and comprises the constant terms of the expansions at critical points of the first kind. This part of the solution can be interpreted as a contribution of 'geometrical optics' taking, however, into account the mutual interference of the wave patches arriving at \(P\) from the immediate neighbourhood of critical points of the first kind.

The rest of van Kampen's paper deals with applications to third order aberrations in systems with a circular aperture. He then found for example that in the case of astigmatism points of the second kind give rise to an asteroid pattern; this pattern was also discussed by Nienhuis (1948).

Although van Kampen's investigations give some valuable results, his analysis cannot be regarded as satisfactory. For example, van Kampen integrates the expansions over infinite ranges, a procedure which introduces non-negligible
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3.24. Images formed by waves on non-uniform amplitude. *

When studying images which are formed by an instrument of the usual design it is as a rule permissible to assume that the waves are of uniform amplitude. Several authors have in recent years considered the possibility of improving the quality of the image by a suitable variation of the amplitude and sometimes also of the phase of the disturbance over the exit pupil. This may be realized in practice, for example by evaporating thin films of metallic or dielectric substances on to the lenses or by means of specially constructed filters.

With the usual approximations, it follows from Huyghens' principle that the complex displacement \( U(x,y) \) in the receiving plane of a system imaging a point source may be written in the form of a double Fourier integral

\[
U(x,y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P(p,q)e^{2\pi i (px + qy)} dp dq, \quad \ldots \ldots (3.30)
\]

where \( P(p,q) \) is the so-called pupil function (sometimes called transmission function), which describes, in terms of the optical direction cosines \( p \) and \( q \) in the image space, the complex displacement over a spherical reference surface centred at \( x = y = 0 \). The integration is only formally carried out over an infinite domain since \( P \) is taken as zero when \( p^2 + q^2 > \rho_m^2 \), \( \rho_m \) being the maximum numerical aperture in the image space. In the case of rotational symmetry (3.30) may be written as

\[
U(r) = 2\pi \int_0^{\infty} P(\rho)J_0(2\pi \rho \rho) \rho d\rho, \quad \ldots \ldots (3.31)
\]

where \( \rho \) is the zonal numerical aperture in the image space and \( r \) is the distance from the optical axis in the receiving plane, measured in wavelengths. (The symbols \( r \) and \( \rho \) now denote different quantities from those in previous sections.)

Luneberg (1944) considered the case of rotational symmetry and studied the effects of variation of amplitude alone. He found that amongst all diffraction patterns of equal energy the highest central maximum is given by the normal pattern \( P(\rho) = \text{constant} \). In particular, any pattern which gives improved resolution must therefore have a lower central maximum. By a variational argument Luneberg deduced an expression for the pupil function in terms of the radius \( r_0 \) of the first dark ring. It appears that for \( r_0 < 0.31(2\pi / \rho_0) \) (i.e. about a half of the radius \( r_A \) of the first dark ring in the Airy pattern) the resulting pattern has a rather low central maximum.

Luneberg also investigated how to choose \( P(\rho) \) so that the fraction of the total illumination

\[
L(r_0) = \frac{\int_{r_0}^{\infty} |U(r)|^2 r \, dr}{\int_0^{\infty} |U(r)|^2 r \, dr} \quad \ldots \ldots (3.32)
\]

* The investigations discussed in this section do not fall under the heading of diffraction theory of aberrations in the customary sense but are included since they are closely related to the problem of optimizing resolution by modifying the waves.
which reaches the circle of radius \( r_0 < r_A \) is as large as possible. He showed that the corresponding pupil function is given as a solution of a certain integral equation. He also discussed the resolution of objects of periodic structure and found that in this case it is impossible to increase the resolving power by varying the amplitude only, but that the contrast of the image may be improved considerably by this means.

Couder (1944) discussed amplitude filters which consist of a plano-concave lens filled with an absorbing liquid. The corresponding pupil function is of the form \( P(\rho) = 10^{-k\rho^a} \). It is perhaps worth noting that Straubel (1902, 1931) who appears to have been the first to study effects of non-uniformity of amplitude on resolution also considered pupil functions of this form. Couder found that with such a filter it is possible to redistribute the illumination in the diffraction pattern so as to facilitate the observation of double stars whose components differ widely in brightness. Amplitude filters were also discussed by Lansraux (1946, 1949).

Boughon, Dossier and Jacquinot (Boughon et al. 1946, Jacquinot et al. 1949) considered applications of amplitude variation to spectroscopy. For diffraction at a slit (3.31) and (3.32) are replaced by the equations

\[
U(x) = \int_{-\infty}^{\infty} P(x') e^{2\pi i x x'} dx', \quad \ldots \ldots \text{(3.33)}
\]

and

\[
L(a) = \frac{\int_{-\infty}^{\infty} [U(x)]^2 dx}{\int_{-\infty}^{\infty} U(x)^2 dx}. \quad \ldots \ldots \text{(3.34)}
\]

The above authors used the analysis of trigonometrical polynomials to determine pupil functions which for a given value of the parameter \( a \) make the fraction of the total illumination \( L(a) \) as large as possible. They also considered distributions for which \( U(x) \) behaves asymptotically like \( x^{-m} \), \( m \) being a positive integer. They showed that by the choice of a suitable trigonometric polynomial as pupil function the effect of diffraction fringes can be reduced, facilitating the detection of satellites to bright spectral lines.

Expansion of pupil functions into series of Hermitian polynomials were considered by Duffieux (1946 b). Expansions into series of Legendre polynomials were suggested by Slevogt (1949).

Lansraux (1947 a, b) discussed the evaluation of the diffraction integral (3.31) for cases where \( P(\rho) \) can be expanded in a series of lambda functions

\[
\Lambda_\sigma(\rho) = [1 - (\rho/\rho_m)^2]^{s-1}. \quad \ldots \ldots \text{(3.35)}
\]

He showed that when

\[
P(\rho) = \sum_{s=1}^{\infty} a_s \Lambda_s(\rho), \quad \ldots \ldots \text{(3.36)}
\]

where the \( a_s \)'s are constants

\[
U(\rho) = \pi \rho_m^2 \sum_{s=1}^{\infty} a_s 2^{s-1} \cdot \frac{J_s(2\pi \rho_m \rho)}{(2\pi \rho_m \rho)^s}. \quad \ldots \ldots \text{(3.37)}
\]

Lansraux asserts that for a small aberration (3.37) is rapidly convergent. That this is not necessarily so can be illustrated by considering the simplest aberration, namely defocusing, when (3.37) reduces to that solution of Lommel which involves the \( U_n \) series. These series converge rapidly only at points in the geometrical shadow (case \( |u/v| < 1 \) in §3.1 above). In the direct beam of light even when very near focus the series (3.37) is therefore not suitable for computations.
Hopkins (1949) extended Lommel's analysis to waves of non-uniform amplitude, the non-uniformity considered being of the form $\alpha + \beta (p/p_o)^2$ where $\alpha$ and $\beta$ are constants. He obtained a solution in terms of Lommel's original $U_n$ and $V_n$ functions and certain closely related functions and gave a number of intensity distribution curves for out-of-focus patterns. He showed that for most ordinary lens systems it is quite justifiable in evaluating the diffraction integral to assume the disturbance over the converging waves to be of uniform amplitude.

Osterberg and Wilkins (1949) studied theoretically the problem of coating the exit pupil with light absorbing and refracting materials so as to reduce the diameter of the central bright disc in the diffraction pattern. They considered pupil functions of the form (3.36) and derived Lansraux' expression very shortly by the application of Sonine's integral formula. In some cases choice can be made of the constants $a_*$ to give pupil functions corresponding to diffraction patterns with prescribed properties. As Osterberg and Wilkins point out, however, it is not possible to go far in this direction because an arbitrary function $U(p)$ cannot, in general, be expressed in series of the type (3.37). The resolution of two particles in a bright field by Sonine type microscope objectives (microscope objectives for which $P(p)$ is given by (3.36)) was discussed in a paper by Osterberg and Wissler (1949).

Wilkins (1950) showed how to coat the exit pupil so as to obtain for any prescribed diameter of the central disc the highest possible central intensity. In Figure 19 we reproduce the intensity curves which he obtained for a few typical cases.
3.25. Other researches.

In his book *L'Intégrale de Fourier et ses Application a l'Optiques* and in a series of papers, Duffieux (1945a, b, 1946b, 1947, 1948) has, partly in collaboration with Lansraux (1945), attempted to apply the methods of Fourier transforms to the study of Fraunhofer diffraction.*

We recall that with a suitable interpretation of the variables \( p, q, x \) and \( y \) the complex disturbances \( P(p, q) \) in the exit pupil and \( U(x, y) \) in the 'pattern at infinity' due to a point source are related by the pair of double Fourier integrals

\[
\begin{align*}
U(x, y) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P(p, q) e^{-2\pi i (px + qy)} \, dp \, dq, \\
P(p, q) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} V(x, y) e^{2\pi i (px + qy)} \, dx \, dy.
\end{align*}
\] (3.38)

If the system images an extended incoherent object in which the intensity distribution is given by \( I(x, y) \), then the corresponding intensity distribution in the image is given by

\[
\Pi(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} I(x', y') E(x - x', y - y') \, dx' \, dy',
\] (3.39)

where

\[
E(x, y) = |U(x, y)|^2.
\] (3.40)

By the application of Parseval's theorem, Duffieux and Lansraux (1945) showed that the double Fourier transform or the space frequency spectrum \( T[\Pi(x, y)] \) is connected with that of the object, namely \( T[I(x, y)] \), by the relations

\[
T[\Pi(x, y)] = T[I(x, y)] \, T[E(x, y)] = D(p, q) \, T[I(x, y)]
\] (3.41)

where

\[
D(p, q) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E(x', y') e^{2\pi i (px + qy)} \, dx \, dy.
\] (3.42)

From (3.41) it follows that \( D(p, q) \) may be regarded as a transmission factor which characterizes the imaging properties of the system when the object is incoherently illuminated.

For an extended coherent object for which the amplitude distribution is denoted by \( I'(x, y) \) the intensity distribution \( \Pi'(x, y) \) in the image is given by

\[
\Pi'(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} I'(x, y) \, U(x - x', y - y') \, dx' \, dy'
\] (3.43)

leading to a relation

\[
T[\Pi'(x, y)] = T[I'(x, y)] \, T[U(x, y)] = P(p, q) \, T[I'(x, y)].
\] (3.44)

(3.44) shows that \( P(p, q) \) may be regarded as a transmission factor when the object is coherently illuminated.

From \( D \) and \( P \) Duffieux and Lansraux derived other transmission factors which have a bearing on the theory of test objects for optical instruments. To one such factor \( d(p) \), defined by

\[
d(p) = \int_{-\infty}^{+\infty} D(p, q) \, dq,
\] (3.45)

is related to resolving power in the image of non-coherent objects. By non-rigorous mathematical arguments Duffieux and Lansraux derived a number

* Additional references to papers dealing mainly with cases where the distribution in the image is a function of one variable only will be found in Duffieux (1946a).

The existence of a Fourier integral relation between the disturbance over a spherical surface filling the exit pupil and the disturbance over its 'focal sphere' was first pointed out by Michelson (1905).
Figure 9. Images in Gaussian plane in presence of third order coma of amount $\phi_{\text{max}} = 0.48\lambda$ and $\phi_{\text{max}} = 1.48\lambda$. Compare with Figures 5 and 6. *After Nienhuis* (1948).

Figure 10. Image in Gaussian plane in presence of third order coma of amount $\phi_{\text{max}} = 7.2\lambda$. Semicircular aperture. *After Nienhuis* (1948).

Figure 11. Images in central plane in presence of third order astigmatism with $\phi_{\text{max}} = 0.16\lambda$ and with $\phi_{\text{max}} = 0.64\lambda$. Compare with Figures 7 and 8. *After Nienhuis* (1948).

Figure 12. Images in presence of third order astigmatism of amount $\phi_{\text{max}} = 2.7\lambda$ in central plane and in plane containing a focal line. *After Nienhuis, reprinted from Zernike* 1948.
Figure 13. Images in presence of third order spherical aberration of amount $\phi_{\text{max}} = 16\lambda$ at marginal focus and at circle of least confusion. After Nienhuis (1948).

Figure 14. Images in plane of paraxial focus, in presence of third order spherical aberration of amounts $\phi_{\text{max}} = 17.5\lambda, 8.4\lambda, 3.7\lambda, 1.4\lambda$. After Nienhuis (1948).

Figure 15. Images in plane of circle of least confusion in presence of third order spherical aberration of amounts $\phi_{\text{max}} = 17.5\lambda, 8.4\lambda, 3.7\lambda, 1.4\lambda$. Scale $3 \times$ as in Figure 14. After Nienhuis (1948).
Figure 16. Images in Gaussian plane in presence of third order coma of amounts \( \Phi_{\text{max}} = 0.3\lambda, 1\lambda, 2.4\lambda, 5\lambda, 10\lambda \). *After* Nienhuis, *reprinted from* Zernike (1948).

Figure 17. Images in central plane in presence of third order astigmatism of amounts \( \Phi_{\text{max}} = 1.4\lambda, 2.7\lambda, 3.5\lambda, 6.5\lambda \). *After* Nienhuis, *reprinted from* Zernike (1948).
of relations connecting the various transmission factors and the intensity distribution in the image. One of these relations obtained by a formal application of Plancherel's theorem connects $P$ with the moments of the energy distribution of the image. Duffieux (1948) stated that these moments when taken about the centre of gravity of the distribution are the same for the geometrical and the physical image. As has been pointed out by E. H. Linfoot*, Duffieux' analysis is rendered invalid by failure to take into account the divergence of certain integrals; that the result also is incorrect can easily be seen by considering the moments of an error-free image. Several other conclusions of Duffieux are based on faulty mathematical arguments and appear to be incorrect.

Kingslake (1948) investigated the diffraction structure of third order comatic images by numerical integrations. He demonstrated that the image is of the size and shape determined by ordinary geometrical optics and that the main effect of diffraction is to break up the image into an elaborate fine structure of dots and lines of light. We give here two of his valuable diagrams (Figures 20 and 21) showing the isophotes for coma of fairly large amounts.

Figure 22. Isophotes in a meridional plane in presence of third order spherical aberration, of amount $\Phi_{\text{max}} = 4\lambda$. After Maréchal (1951).

Figure 23. Isophotes in a meridional plane in presence of third order spherical aberration, of amount $\Phi_{\text{max}} = 6\lambda$. After Maréchal (1951).
Maréchal (1947b, 1948) designed and constructed a machine for rapid evaluation of double integrals of the type occurring in the investigations of aberration diffraction effects. To determine the intensity at each point of the image, operations taking about 8–10 minutes are needed. The precision of the apparatus is quite adequate for practical purposes, the estimated inaccuracies $\Delta I$ being as follows ($I = 1$ at the centre of the Airy pattern): $\Delta I < 0.01$ when $I = 1$, $\Delta I < 0.003$ when $I = 0.1$, $\Delta I < 0.001$ when $I = 0.01$. With the help of this integrator Maréchal (1948, 1951) obtained a number of important isophote diagrams some of which are shown in Figures 22–25. Most of these diagrams show the effects of aberrations in the range $1\lambda$ to $10\lambda$ for which available series expansions are not well adapted. The calculations performed with the aid of this integrator were also compared by Maréchal with photometric measurements of the intensity distribution in actual images; good agreement was found.

* I am greatly indebted to Dr. Maréchal for giving me access to these figures before the publication of his own paper.
It is well known that in the approximations of Kirchoff's scalar theory, the disturbance produced by the passage of spherical waves through an aperture opening may be regarded as due to the combined effect of 'geometrical waves' and 'diffraction waves' emerging from the aperture edge.* Nienhuis (1948) assumed that this result holds (and also that the 'diffraction waves' possess analogous properties) when the incident waves are not strictly spherical† and

Figure 25. Isophotes in plane of astigmatic focal line, showing combined effect of spherical aberration (2λ), coma (1λ) and astigmatism (1λ) of third order. *After Maréchal (1951).

applied it to discuss some general features of diffraction images in the presence of third order aberrations.

In the case of astigmatism, Nienhuis showed by a geometrical argument that the 'diffraction waves' from a circular edge give rise to an asteroid pattern. This pattern is partly obstructed by the effect of the 'geometrical waves'. Its

* Ramachandran (1945) showed that the integral expressing the edge effect can be derived in a very simple manner if the variation of the amplitude of the secondary wavelets with direction is neglected. This idea was also used by Kathawate (1945) to discuss the effect of diffraction of spherical waves at boundaries and obstacles of various forms.

† Many assertions in Nienhuis' thesis and in other literature give the impression that the transformation of the diffraction integral into an integral expressing the edge effect and a term expressing the geometrical effect has been carried out in the general case, but an examination of the relevant papers reveals that the published proofs relate to spherical and plane waves only.
shape is in the first approximation independent of the focal setting and is therefore more apparent in a receiving plane through one of the focal lines (see Figure 12).* In the presence of coma, rays from opposite points of each aperture zone meet the Gaussian plane at the same point. Nienhuis found that the fringe structure of the comatic pattern can be ascribed to the interference of these rays, in agreement with his observation that the fringes disappear when half of the light is blocked out by using a semicircular aperture (Figure 10). For spherical aberration Nienhuis found that in the receiving plane through the paraxial focus it is mainly the edge effects which determine the structure of the pattern whilst in other planes the interference of the ‘geometrical waves’ plays a considerable part.

In connection with this work it is of interest to note that Durand (1949) studied in a comprehensive manner the fringe structure of third order aberration figures. Though Durand’s analysis does not take account of diffraction it has many points of contact with the work of Nienhuis. Durand’s results may be expected to give good approximations to the appearance of the image whenever edge effects do not play an important part. This is particularly well illustrated by the case of coma (Figure 26).

Françon (1948 a, b) discussed the effects on the image due to a certain type of non-homogeneity in the glass of an objective. He found that in the presence of filaments or veins whose refractive index differs slightly from the bulk of the glass, the effects are negligible if the path differences they introduce are less than \( \lambda/8 \) for \( p = 50 \), \( \lambda/6 \) for \( p = 100 \) and \( \lambda/4 \) for \( p = 200 \), \( p \) being the ratio of the

* Some of Nienhuis’ photographs of diffraction images are also given by Zernike (1948). The corresponding values of the aberration constants differ in several cases and some appear to be incorrect. In particular it appears that the aberration constants for the first two comatic patterns of Figure 16 should be larger. Values of \( \Phi_{\text{max}} \) based on Nienhuis’ data are given in Figures 16 and 17.
diameter of the objective to the width of the filament. When one observes two
neighbouring objects of widely differing brightness the tolerances are much more
severe.

Epstein (1949) took up once more the problem treated by Lommel (1885)
(see §3.1) and found methods for the numerical evaluation of the associated
diffraction integral, which possess some advantages over those of Conrady and
Buxton.

\[ \frac{\nu}{f} = \frac{a^2}{f} \Delta f, \quad \frac{v}{f} = \frac{2\pi a r}{f}, \]

where \( a \) = radius of exit pupil, \( f \) = focal length, \( \Delta f \) = amount of defocusing.

Contour lines for the fraction of the total illumination inside circles of radius \( r \), centred on
axis, in receiving planes near focus of spherical waves issuing from a circular aperture
(a) according to diffraction theory, (b) according to geometrical optics. After Wolf (1950).

The distribution of the total illumination in out-of-focus images formed by
spherical waves issuing from a circular aperture was investigated by Wolf (1951).
He derived expressions for the fraction of the total illumination present inside
circles of given radii in the receiving plane and applied them to study the
distribution in a few selected planes near the focus. Wolf also gave a diagram
(see Figure 27) showing the corresponding three-dimensional distribution and
compared it with the prediction of geometrical optics. His results, together with
those of Lommel, Zernike and Nijboer, provide a mathematical basis for a
quantitative discussion of the imaging properties in optical systems where, as for example in a well corrected refracting telescope objective, the chromatic variation of focus is the only appreciable aberration.

Mme. Regnier (1950) studied the influence of spherical aberration on the limiting visibility of Foucault test objects. She used a relation due to Duffieux and Lansraux between the pupil function \( P(p, q) \), the transmission factor \( d(p) \) and the limiting visibility, and calculated \( d(p) \) for spherical aberration of amounts \( h/4, \lambda/2, 3h/4, \lambda \) and for different forms of exit pupil. The results yield conclusions about the dependence of the limiting visibility of test objects on the aberration of the objective.

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