Babinet's principle in the Fraunhofer diffraction by a finite thin wire

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ABSTRACT

The scattered waves by a thin finite wire are evaluated by using the Rayleigh-Sommerfeld integral in the Fraunhofer approximation. The scattered fields by the complementary thin wire are also obtained with the aid of the Babinet’s principle. The scattering integrals are evaluated directly. It is shown that Babinet’s principle holds excellently for this problem. The scattered fields are examined numerically.

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1. Introduction

The principle of Babinet is a useful tool, which provides flexibility in the solution of aperture diffraction problems in optic and electromagnetics. This principle was proposed by Babinet in 1837 [1] and states that the sum of the scattered fields by an obstacle and its complementary aperture leads to the unobstructed incident wave [2]. In fact this model is in harmony with the boundary diffraction wave (BDW) theory. The theory of BDW examines the scattering process by an aperture as the sum of two waves [3–5]. The first one is the geometrical optics field that passes through the aperture, unobstructed by the discontinuities. The second wave is the diffraction field, which is radiated by the edges of the aperture. This approach is based on the proposal of Young [6]. The diffracted waves that are created by an obstacle and its complementary aperture have a phase difference of π. For this reason when they are added to each other, they will vanish. The total field becomes the geometrical optics wave, which is directly equal to the incident field.

The rigorous form of the Babinet’s principle was first derived by Copson [7,8] and Meixner [9] for the perfectly conducting planes. In 1957, Neugebauer extended the principle for the scattering problem of electromagnetic waves by apertures in absorbing screens [10]. The limitations of the Babinet’s principle were put forward by Boersch [11] and Hosemann and Joerchel [12]. Lipson and Walkley experimentally studied these limitations in the context of the diffraction of light by a circular aperture [13]. Andrews and Margolis [14] examined the Babinet’s principle for the diffraction of the electromagnetic waves by taking into account the edge diffracted in the complementary half-planes. Totzeck and Krumbügel utilized the method of Andrews and Margolis with the BDW theory and applied for the diffraction of weak phase objects [15]. Jiménez and Hita introduce a proof of the Babinet’s principle by using the scalar diffraction theory in the Fraunhofer approximation [16]. However, the investigation of Ganci on the diffraction of waves by a thin wire showed that the Babinet’s principle leads to erroneous results in the Fraunhofer region [17]. He proposed a Gaussian beam as the incident wave on the wire and obtained the scattering integral in terms of the actual and complementary wires. The evaluation of the actual and complementary integrals yielded two different results. Thus, Ganci concluded that the Babinet’s principle is incorrect.

The aim of this study is to investigate the scattering problem of waves by a thin wire, located on the x-axis. We will obtain the scattering integral of the wire by using the Rayleigh-Sommerfeld integral in the Fraunhofer approximation. We will also derive the scattering integral for the complementary thin wire, which extends to ±∞ on the x-axis. We will evaluate the two integrals and interpret the results in terms of the Babinet’s principle.

A time factor of exp(jwt) is taken into account and suppressed throughout the paper. w is the angular frequency.

2. Theory

We take into a thin wire, located at \( L = \{ x, y, z \} : x \in (- h, h) : y = 0, \ z = 0 \), as given in Fig. 1. A Gaussian beam of

\[
\begin{align*}
\psi(x, y, z) &= \psi_0 \exp \left[ -j \kappa(z + z_0 - j b) \right] \exp \left[ -j k \left( \frac{x^2 + y^2}{2(z + z_0 - j b)} \right) \right] 
\end{align*}
\]

(1)

is illuminating the wire [18]. 2b is the beam waist at \( z = z_0 \). \( \psi_0 \) is the complex amplitude.
The Rayleigh-Sommerfeld integral can be written as

\[ u_1(x, y, z) = \frac{j k}{2\pi} \int_A u(x', y') \exp(-jkR) \, dx' \, dy' \quad (2) \]

for the scattered waves by an aperture of \( A \) in the plane of \( z = 0 \). \( k \) is the wavenumber. \((x', y', z')\) and \((x, y, z)\) are coordinates of the integration and observation points. \( u_1 \) is the scattered wave. \( R \) has the expression of \( \sqrt{(x - x')^2 + (y - y')^2 + z^2} \) and represents the distance between the integration and observation points. \( R \) can be approximated as

\[ R \approx z + \frac{x'^2 + y'^2}{2z} - \frac{xx' + yy'}{z} \quad (3) \]

and

\[ R \approx z \quad (4) \]

at the phase and amplitude functions of the integral in the Fraunhofer region. Thus Eq. (2) reads

\[ u_1(x, y, z) = \frac{j}{\lambda} \exp\left[-jk\left(z + \frac{x^2 + y^2}{2z}\right)\right] \int_A u(x', y') \exp\left(jk\frac{xx' + yy'}{z}\right) \, dx' \, dy' \quad (5) \]

for \( \lambda \) is the wavelength. Suppose that \( A_c \) is the complementary aperture of \( A \) in the plane of \( z = 0 \). Then the relation of

\[ u_1(x, y, z) = \frac{j}{\lambda} \exp\left[-jk\left(z + \frac{x^2 + y^2}{2z}\right)\right] \int_{A_c} u(x', y') \exp\left(jk\frac{xx' + yy'}{z}\right) \, dx' \, dy' \quad (6) \]

can be proposed according to the Babinet's principle. \( A_{\infty} \) is the full aperture on \( z = 0 \).

The scattered fields by the thin wire, given in Fig. 1, can be given by the expression of

\[ u_{s1}(x, y, z) = \frac{j}{\lambda} \exp\left[-jk\left(z + \frac{x^2 + y^2}{2z}\right)\right] \int_{x' = -h}^{h} \int_{y' = -\infty}^{\infty} u_i(x', y') \exp\left(jk\frac{xx' + yy'}{z}\right) \, dx' \, dy' \quad (7) \]

where \( u_i(x', y') \) can be defined as

\[ u_i(x', y') = u_0 \exp\left[-\frac{jk(z_0 - jb)}{z_0 - j} \right] \exp\left[-jk\frac{x'^2 + y'^2}{2(z_0 - jb)}\right] \quad (8) \]

from to Eq. (1). Eq. (7) can be arranged as

\[ u_{s1}(x, y, z) = f(x, y, z, \alpha) \int_{x' = -h}^{h} \int_{y' = -\infty}^{\infty} \exp\left[-jk\frac{x'^2 + y'^2}{2\alpha}\right] \exp\left(jk\frac{xx' + yy'}{z}\right) \, dx' \, dy' \quad (9) \]

for \( \alpha \) is \( z_0 - jb \). \( f(x, y, z, \alpha) \) has the expression of

\[ f(x, y, z, \alpha) = \frac{j u_0}{\alpha} \exp\left[-jk\left(z + \frac{x^2 + y^2}{2z} + \alpha\right)\right] \quad (10) \]

The scattered field by the complementary wire, which is located at \( L_C = \{(x, y, z) \mid x \in (-\infty, -h) \cup (h, \infty), y = 0, z = 0\} \) can be written as

\[ u_{s2}(x, y, z) = u_F - f(x, y, z, \alpha) \int_{y' = -\infty}^{\infty} \int_{y' = -\infty}^{\infty} \exp\left[-jk\frac{x'^2 + y'^2}{2\alpha}\right] \exp\left(jk\frac{xx' + yy'}{z}\right) \, dx' \, dy' \quad (11) \]

for \( u_i \) is the incident field, given by Eq. (1). \( u_F \) can be defined by the equation of

\[ u_F(x, y, z) = f(x, y, z, \alpha) \int_{x' = -h}^{h} \int_{y' = -\infty}^{\infty} \exp\left[-jk\frac{x'^2 + y'^2}{2\alpha}\right] \exp\left(jk\frac{xx' + yy'}{z}\right) \, dx' \, dy' \quad (12) \]

\( u_{s1} \) must be equal to \( u_{s2} \) according to the Babinet’s principle. In the next section, we will evaluate directly the integrals, in Eqs. (9) and (11). The resultant fields must be equal to each other if the Babinet's principle is correct.

3. Evaluation of the scattering integrals

We will evaluate \( u_{s1} \) and \( u_{s2} \) by the aid of the relations of

\[ \int_{-\infty}^{\infty} \exp[-j\beta(Ax^2 - Bx)] \, dx = \exp\left(-\frac{\pi}{4}\right) \sqrt{\pi} \exp\left(j\beta B^2 \frac{A}{4A}\right) \quad (13) \]

\[ \int_{-\infty}^{\infty} \exp[-j\beta(Ax^2 - Bx)] \, dx = \exp\left(-\frac{\pi}{4}\right) \sqrt{\pi} \exp\left(j\beta B^2 \frac{A}{4A}\right) \quad (14) \]

\[ \int_{-\infty}^{\infty} \exp[-j\beta(Ax^2 - Bx)] \, dx = \int_{-\infty}^{\infty} \exp[-j\beta(Ax^2 - Bx)] \, dx - \int_{-\infty}^{\infty} \exp[-j\beta(Ax^2 - Bx)] \, dx \quad (15) \]
and
\[ \int_{-\infty}^{-a} \exp[-j\beta(Ax^2 - Bx)]dx = \int_{-a}^{\infty} \exp[-j\beta(Ax^2 - Bx)]dx - \int_{-\infty}^{-a} \exp[-j\beta(Ax^2 - Bx)]dx \] (16)
where \( a, A, B \) and \( \beta \) are constant numbers [19,20]. \( F[x] \) is the Fresnel function, which can be defined as
\[ F[x] = \frac{\exp(j\pi/4)}{\sqrt{\pi}} \int_{x}^{\infty} \exp(-jt^2)dt \] (17)
The equation of
\[ F[-x] = 1 - F[x] \] (18)
is valid for the Fresnel functions. Eq. (9) can be rewritten as
\[ u_{12}(x, y, z) = f(x, y, z, \alpha) \int_{x' = -h}^{h} \exp \left[ -jk \left( \frac{1}{2\alpha} x'^2 - \frac{x'}{z} \right) \right] dx' \]
\[ \int_{y' = -\infty}^{\infty} \exp \left[ -jk \left( \frac{1}{2\alpha} y'^2 - \frac{y'}{z} \right) \right] dy' \] (19)
which can be evaluated as
\[ u_{12}(x, y, z) = -j\pi \sqrt{\frac{2\alpha}{k}} \exp \left( jk\frac{x^2 + y^2}{2z^2} \right) f(x, y, z, \alpha) \]
\[ \left\{ \sqrt{\frac{2\alpha}{k}} - F \left[ \frac{k}{\sqrt{2\alpha}} (h - \frac{\alpha x}{z}) \right] \right\} \] (20)
according to Eqs. (13), (14) and (18). \( u_{12} \) reads
\[ u_{12}(x, y, z) = u_{IF} - u_C \] (21)
for \( u_C \) has the expression of
\[ u_C = f(x, y, z, \alpha) \int_{y' = -\infty}^{\infty} \left( \int_{x' = -h}^{h} \exp \left[ -jk \frac{x'^2 + y'^2}{2\alpha} \right] dx' \right) dy' \]
\[ \exp \left( jk \frac{x^2 + y^2}{z} \right) \] (22)
when the relation of
\[ \int_{x' = -\infty}^{\infty} \int_{x' = -h}^{h} \int_{x' = -h}^{h} \] (23)
is taken into account. \( u_C \) can be easily evaluated as
\[ u_{IF}(x, y, z) = -j2\pi\alpha \exp \left( jk\frac{x^2 + y^2}{2z^2} \right) f(x, y, z, \alpha) \] (24)
with respect to Eq. (13). The expression of
\[ u_C = u_{IF} - u_{12} \] (25)
can be directly written according to Eq. (22). Thus \( u_{12} \) is found to be
\[ u_{12} = u_{11} \] (26)
when Eqs. (21) and (25) are considered. Eq. (26) proves that the Babinet’s principle holds for the Fraunhofer approximation for incident fields other than plane waves.

4. Conclusion

In this paper, we investigated the Fraunhofer diffraction of a Gaussian beam, incident on a finite thin wire. We also considered the scattered waves by the complementary wire. The application of the Babinet’s principle shows that the same scattered fields can be obtained by using the wire or its complementary geometry. Such a result proves the validity of the Babinet’s principle in the Fraunhofer approximation, the correctness of which was discussed by some authors [13,17]. In fact the Babinet’s principle is based on the addition and subtraction properties of the definite integrals mathematically. For this reason its correctness is an expected result.

References