we have an ideal situation, wherein both spatial and temporal coherence properties of the gas laser and its inherent high illumination are called upon to advantage.

In the schemes discussed, the effect of increasing the size of the source can be simulated by use of a gas laser. The light from the gas laser is focused by a microscope objective on a ground glass rotating in its own plane (Fig. 6). When the focus coincides with the plane of the ground glass, the effective size of the source is determined by the average size of the grain on the ground glass. Thus high-contrast fringes can be obtained over a considerable width of the beam. To increase the effective size of the source, the microscope objective may be thrown slightly out of focus; then, however, the width over which high-contrast fringes are obtained is very much reduced.

CONCLUSIONS

The contrast of the interference fringes obtained with a Twyman–Green interferometer is determined by the spatial coherence of the illuminating light source if one of the plane mirrors is replaced by a right-angle prism, cube-corner prism, or a lens plane-mirror combination. This procedure is useful for demonstrating the spatial coherence properties of a source as a function of its angular extension. These properties are demonstrated much more easily by the use of a gas laser source. By use of a ground glass rotating in its own plane and an out of focus image thrown on it, the degree of spatial coherence can be controlled at will.

Total Reflection: A New Evaluation of the Goos–Hänchen Shift

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Several different expressions for the Goos–Hänchen shift have been given in the literature. By looking at the problem from the point of view of the conservation of energy, new and more accurate expressions can be derived. For light waves, classical optics is used; for waves associated with particles, a quantum-mechanical approach is used. A comparison is made of the results of the present work with the earlier theories.

1. INTRODUCTION

It is a widely held opinion that total reflection of light is only a special case of partial reflection. Various experiments1 have proved the existence of a wave in the medium of lower index of refraction (usually air). The theory confirms this experimental fact. The intensity of this wave, measured along the normal to the boundary surface, decreases very rapidly and practically disappears within a distance of a few wavelengths. For this reason, the wave is undetected in most experiments.

In 1943, Goos and Hänchen2 devised an experiment to show what happens at total reflection if the incident wave penetrates into the medium of lower index and re-emerges into the medium of higher index (see Fig. 1). The reflected beam \( R_1 \) is the one expected if total reflection behaves as a special case of partial reflection. The Goos and Hänchen experiment was devised assuming the behavior shown for reflected beam \( R_{II} \). The shift \( d \) was expected and was observed by these authors. In fact, they not only succeeded in demonstrating the phenomenon, but also were able, with the help of a clever experimental trick, to measure the shift quantitatively.2,3
Soon after Goos and Hänchen had finished their first work, Artmann proposed a theory of the phenomenon. Starting from the Fresnel–Maxwell equations, he considered only the mathematical expression for the incident and totally reflected beams. From the difference of phase between these two beams, he was able to account for the observed shift. With this theory he predicted two expressions for the shift, one for polarization of the light parallel to the plane of incidence, the other for perpendicular polarization. Goos and Hänchen made new measurements and confirmed the fact that there was a dependence on the polarization of light. Several months later, Fragstein published another theory based on the very important work of Picht and Schaefer. By making the approximations contained in the theory of Schaefer and Pich, Fragstein found the same two expressions for the shift as had Artmann.

According to Fragstein, the wave behaves qualitatively as indicated by Goos and Hänchen, or more precisely, as indicated by Schaefer and Pich. For many reasons, experimental measurements of high accuracy are very difficult. Therefore, both Fragstein and Artmann have undertaken rather intricate calculations to evaluate the experimental side effects and have engaged in a controversy about them in several papers. However, neither of these authors admits explicitly that his expressions for the shift are approximate and are valid only for angles very near the critical angle of total reflection, as will be shown later.

Several other authors mention the Goos–Hänchen effect as a proof that the wave travels, in part, in the medium of lower refractive index. On this basis, a new evaluation of the shift can be developed without making any of the previous approximations.

In this paper we assume that the cause of the shift is that for a finite incident plane wave suffering total reflection, on the average some energy enters into the medium of lower index on one side of the beam and comes back into the medium of higher index on the other side of the beam. This surprising concept is due to Picht. The Goos–Hänchen shift is identified with a translation of the beam and the principle of conservation of energy is used to establish the quantitative expression for the shift. On this basis two expressions, slightly different from those previously proposed, are obtained. The three theories agree exactly when the angle of incidence is the critical angle.

Using quantum mechanics, Hora carried out the treatment of Artmann for a beam of particles totally reflected at a potential barrier. He obtained an expression for the shift with, indeed, the same approximation. It is likewise possible in this case to carry out the more precise new evaluation.

2. EVALUATION OF THE GOOS–HÄNCHEN SHIFT FOR A BEAM OF LIGHT

Let us consider first an infinite incident plane wave, which impinges on the surface separating two media, with respective indices of refraction \( n_1 \) and \( n_2 (n_1 > n_2) \). In the medium of index \( n_1 \) the angle of incidence of the wave is assumed to be greater than the critical angle, so that the condition of total reflection is fulfilled. In this case a rigorous treatment of the Fresnel–Maxwell equations shows that there is a so-called surface wave (or evanescent wave) in the medium of index \( n_2 \). The time-averaged flux of energy carried by this surface wave across a plane parallel to the plane separating the two media can be shown to be zero. In many textbooks it is concluded from this that the surface wave is a standing wave. Thus, once the surface wave is established, it should exist without dissipating any energy, as long as the infinite incident parallel beam of light continues. However, it is easy to show that the time-averaged flux of energy carried by the surface wave, across a plane perpendicular both to the plane separating the two media and to the plane of incidence, is different from zero. Thus there arises the apparent contradiction that the intensity of the reflected beam is the same as the intensity of the incident beam and yet there is, in the medium of lower index, a flux of energy parallel to the surface of separation. In the central region of the wave, with either the Picht or with the Schaefer and Pich conception, a first approximation,
the problem is that of an infinite plane wave. In such a case, Arzelies\(^7\) has shown how to compute the energy flux. We will also assume that the over-all phenomenon of transferring some energy from one side of the beam to the other side results in a translation. The process just described must not be confused with the instantaneous exchange of energy between the two media, which occurs continually at each point of the surface of separation, which has a time-average of zero, as is shown by Arzelies\(^7\) and by Born and Wolf.\(^8\) The explanation and mathematical treatment of the transfer of energy from one medium to the other on the two sides of the beam will not be presented in this paper. The reader may refer to Picht\(^5\)\(^-\)\(^8\) and Schaefer and Pich.\(^9\)

The postulation of translation implies that the profile of the totally reflected beam is the same as the profile of the incident beam. Remembering that the incident wave behaves in its central part, in first approximation, as a perfect plane wave, we can see that the energy transferred on the left of the beam (see Fig. 2) from the medium \(1\) to the medium \(2\) (of lower index, \(\sin^2\theta = K_2/\mu_2\)), is the energy needed in the central part of the beam to establish the surface wave in medium \(2\) corresponding to the plane wave in medium \(1\) (of higher index, \(\sin^2\theta = K_1/\mu_1\)). This surface wave is not a standing wave, as is often stated. It travels in a direction parallel to the plane separating the two media. In the case presented in Fig. 2, the surface wave travels from left to right. Now, as explained by Picht, on the right the energy flows back from medium \(2\) to medium \(1\). This produces the Goos–Hänchen shift, though Picht did not explicitly predict the shift.

Then from the preceding statement about translation and from the conservation of energy, the time-average flux of energy \(\Phi_1\) for the plane wave across a strip whose width is the Goos–Hänchen shift must be equal to the time-average flux of energy \(\Phi_2\) in the entire medium of lower index, parallel to the plane of separation (see Fig. 2). This statement is obvious when the right side of the reflected plane wave in Fig. 2 is considered. If there were no shift, the right side would be \(Ax\). Because of the shift it is \(A'x\), but the energy of the plane wave, between \(Ax\) and \(A'x\), can only be the energy of the surface wave coming back into the medium of higher index. This provides a straightforward way of evaluating the shift.

To find the mathematical expression for the shift, the equations of Arzelies\(^7\) for the fluxes will be used. We shall not reproduce all the rather lengthy but straightforward computations of Arzelies. See Ref. 16 (which will be denoted RV in the text) and Ref. 17 (denoted OP). In the following expressions, all the properties of the medium of higher refractive index are characterized by the subscript \(1\), and of the medium of lower index by subscript \(2\). When there is no subscript, the ratio of the value in medium \(2\) to the value in medium \(1\) is meant. For example, \(n = n_2/n_1\), \(K = K_2/K_1\), etc. The \(K\)'s are the dielectric constants; the \(\mu\)'s are the magnetic permeabilities; \(M\) stands for the magnitude of the magnetic vector, \(E\) for the magnitude of the electric vector, and \(D\) for the magnitude of the displacement vector.

**Case a. The Electric Vector of the Incident Finite Plane Wave is Perpendicular to the Plane of Incidence**

The magnitude of the magnetic vector of the surface wave at the level of the plane of separation between the two media is (RV, p. 28):

\[
M_x^2 = \frac{64\pi D_2^2 \cos^2i(2\sin^2i - K_\mu)}{K_{1\mu}}. 
\]

(1)

Here and hereafter, the magnetic vector itself is meant, and not one of its components. But for the incident wave, we have the relation (RV, p. 9)

\[
M_1 = [4\pi/(K_{1\mu})^4]D_1. 
\]

(2)

The time-average flux of energy across a plane surface perpendicular to \(Ox\), of width \(L\) in the direction \(Oy\), indefinite in the direction \(Os\), and not defined for \(Os<0\) (see Fig. 3) is (OP, p. 218)

\[
\Phi_2 = \frac{L}{32\pi^2K_2} \left( \frac{1 - \gamma_2^2}{1 + \gamma_2^2} \right) M_2^2. 
\]

(3)

where \(\gamma_2\) is the damping factor of the surface wave.

In Arzelies' paper, \(\lambda_2\) is used instead of \(\lambda_2\), but his convention is \(\lambda_0 = \lambda_{\text{vaccum}}/n_0\), which is defined as \(\lambda_2\) in this paper.

---

Using the Poynting vector, we find for the time-
average flux of energy of the plane wave, across a surface of
area \( Ld \) perpendicular to the direction of propagation of
that wave, the expression

\[
\Phi_1 = \left( \frac{Ld}{8\pi} \right) (\mu_1/K_1)^4 M_i^2,
\]

where \( d \) is the Goos–Hänchen shift. To evaluate \( d \) we
write \( \Phi_1 = \Phi_2 \). Taking account of the following relations
(RV, pp. 26–28)

\[
\gamma^2 = 1 - n^2 / \sin^2 i,
\]
\[
\sin^2 i > n^2 = K_\mu,
\]
\[
V_1 = 1/(K_\mu_1)^4, \quad V_2 = 1/(K_\mu_2)^4,
\]

where \( V_i \) is the velocity of light in the medium \( i \), we find

\[
d = d_1 = \frac{1}{\pi} \frac{\mu \sin i \cos^2 i}{\cos^2 i + \sin^2 i - n^2} \frac{\lambda_1}{(\sin^2 i - n^2)^4},
\]

(5)

However, if we write with Goos and Hänchen \( \lambda_1 = \lambda_\text{neum}\/n_1 \), we have \( \lambda_2 = \lambda_1 / n \). Thus in terms of the
Goos and Hänchen conventions

\[
d_1 = \frac{1}{\pi} \frac{\mu \sin i \cos^2 i}{\cos^2 i + \sin^2 i - n^2} \frac{\lambda_1}{(\sin^2 i - n^2)^4},
\]

(6)

Now for most glasses \( \mu = \mu_2/\mu_1 = 1 \); and furthermore,
the observation of the shift is easy only near the critical
angle of total reflection, so that \( \sin^2 i \approx n \). Thus (6) re-
duces to

\[
d_1 \approx (\sin i / \pi) \left[ \lambda_1 / (\sin^2 i - n^2)^4 \right],
\]

(7)

or, because \( \sin^2 i = n_2 / n_1 \),

\[
d_1 \approx (1/\pi n_1) \left[ n_2 \lambda_1 / (\sin^2 i - n^2)^4 \right].
\]

(8)

Equation (7) is the “exact” expression for the shift
given by Artmann and by Fragstein, and Eq. (8) is the
corresponding approximation valid when the value
of the angle of incidence \( i \) is greater than, but close to,
the critical angle of total reflection.

Case b. The Electric Vector of the Incident Plane
Wave is Parallel to the Plane of Incidence

As shown by Arzelies and others, we need only
permute \( \mu \) and \( K \) to treat Case b when we know the
result for Case a.

Thus, using the Goos and Hänchen conventions, we find

\[
d_{11} = \frac{1}{\pi} \frac{K \sin i \cos^2 i}{K^2 \cos^2 i + \sin^2 i - n^2} \frac{\lambda_1}{(\sin^2 i - n^2)^4},
\]

(9)

But \( n^2 = K_\mu \), and for glasses \( n^2 = K \). If we assume, as
before, \( \sin^2 i \approx n \), then we obtain

\[
d_{11} \approx (\sin i / \pi n^2) \left[ \lambda_1 / (\sin^2 i - n^2)^4 \right],
\]

(10)

or

\[
d_{11} \approx (1/n^2)(1/\pi n_1) \left[ n_2 \lambda_1 / (\sin^2 i - n^2)^4 \right].
\]

(11)

Equation (10) is the “exact” expression for the shift
given by Artmann and Fragstein, and Eq. (11) is the
corresponding approximation, valid under the same
conditions as those of Eq. (8).

Comparison Between the Artmann–Fragstein
Expressions and the Present Results

We see at once that, for an angle of incidence \( i \) equal
to \( \pi / 2 \), the Artmann–Fragstein Eqs. (7) and (10) give
values for the shift which are, respectively,

\[
d_1 = \pi^{-1} \left[ \lambda_1 / (1 - n^2)^4 \right]
\]

and

\[
d_{11} = (\pi n^2)^{-1} \left[ \lambda_1 / (1 - n^2)^4 \right].
\]

These results are obviously unrealistic, because in
this case the incident and reflected beams are colinear
and both \( d_1 \) and \( d_{11} \) must be equal to zero. The new
expressions (6) and (9) have the proper behavior as
\( i \) tends towards \( \pi / 2 \).

At first sight the Artmann derivation seems strictly
correct; it is thus surprising that it fails as \( i \) tends
towards \( \pi / 2 \) and that the new evaluation leads to a
different expression.

In their work, Goos and Hänchen assumed the
empirical relation

\[
d_{(1,1)} = k_{(1,1)} n_2 \left[ \lambda_1 / (\sin^2 i - n^2)^4 \right],
\]

(12)

and from their measurements they computed \( k_1 \) and
\( k_{11} \). On the other hand, we find

\[
k_1 = \frac{1}{\pi n_1}, \quad k_{11} = \frac{1}{\pi n_2} = \frac{1}{\pi n_1}, \quad \frac{n^2}{n_1 n_2}
\]

from the simplified Artmann–Fragstein expressions;

\[
k_1 = \frac{1}{\pi n_2}, \quad k_{11} = \frac{1}{\pi n_2} = \frac{1}{\pi n_1}, \quad \frac{n^2}{n_1 n_2}
\]

from the “exact” Artmann–Fragstein expressions; and

\[
k_1 = \frac{1}{\pi n^2}, \quad k_{11} = \frac{1}{\pi n^2} = \frac{1}{\pi n_1}, \quad \frac{n^2}{n_1 n_2}
\]

from the present treatment, taking \( \mu = 1 \) and \( K = n^2 \).

In the experiments performed with polarized light, a
flint glass was used with \( n_1 = 1.762 \) for the mercury
line (546 m\( \mu \)), and the second medium was air
(\( n_2 = 1.000 \)). On the basis of these values, the theoretical
values have the right order of magnitude.

However, if the accuracy claimed by Goos and Hänchen
is reliable, there is a definite discrepancy between experiment and the theories. Without challenging the experimental results, which are very good considering the order of magnitude of the actual shift—only a few microns—some remarks can be made about the accuracy of the results about a smooth curve is greater than 10% compared to the ideal optics assumed in the theoretical treatment because of the difficulty of removing contaminant films and of getting a good optical homogeneity, a perfect surface polish and flatness, and parallelism of the plates used.

| $i - i_c$ | Goos and Hänchen ($\pm 10\%$) | Artmann-Fragstein approximate expression | Artmann-Fragstein complete expression | New theory with $\mu = 1$, $K = n^2$
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It would be interesting to perform some new experiments with a laser. Collimation difficulties would be surmounted and the wavelength sharpness increased. Large values of $i - i_c$ should then be considered, as well as small ones.

3. QUANTUM MECHANICAL TREATMENT OF THE GOOS-HÄNCHEN SHIFT FOR A BEAM OF PARTICLES

In 1960, Hora\textsuperscript{14} derived an expression for the Goos-Hänchen shift suffered by a beam of particles. He used quantum mechanics and followed a method equivalent to the Artmann treatment in classical optics. It is possible to extend the same reasoning used above to the quantum-mechanical treatment and to obtain thereby a new expression for the shift. Figure 4 shows the two sets of coordinate axes which will be used.

The time-dependent Schrödinger equation for the problem is

$$-(\hbar^2/2m)\nabla^2\psi + \psi = i\hbar(\partial\psi/\partial t),$$

where $v = V$ in region I ($x < 0$) and $v = V'$ in region II ($x > 0$). $V$ and $V'$ are two constants, chosen such that $V' > V$. Only the eigenvalues $E > V'$ will be considered. For such values for the energy it can be shown that critical angle $i_c$ is given by

$$\sin i_c = [(E - V')/(E - V)]^{1/2}.$$  

(14)

Two cases must be considered.

Case a. $i < i_c$, or Partial Reflection

The incident wave is

$$U_1 = F \exp[i(2\pi/\hbar)(x\psi_x + y\psi_y)].$$

(15)

The reflected wave is

$$U_2 = G \exp[i(2\pi/\hbar)(-x\psi_x + y\psi_y)].$$

(16)

The refracted wave is

$$U' = H \exp[i(2\pi/\hbar)(x\psi'_x + y\psi'_y)].$$

(17)

The three wavefunctions above are the time-independent forms of the most general wavefunctions.\textsuperscript{21}

\textsuperscript{21} We have used $j$ for $\sqrt{-1}$ in order to avoid the confusion with the symbol $i$ used for the angle of incidence.
Here

\[ p_\pm = \left[ 2m(E - V) \right]^\frac{1}{2} \cos \theta, \]

\[ p_\pm' = \left[ 2m[(E - V') - (E - V) \sin^2 \theta] \right]^\frac{1}{2}, \]

\[ p_\beta = \left[ 2m(E - V) \right]^\frac{1}{2} (- \sin \theta). \]

Because of the requirement for continuity of the wavefunction and of its x derivative at \( x = 0 \), the following equalities are obtained

\[ G = \frac{\cos \theta - (\sin^2 \theta - \sin \theta \sin \theta)}{F \cos \theta + (\sin \theta - \sin \theta)}, \]

\[ H = \frac{2 \cos \theta}{F \cos \theta + (\sin^2 \theta - \sin \theta)}. \]

We notice that there is no special difference of phase between the incident, the reflected, and the refracted waves. The conservation of the number of particles can be verified without the introduction of any phase shift between the incident, reflected, and refracted beams.

**Case b. \( i > i_0 \), or Total Reflection**

The incident wave is

\[ U_1 = F \exp\left[ j(2\pi/h)(xp_x + yp_y) \right]. \]

The reflected wave is

\[ U_2 = G \exp\left[ j(2\pi/h)(-xp_x + yp_y) \right]. \]

The refracted wave is

\[ U' = H \exp\left[ -(2\pi/h)xp_x \right] \exp\left[ j(2\pi/h)yp_y \right]. \]

Using, as before, both the boundary conditions and the relations \( p_x = \left[ 2m(E - V) \right]^\frac{1}{2} \cos \theta \), \( p_x' = \left[ 2m[(E - V) \sin^2 \theta - (E - V')] \right]^\frac{1}{2} \), \( p_y = \left[ 2m(E - V) \right]^\frac{1}{2} (- \sin \theta) \), we get

\[ H/F = 2 \cos \theta / [\cos \theta + j(\sin^2 \theta - \sin^2 \theta)], \]

and

\[ G/F = \exp(-j\chi), \]

where

\[ \chi = \arctan[(\sin^2 \theta - \sin^2 \theta)/(1 - \sin^2 \theta)]. \]

**Evaluation of the Shift by Hora’s Method**

This method is strictly the quantum-mechanical equivalent of Artmann’s evaluation in classical optics. Instead of one incident plane wave, Hora considers a continuous family of plane waves. Each one is characterized by an angle \( \theta \) such that \( i_0 - \epsilon < \theta < i_0 + \epsilon \), but also such that every \( \theta > i \). This family forms a Debye-Picht cylindrical wave. Hora considers the whole family at time \( t = 0 \). By choosing a certain form for the constant \( F \), he obtains

\[ \psi_0 = c \left[ \left( \frac{2m(E - V)}{a} \right)^\frac{1}{2} \right] \int_{i_0 - \epsilon}^{i_0 + \epsilon} \exp\left[ \frac{2\pi}{h}(x \cos \theta - y \sin \theta) \left( \frac{2m(E - V)}{a} \right)^3 \right] d\theta. \]

Assuming \( \epsilon \) very small, he shows that this family behaves like the well-known wavepacket. The new form for the constant \( F \) enables us to show, by giving the constant \( a \) the value \( a = (\tan \epsilon) \left[ \left( \frac{2m(E - V)}{a} \right)^3 \right] \), that the flux of incident particles is independent of \( \epsilon \).

The reflected beam is

\[ \psi_R = c \left[ \left( \frac{2m(E - V)}{a} \right)^\frac{1}{2} \right] \int_{i_0 - \epsilon}^{i_0 + \epsilon} \exp(-j\eta) \]

\[ \cdot \exp\left[ -j(2\pi/h)(x \cos \theta + y \sin \theta) \left( \frac{2m(E - V)}{a} \right)^3 \right] d\theta. \]

Then he considers the expansion of \( \chi(\sin \theta) \)

\[ \chi(\sin \theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\partial^2 \chi(\sin \theta)}{\partial(\sin \theta)^n} \right)_{i_0} (\sin \theta - \sin \theta), \]

keeping only the first two terms of the expansion. But if

\[ R = \exp\left[ -j \left( \chi(\sin \theta) - \left( \frac{\partial \chi(\sin \theta)}{\partial(\sin \theta)} \right)_{i_0} \right) \chi(\sin \theta) \right], \]

we have

\[ \psi_R(x,y) = R \psi_0 \left( -x; y + \frac{h}{2\pi \left( \left[ \left( \frac{2m(E - V)}{a} \right)^3 \right] \frac{\partial \chi(\sin \theta)}{\partial(\sin \theta)} \right)_{i_0} \right). \]

For \( \eta = 0 \), comparison with the expression (32) for the incident beam shows that

\[ \psi_0(x,y) = R \psi_0 \left( -x; y + \frac{h}{2\pi \left[ \left( \frac{2m(E - V)}{a} \right)^3 \right] \frac{\partial \chi(\sin \theta)}{\partial(\sin \theta)} \right)_{i_0} \].
In the $Oy$ direction the shift is:

$$d_y = \frac{\hbar}{2\pi [2m(E-V)]^{1/2}} \int_0^1 \frac{\partial \chi}{\partial t} \cos \theta_0 \, dt.$$  

(39)

In the direction normal to the direction of propagation of the beam, the shift which fulfills the definition given by Goos and Hänchen is

$$d = d_y \cos \theta_0 = \frac{\hbar}{2\pi [2m(E-V)]^{1/2}} \int_0^1 \frac{\partial \chi}{\partial t} \, dt.$$  

(40)

but

$$\left( \frac{\partial \chi}{\partial t} \right) \bigg|_{\theta_0} = \frac{2 \sin \theta_0}{\sin^2 \theta_0 - \sin^2 \theta_1},$$  

(41)

and by definition

$$\frac{\hbar}{[2m(E-V)]^{1/2}} = \lambda (\text{de Broglie}).$$

Thus

$$d = \frac{\sin \theta_0}{\pi} \frac{\lambda}{\sin^2 \theta_0 - \sin^2 \theta_1}$$  

(42)

and we find exactly the equivalent of the Artmann-Fragstein expression [Eq. (7)].

**Evaluation of the Shift by the Present Method**

As for electromagnetic waves, to a first approximation a wavepacket behaves like an infinite plane wave in the central part of the packet. It is in this region that we have to evaluate the fluxes $\Phi_1$ and $\Phi_2$. In order to carry out the calculations, we make use of the fact that, in quantum mechanics, the current density associated with a wave function $\psi$ is given by

$$J = -j(\hbar/2m) [\psi^* \nabla \psi - \nabla \psi^*].$$  

(43)

Using this operator we can compute the flux $\Phi_2$ of particles across the half-plane $y=0$, $x>0$. In the half-space $x>0$, when total reflection occurs, the time-dependent wave function is

$$\psi' = H \exp \left(-\frac{2\pi}{\hbar} \{2m[(E-V) \sin^2 \theta - (E-V')]\}^{1/2} \right) \times \exp \left( j \frac{2\pi}{\hbar} \{2m(E-V)\}^{1/2} \cdot y(-\sin \theta) \right) \times \exp \left( -j \frac{2\pi E t}{\hbar} \right).$$  

(44)

For computing the flux $\Phi_2$ we need only consider the component $J_y$ of the current density $J$. Then the flux sought is given by

$$\Phi_2 = \int_0^\infty J_y(\psi') dx,$$  

(45)

so that

$$\Phi_2 = (\cos \phi) H*H$$

$$\times \frac{[2(E-V)/m]^{1/2}}{2(2m/\hbar)^2 [(E-V)(\sin \phi - (E-V'))]^1} \exp \left[-j(2\pi E t/\hbar)\right].$$  

(46)

Next we have to compute (\Phi_2), the flux of particles across a strip of width $d$ (see Fig. 2). Strictly speaking, the wave considered here is the totally reflected wave, but the incident wave gives the same result. We shall write for the incident plane wave the expression

$$\psi_1 = F \exp \left( j(2\pi/\hbar) \{2m(E-V)\}^{1/2} \psi \right) \times \exp \left[-j(2\pi E t/\hbar)\right].$$  

(48)

(See Fig. 4 for the meaning of the new axes used. For computational convenience the axes $(\tilde{O}X, \tilde{O}Y)$ are used here, instead of $(\tilde{O}x, \tilde{O}y)$ which were used before.)

Considering $J_x(\psi_1)$, we find

$$\Phi_1 = F*F \{2(E-V)/m\}^{1/2} \lambda.$$

(49)

From the condition $\Phi_1 = \Phi_2$ we obtain

$$d = \frac{(-\sin \theta) \hbar}{4\pi [2m(E-V)]^{1/2}} \frac{H*H}{P*F} \frac{1}{(\sin^2 \theta - \sin^2 \theta_1)^1}.$$  

(50)

Equation (29) gives

$$H*H/F*F = 4 \cos \theta / (\cos^2 \theta + \sin^2 \theta - \sin^2 \theta_1).$$

Now since $\lambda (\text{de Broglie}) = \hbar/\{2m(E-V)\}^{1/2}$, we find:

$$d = \frac{\sin \theta}{\pi} \frac{\cos^2 \theta}{\cos^2 \theta + \sin^2 \theta_1} \frac{\lambda}{(\sin^2 \theta - \sin^2 \theta_1)^1}.$$  

(51)

which is exactly equivalent to the expressions (6) and (9) found for a beam of light.

**Comparison Between the Two Evaluations**

When the angle of incidence $\theta$ of the beam of particles tends toward $\pi/2$, the Hora expression (42) gives a nonzero shift. That result is as unrealistic as it was for electromagnetic waves. The new expression (51) has the proper behavior. As in the classical treatment above, no approximation is made during the mathematical process of the present evaluation. The result is simply a consequence of the conservation of the number of monoenergetic particles. On the other hand, in the Hora

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method it is assumed, in the expansion for $\chi(s_i)$, that only the first two terms are important. This assumption appears to be the source of the difficulty. Study of the function $\chi(s_i)$ shows that it has two singular points, one for $s_i = s_{i_0}$, the other for $s_i = 1$. At these points all the derivatives are infinite. Thus we must be very careful in expanding that function. However, the expressions for the derivatives of $\chi(s_i)$ are so involved that it has not been possible to reach Eq. (51) by extending Hora's procedure.

The Hora-Fragstein method seems also to indicate a general translation. This does not seem to be physically correct because, if it were, it should be possible to demonstrate such a translation starting directly from the expression for $\chi$ without having to consider its expansion. If we examine $\chi$ only [Eq. (31)], such a translation is not apparent. When $i = i_0$, $\chi = 0$, but the translation should be infinite. When $i = \pi/2$, $\chi = \pi$, but the translation should be equal to zero. Finally, none of the classical computations on total reflection for an infinite plane wave appear to support such an idea. Instead, as the Picht$^8$ and Schaefer and Pich$^9$ treatments indicate, for electromagnetic waves, a transfer of energy takes place from one side of the beam to the other. The over-all result appears as a translation, but it does not imply the translation of each component of the beam. For a beam of particles, the equivalent statement is that there must be a transfer of particles from one side of the beam to the other side, but this does not imply that every particle suffers the translation. In fact, the uncertainty principle applied to the position of the particles complicates such a “classical picture” and makes it more or less meaningless.

4. SUMMARY

The treatments by Picht$^8$ and Schaefer and Pich$^9$ of total reflection of a limited plane wave (or nearly plane wave) show that, on the average, a transfer of energy takes place from one side of the beam to the other. Starting from this peculiar feature, the Goos-Hänchen shift can be explained. Application of the principle of conservation of energy (or of particles) results in quantitative evaluation of the shift. The shift can then be calculated without the introduction of approximations. The expressions arrived at by this procedure behave properly as the angle of incidence $i$ of the beam varies. They are slightly different from the ones proposed before, which do not behave properly as $i$ tends toward $\pi/2$. The experimental observations of the shift and the new theory presented here form a strong support for the view that, when total reflection occurs, there is an exchange of energy between the two media involved and a transfer of energy (or of particles) from one side of a limited beam to the other side.

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