The upper (lower) sign holds when approaching \( \varphi = \pi \) from the illuminated (shadowy) side. The discontinuity of \( v_B \) on the boundary of the shadow is just compensated by the discontinuity of \( v^* \) in such a way that the sum \( v = v^* + v_B \) is continuous. Further we can confirm a general result of Sommerfeld, according to which on the boundary of the shadow itself for large \( \varphi \) the relation asymptotically holds

\[
v = \frac{1}{2} v^* (\varphi > 1), \quad \varphi = \pm \pi + 2\pi n N. \tag{37}
\]

That not only for the function \( v_B \), but also for all its derivatives the discontinuities on the boundary of the shadow are compensated for by those of \( v^* \), can be confirmed by using Eq. (28). We do not want, however, to enter the details of this proof here.

Numerical estimates of \( A_2(\varphi) \) gave the result that already the second term of the series (35) can be neglected in all practical cases so that the proposed task of finding a representation of the diffraction wave, which can be used for the transition from light to shadow and at large distances, can be considered as practically solved by the principal term of our series, as given in (35).

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**On the Anomalous Propagation of Phase in the Focus**

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(Received October 11, 1938)

The problem of anomalous phase propagation of a spherical wave at the focus has been discussed for the case of a diffracting aperture of arbitrary shape. The solution given by Kirchhoff's integral has been split up into an "incident light wave," which shows the distribution of light to be expected according to geometrical optics and a "diffracted wave," which may be thought of as due to scattering of the incident wave at the diffracting edge. A sudden change of phase by \( \pi \) has been shown to occur already in the incident wave. Thus we may, in this sense, consider this phenomenon as a geometric optical one.

The case of a circular diffracting aperture, the focus lying on the normal through its center, which has been treated usually, appears to be not very suitable for an experimental investigation of the discussed phenomenon. It is this particular shape of the diffracting edge, which produces diffraction phenomena of considerable light intensity along the optical axis. These, however, are not because of the existence of a focus, but only because of the particular shape of the diffracting aperture.

**I. Introduction**

Sommerfeld's first great scientific achievement was the solution of the problem of diffraction at a perfectly conducting half-plane by the methods of exact analysis. On the occasion of his jubilee it may thus be appropriate to present a note, which deals with a related problem and is based on a paper I wrote while staying at his institute at Munich. The problem in question is the anomalous phase propagation of a spherical wave at the focus, which was dis-

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2. A. Rubinowicz, Ann. d. Physik 53, 257 (1917) and 73, 339 (1924), to be referred to as I and II in the text.
3. For extensive literature see F. Reiche, Ann. d. Physik 29, 65, 401 (1909) and J. Picht, Optische Abbildung (Braunschweig, 1931).
4. An elementary suggestive representation by means of Fresnel's zones has, however, been given by A. D. Fokker, Physica 3, 334 (1923); 4, 166 (1924).
and, besides, generalizing the treatment so as to apply to an arbitrary diffracting aperture.

We start from Kirchhoff's theory of diffraction but generalize the treatment so as to apply it to an arbitrary diffracting aperture.

At all those points where light should be present according to the laws of geometrical optics, the incident wave \( u_I \) is given by the solution of \( \Delta u + k^2 u = 0 \) for direct propagation of light, that is, if the time factor is \( e^{-i\omega t}, e^{i\omega t} R \) for a point source; \( R \) is the distance from the source. It is supposed to vanish everywhere else. The wave function \( u_I \) will thus decrease discontinuously by \( e^{\alpha R}/R \) while passing from the "light cone" across the "boundary of the shadow" into the "shadow" (Fig. 1).

Concerning the shape of the geometrical boundary of the shadow we note that it is given by the surface of the cone \( K \), which is generated by the lines joining the source of light \( L \) and the points of the diffracting edge \( D \). The part of the cone between \( L \) and \( D \) has to be regarded as cut off.

The diffraction wave \( u_D \) can be considered as due to scattering of the incident light wave \( u_I \) at the different points of the diffracting edge \( D \); it is composed of the secondary elementary waves produced there. Since we expect \( u = u_I + u_D \) taken as a whole to be continuous on the boundary of the shadow, \( u_D \) must be discontinuous in such a way as to compensate the discontinuity of \( u_I \). It ought thus to increase suddenly by \( e^{i\alpha R}/R \) at the passage from the light cone into the shadow.

Let us suppose that the diffraction of a convergent spherical wave may be treated just like that of a divergent spherical wave considered above by representing the corresponding wave function \( u \) as a sum of \( u_I \) and \( u_D \). Before determining these functions we have again to ascertain the boundary of the geometrical shadow. Obviously this will be formed by the double cone \( K_1 + K_2 \), generated, as above, by joining its vertex, now the focus \( F \), with \( D \) (Fig. 2), which simultaneously cuts off the remaining part of the cone \( K_1 \).

Let us represent the incident light wave by \( u_I = e^{-\alpha R}/R \) (\( R \) now denoting the distance from the focus \( F \)) in that portion of space which is bounded by the cone \( K_1 \) and the screen \( S \) and contains also the interior of the cone \( K_1 \). The minus sign had to be put into the exponent because the wave is now convergent; the time factor \( e^{i\omega t} \) is the same as above. When passing through the cone \( K_1 \) into the geometrical shadow \( u_I \), it is again supposed to disappear.

Let us now further assume that a diffracted wave \( u_D \) is produced by scattering of the incident wave \( u_I \) at the diffracting edge \( D \). If \( u_I + u_D \) is to be continuous at the boundary of the shadow, \( u_D \) ought to increase suddenly by \( e^{i\alpha R}/R \) when passing through the cone \( K_1 \) into the geometrical shadow. Now let us still make the plausible assumption that the elementary wavelets scattered at the different points of the diffracting edge, and with them also the diffracted wave \( u_D \) consti---
tuted by them, do not show any peculiar irregularity in their behavior while passing through the focus \( F \). Then the discontinuity of \( u_0 \) ought to be continued regularly on \( K_2 \). Since \( R \) is always positive according to our notation, a sudden increase of \( u_0 \) by \( e^{ikR}/R \) on the cone \( K_1 \) will correspond to that by \( e^{i\alpha R}/R \) on \( K_2 \). The direction of the passages through the surfaces is that indicated by the arrows \( A \) and \( B \) in Fig. 2. Otherwise the diffracted wave would be discontinuous at the focus \( F \) and travel along the cone \( K_2 \) towards the focus, that is, in a direction opposite to that in which it should go.

And now to our decisive remark: Outside the double cone \( K_1 + K_3 \), that is, within the geometrical shadow, we have \( u = u_0 \), since for a passage through \( K_1 \) out of the light cone, like that shown by the arrow \( A \), the discontinuity of \( u_1 \), is just compensated by that of \( u_0 \). Now, \( u_0 \) increases discontinuously at the passage through \( K_2 \) indicated by arrow \( B \). To compensate this discontinuity on \( K_2 \) we shall have to put \( u = u_0 - e^{i\alpha R}/R \) inside \( K_2 \). Hence we see that the incident light must be represented by \( u_{1a} = e^{i\alpha R}/R \) inside the cone \( K_2 \). The minus sign of \( u_{1a} \) indicates that \( u_{1a} \) and \( u_{1b} \) differ in phase by \( \pi \). We may thus summarize our results in the statement: The incident light wave is represented by a convergent spherical wave inside \( K_1 \) and by a divergent spherical wave inside \( K_2 \), which differ in phase by half a period.

Since, in general, the diffraction wave vanishes in the limiting case of infinitely small wave-lengths, i.e., for \( k \to \infty \), while the incident wave still persists, the phase anomaly will have to be considered as a phenomenon, which has to be taken into account in the geometric optical treatment of the problem as a sudden change in phase by half a period. It must, however, be said that this statement cannot apply at all those points of space, where, independently of the value of \( k \), the intensity of the diffracted wave is of the same order of magnitude as that of the incident light wave. This is only possible at points, where the secondary waves from at least some finite portion of \( D \) arrive with all their phases equal. In general the focus \( F \) will be such a point, but in some particular cases still other points of this kind exist. In the case exclusively considered hitherto, for instance, where the diffracting aperture is circular and the focus lies on the normal through its center, all the points on the optical axis thus formed will be of that kind.

In the following paragraphs a mathematical formulation and some extension of the propositions made will be given.

II. INCIDENT AND DIFFRACTED WAVE

The distribution of light, the case of diffraction of a convergent spherical wave, at those points of space which lie in front of the screen \( S \) and the surface, extended over the diffracting edge \( D \), say \( f \), is given by Kirchhoff's integral:

\[
\frac{1}{4\pi} \int_{D} \left( \frac{e^{-ikr}}{\rho} \frac{\partial e^{ikr}}{\partial n} - \frac{e^{ikr}}{r} \frac{\partial e^{-ikr}}{\partial n} \right) \, df, \tag{1}
\]

where \( \rho \) is the distance of the surface element \( df \) from the focus \( F \), \( r \) the distance of \( df \) from the point \( P \), at which we calculate \( u_{K} \) and \( n \) the normal of \( f \) in the direction away from the focus \( F \). No restricting assumptions will be made at present concerning the shape of the diffracting edge \( D \).

To transform this expression we shall first represent the incident light wave, say \( u_1 \), by means of surface integrals. Making use of the same notation as in Section I, we put

\[
u_1 = u_{1a} + u_{1b}, \tag{2a}
\]

where \( u_{1a} \) and \( u_{1b} \) are defined as follows:

\[
u_{1a} = e^{i\alpha R}/R \text{ inside the region bounded by } K_1 \text{ and } f, \tag{2b}
\]

\[= 0 \text{ at all other points.}
\]

\[
u_{1b} = -e^{i\alpha R}/R \text{ inside the cone } K_2, \tag{2c}
\]

\[= 0 \text{ at all other points.}
\]

To obtain integral representations for (2b) and (2c) we make use of the following well-known theorem: Let \( u^* \) be a regular solution of \( \Delta u + k^2 u = 0 \) inside the region enclosed by the surface \( \Phi \). Then the integral

\[
\frac{1}{4\pi} \int_{\Phi} \left( \frac{\partial u^*}{\partial n} - \frac{e^{ikr}}{r} \frac{\partial u^*}{\partial n} \right) \, df \tag{3}
\]

represents the function \( u^* \) at the point \( P \) (whose
distance from $df$ is denoted by $r$ if $P$ lies inside
Φ, and vanishes for $P$ outside $Φ$. $n$ is the external
normal of $Φ$.

Using (3) we get immediately

$$u_1 = \frac{1}{4\pi} \int_{f+K_1+u_1} \left( \frac{e^{-ik\rho}}{\rho} \frac{\partial}{\partial n} \frac{e^{ikr}}{r} \right) df,$$

(4a)

$$u_2 = \frac{1}{4\pi} \int_{K_2+u_2} \left( \frac{e^{ik\rho}}{\rho} \frac{\partial}{\partial n} \frac{e^{ikr}}{r} \right) df,$$

(4b)

where $Ω_1$ and $Ω_2$ are spherical surfaces round $F$ as its
center. They had to be introduced on account of the singularity of $e^{±ik\rho}/\rho$ at $F$, which is thus cut off.

Concerning the integral representations (4a, b) we have to make the following remarks:

First the integral over $f$ in (4a) is identical with
$u_K$.

Secondly the integrals over $K_1$ and $K_2$ in
(4a, b), which we shall denote by $-u_{D_1}$ and $-u_{D_2}$, yield the expressions

$$u_{D_1} = \frac{1}{4\pi} \int_{Ω_1} \frac{e^{ik(r+\rho)}}{\rho} \left( \frac{ik}{r} - \frac{1}{r^2} \right) \cos(n,r) df,$$

(5a)

$$u_{D_2} = \frac{1}{4\pi} \int_{Ω_2} \frac{e^{ik(r+\rho)}}{\rho} \left( \frac{ik}{r} - \frac{1}{r^2} \right) \cos(n,r) df,$$

(5b)

since $(\partial/\partial n)(e^{±ik\rho}/\rho)$ vanishes on $K_1$ and $K_2$ and
further

$$\frac{\partial}{\partial n} \frac{e^{ikr}}{r} = e^{ikr} \left( \frac{ik}{r} - \frac{1}{r^2} \right) \cos(n,r).$$

$n$ is here the external normal of $K_1$ or $K_2$. Its direction is reversed as we pass along a generator of our conic focus $F$.

Finally we note that the two integrals over $Ω_1$ and $Ω_2$ yield $-r_0 + e^{ik\rho}/4\pi K$, where $\omega$ is the solid angle subtended by these surfaces.

Now, on the basis of these remarks and Eq.
(2a) we obtain adding (4a) and (4b) :

$$u_K = u_1 + u_{D_1} + u_{D_2},$$

where the sum $u_{D_1} + u_{D_2}$ can be reduced to a line
integral over the diffraction edge $D$. To perform this transformation we shall introduce on our
double cone $K_1 + K_2$, an orthogonal coordinate
system, given by the distance $\xi$ from the focus $F$ and
the lines of intersection $\sigma$ of $\xi = \text{const.}$ with $K_1$
and $K_2$. The sign of $\xi$ may be chosen so that
$\xi = -\rho$ on $K_1$ and $\xi = +\rho$ on $K_2$.

To obtain a suitable expression for $df$ we consider the area subtended on our cone $K_1 + K_2$ by an element $ds$ of the diffracting edge $D$. Let us denote by $ρ_s$ the distance of $ds$ from the focus $F$.

Then the line element of $ds$ (Fig. 3) on our area is given by $ds = (ξ/ρ_s) \sin(ρ_s, ds)$ and therefore
$$df = dξdσ, \quad dy = (ξ/ρ_s) \sin(ρ_s, ds)dξds.$$ The sign of $df$ represented in this way will be positive on $K_2$, and negative on $K_1$, while it is always positive in the integrals (5a, b) for $u_{D_1}$ and $u_{D_2}$.

But in these integrals also $\cos(n, r)$ changes its
sign at the focus $F$, where, as we have said, the direction of $n$ is suddenly reversed. Choosing now that direction of $n$, which coincides with its former direction on $K_1$, opposite to it on $K_2$, we shall have to put, with these new assumptions concerning $n$

$$-\cos(n, r)(ξ/ρ_s) \sin(ρ_s, ds)dξds$$

instead of $\cos(n, r)df$ in our integrals. Considering further that now

$$\cos(n, r)(r_s/r) \cos(n, r_s),$$

where $r_s$ is the distance of the point $P$ from $ds$, we obtain the expression

$$u_D = -\frac{1}{4\pi} \int_{Ω_1} ds \sin(ρ_s, ds) \cos(n, r_s),$$

$$\times r_s \int_{ρ_s}^{r_0} dξ e^{ik(ξ+ρ)} \left( \frac{ik}{r^2} - \frac{1}{r^2} \right)$$

for the sum $u_D = u_1 + u_{D_1}$. Since now

$$r_s^2 = r_0^2 + (ξ + ρ_s)^2 - 2r_0(ρ_s + ξ) \cos(ρ_s, ρ_0)$$

we get finally, on following the method given in
(1) (p. 261)

$$u_D = \frac{1}{4\pi} \int_{Ω_1} \frac{e^{-ikξ}}{ρ_s} \cos(n, r_s)$$

$$\times \sin(ρ_s, ds)ds.\quad (6)$$

Kirchhoff's diffraction integral may thus be given the form

$$u_K = u_1 + u_D,\quad (7)$$
where \( u_I \) represents a wave of the kind to be expected according to geometrical optics and \( u_D \) a wave, which may be thought of as due to scattering of the incident wave by the diffracting edge. The secondary waves, which originate at the different elements \( ds \) of the edge, are seen to be asymmetric on account of the direction factor

\[
\frac{\cos (n, r_s)}{1 - \cos (r_s, \rho_s)}. \tag{8}
\]

This becomes infinite on that generator of the double cone \( K_1 + K_2 \), which belongs to the element \( ds \), since there \( \cos (r_s, \rho_s) = 1 \).

III. APPROXIMATE REPRESENTATION OF THE DIFFRACTION WAVE

If we want to discuss the distribution of light for the particular case of diffraction of a convergent spherical wave, we have to resort to appropriate approximate formulae for the diffraction wave \( u_D \) (or for the complete wave function \( u_K \)). For all those points of space, at which the method of stationary phase can be applied to the integral (6) either directly or indirectly, such formulae are readily obtained in the same way as for a divergent spherical wave (cf. II). The necessary condition for this is that the phase factor \( k(r_s - \rho_s) \) varied rapidly enough as we pass along the diffracting edge \( D \). Then only the immediate vicinity of those points \( P_s \), where the relatively slowest variation of phase occurs, that is, where

\[
d(r_s - \rho_s)/ds = 0 \tag{9}
\]

or

\[
\cos (r_s, ds) = \cos (\rho_s, ds), \tag{9a}
\]

will yield a noticeable contribution to the integral (6).

Hence it appears that the light scattered by a particular element \( ds \) of the edge is spread over a circular cone, whose vertex lies in that element. There is also the line joining the element \( ds \) and the focus \( F \) among the generators of this cone. Its direction coincides with that of the light ray of the convergent spherical wave, incident upon \( ds \). It is in such half-cones that light is reflected in the points of a reflecting curve, e.g., a wire, according to the concepts of geometrical optics. We may thus call this half-cone the reflection cone.

If we want to apply directly the method of stationary phase to the integral (6), we have to assume that apart from the factor \( \exp [ik(r_s - \rho_s)] \) the integrand varies so slowly that it may be taken out of the integral. This assumption is, however, not fulfilled in the points near the boundary of the shadow, where the direction factor (8) becomes infinite. Therefore we cannot apply this method directly at those very points in which we are most interested, at the treatment of diffraction problems. But fortunately there is an indirect way of deriving approximate formulae, which can be applied even at points in the boundary of the shadow itself. By following the method applied in II, (p. 352) we obtain for the contribution \( u_D |_s \), of the effective region round the point \( P_s \) to \( u_D \) the expression

\[
u_D |_s = \frac{1}{2i r_p (\xi)^{1}} \frac{(R + r_s - \rho_s)}{1 - \cos (\rho_s, \rho_s)} \cos (n, r_s)
\]

\[
\times \sin (\rho_s, ds) e^{i(-kB + 3\pi/4)}
\]

\[
\times \int_{-\infty}^{\infty} e^{i(\pi/2)\xi^3} d\xi, \tag{10}
\]

if the points considered lie inside the cone \( K_1 \),

\(^5\) The frequent appearance of Fresnel's integrals in Kirchhoff's theory of diffraction is due to the fact that they occur in all those cases where the incident light is reflected by the diffracting edge.
Here we have put
\[
\tilde{\xi}'' = \frac{d^2}{ds^2} (r_s - \rho_s) = \sin^2 (\rho_s, ds) \left( \frac{1}{r_s} - \frac{1}{\rho_s} \right) \\
+ \left( \cos (\rho_s, H) - \cos (\rho_s, H) \right) \frac{1}{H},
\]
(11)

where \( H \) is the radius of curvature of the diffracting edge in the point \( P_s \), taken as a vector in the direction of the principal normal at \( P_s \).

If we want to calculate \( u_D \) by means of (10) at some point of space, we have first to ascertain from which points \( P_s \) of the diffracting edge the incident light is reflected in cones, to the point of space considered. \( u_D \) will then be given as the sum of the respective \( u_D|_{r_s} \). For a circular diffracting aperture, the focus \( F \) lying on the normal through its center, formula (10) may be found in a paper by Schwarzschild.\(^4\) Then only two points \( P_s \) need be considered and \( u_K \) consists simply of the incident wave and the waves, reflected at these two points.

For the derivation of (10) it has been supposed that in the vicinity of \( P_n \), the phase may be represented with sufficient accuracy by
\[
(r_s - \rho_s) = (r_s - \rho_s) + \frac{d^2}{ds^2} (r_s - \rho_s) (s - s_s) \frac{1}{2} \\
= (r_s - \rho_s) + \frac{1}{2} \tilde{\xi}'' (s - s_s)^2
\]
in which use is made of (9). This approximation fails, however, for \( \tilde{\xi}'' \) vanishes on the reflection cone, as appears already, from the occurrence of \( (\tilde{\xi}'')^2 \) in the denominator of (10). The points \( \tilde{\xi}'' = 0 \) are distinguished as forming the intersection lines of two “infinitely neighboring” reflection cones. The images of the incident convergent light wave at its reflection from a definite element \( ds \) of the diffracting edge lie on these lines, which may thus be referred to as focal lines. They separate the reflection cone into two domains, one of positive and one of negative \( \tilde{\xi}'' \). It has, however, been supposed that \( \tilde{\xi}'' > 0 \) at the derivation of (10). Now, on investigating the case \( \tilde{\xi}'' < 0 \), we find that there is a change of phase by \( \pi/2 \) occurring at the passage across the focal line \( \tilde{\xi}'' = 0 \). The phase anomaly is thus shown, not only by the incident, but also, eventually, so to say on a small scale, by the diffracted wave.\(^8\)

The method of stationary phase fails near the focus \( F \). In the focus itself, where \( r_s = \rho_s \), all the wavelets originating at different points of the diffracting edge arrive with equal phase. Also near the focus their phases will differ little, so that it will be necessary to consider the diffracting edge as a whole and not only some small effective regions. A general representation like that possible at those points of space, where the method of stationary phase applies, does no longer seem to be feasible. Besides, it will be advisable to use \( u_K \) and not \( u_D \) near \( F \), where the latter becomes infinite according to (2) and (7).

We have still to say a few words about the special case of a circular diffracting aperture, the focus of the convergent spherical wave lying on the normal through the center of the circle. For every point on the optical axis thus formed not only the phase \( r_s - \rho_s \), but the whole integrand in (6) will be constant. Then the amplitude of \( u_D \) does no longer depend upon the wave-length and will be comparable with that of the incident light.

These peculiar conditions on the optical axis are due, however, to the particular shape of the diffracting edge and not to the existence of a focus. This large amplitude of \( u_D \) is also responsible for the fact that points on the axis are favored at all observations. The large focal length and the smallness of the aperture used at most observations will add to this peculiarity, since the phase variations of the elementary wavelets for a point at a given distance from the axis decrease with the distance from and the smallness of the diffracting aperture. These two circumstances favor an extension of the conditions on the axis into the neighboring portions of space.

\(^4\) (10) enables us to treat approximately diffraction—though only if it is an extrafocal one—by more than one diaphragm, a problem rather difficult to solve by Fresnel-Kirchhoff’s original methods.


\(^8\) This anomaly of phase on a small scale occurs, of course, also at the diffraction of a divergent spherical wave, which has been treated in II.