XII. On Methods for detecting small Optical Retardations, and on the Theory of Foucault's Test. By Lord Rayleigh, O.M., F.R.S.*

As was, I think, first emphasized by Foucault, the standard of accuracy necessary in optical surfaces is a certain fraction of the wave-length (λ) of the light employed. For glass surfaces refracting at nearly perpendicular incidence the error of linear retardation is about the half of that of the surface; but in the case of perpendicular reflexion the error of retardation is the double of that of the surface. The admissible error of retardation varies according to circumstances. In the case of lenses and mirrors affected with "spherical aberration," an error of \( \frac{1}{2} \lambda \) begins to influence the illumination at the geometrical focus, and so to deteriorate the image. For many purposes an error less than this is without importance. The subject is discussed in former papers†.

But for other purposes, especially when measurements are in question, a higher standard must be insisted on. It is well known that the parts of the surfaces actually utilized in interferometers, such as those of Michelson and of Fabry and Perot, should be accurate to \( \frac{1}{10} \lambda \) to \( \frac{1}{20} \lambda \), and that a still higher degree of accuracy would be advantageous. Even

* Communicated by the Author.
under difficult conditions interference-bands may be displayed in which a local departure from ideal straightness amounting to \( \frac{1}{20} \) of the band period can be detected on simple inspection. I may instance some recent observations in which the rays passing a fine vertical slit backed by a common paraffin-flame fell upon the object-glass of a 3-inch telescope placed some 20 feet away at the further end of a dark room. No collimator was needed. The object-glass was provided with a cardboard cap, pierced by two vertical slits, each \( \frac{1}{10} \) inch wide, and so placed that the distance between the inner edges was \( \frac{1}{8} \) inch. The parallelism of the three slits could be tested with a plumb-line. To observe the bands formed at the focus of the object-glass, a high magnifying-power is required. This was afforded by a small cylinder lens, acting as sole eyepiece, whose axis is best adjusted by trial to the required parallelism with the slits. Fairly good results were obtained with a glass tube of external diameter equal to about 3 mm., charged with water or preferably nitro-benzol. Latterly, I have used with advantage a solid cylinder lens of about the same diameter kindly placed at my disposal by Messrs. Hilger. With this arrangement a wire stretched horizontally across the object-glass in front of the slits is seen in fair focus. When the adjustment is good, the bands are wide and the blacknesses well developed, so that a local retardation of \( \frac{1}{20} \lambda \) or less is evident if suitably presented. The bands are much disturbed by heated air rising from the hand held below the path of the light.

The necessity for a high magnifying-power is connected with the rather wide separation of the interfering pencils as they fall upon the object-glass. The conditions are most favourable for the observation of very small retardations when the interfering pencils travel along precisely the same path, as may happen in the interference of polarized light, whether the polarization be rectilinear, as in ordinary double refraction, or circular, as along the axis of quartz. In some experiments directed to test whether "motion through the æther causes double refraction" *, it appeared that a relative retardation of the two polarized components could be detected when it amounted to only \( \lambda/12000 \), and, if I remember rightly, Brace was able to achieve a still higher sensibility. The sensibility would increase with the intensity of the light employed and with the transparency of the optical parts (nicols, &c.), and it can scarcely be said that there is any theoretical limit.

* Phil. Mag. vol. iv. p. 678 (1902); 'Scientific Papers,' vol. v. p. 66.
Another method by which moderately small retardations can be made evident is that introduced by Foucault* for the figuring of optical surfaces. According to geometrical optics rays issuing from a point can be focussed at another point, if the optical appliances are perfect. An eye situated just behind the focus observes an even field of illumination; but if a screen with a sharp edge is gradually advanced in the focal plane, all light is gradually cut off, and the entire field becomes dark simultaneously. At this moment any irregularity in the optical surfaces, by which rays are diverted from their proper course so as to escape the screening, becomes luminous; and Foucault explained how the appearances are to be interpreted and information gained as to the kind of correction necessary. He does not appear to have employed the method to observe irregularities arising otherwise than in optical surfaces, but H. Draper, in his memoir of 1864 on the Construction of a Spherical Glass Telescope †, gives a picture of the disturbances due to the heating action of the hand held near the telescope mirror. Töpler's work dates from the same year, and in subsequent publications ‡ he made many interesting applications, such as to sonorous waves in air originating in electric sparks, and further developed the technique. His most important improvements were perhaps the introduction of a larger source of light bounded by a straight edge parallel to that of the screen at the observing end, and of a small telescope to assist the eye. Worthy of notice is a recent application by R. Cheshire § to determine with considerable precision for practical purposes the refractive index of irregular glass fragments. When the fragment is surrounded by liquid || of slightly different index contained in a suitable tank, it appears luminous as an irregularity, but by adjusting the composition of the liquid it may be made to disappear. The indices are then equal, and that of the liquid may be determined by more usual methods.

We have seen that according to geometrical optics ($\lambda = 0$)

† 'Smithsonian Contribution to Knowledge,' Jan. 1864.
§ Phil. Mag. vol. xxxii, p. 409 (1916).
|| The liquid employed was a solution of mercuric iodide, and is spoken of as Thoulet's solution. Liveing (Camb. Phil. Proc. vol. iii. p. 258, 1879), who made determinations of the dispersive power, refers to Sonstadt (Chem. News, vol. xxix. p. 128, 1874). I do not know the date of Thoulet's use of the solution, but suspect that it was subsequent to Sonstadt's.
the regular light from an infinitely fine slit may be cut off suddenly, and that an irregularity will become apparent in full brightness however little (in the right direction) it may deflect the proper course of the rays. In considering the limits of sensibility we must remember that with a finite $\lambda$ the image of the slit cannot be infinitely narrow, but constitutes a diffraction pattern of finite size. If we suppose the aperture bounding the field of view to be rectangular, we may take the problem to be in two dimensions, and the image consists of a central band of varying brightness bounded by dark edges and accompanied laterally by successions of bands of diminishing brightness. A screen whose edge is at the geometrical focus can cut off only half the light and, even if the lateral bands could be neglected altogether, it must be further advanced through half the width of the central band before the field can become dark. The width of the central band depends upon the horizontal aperture $a$ (measured perpendicularly to the slit supposed vertical), the distance $f$ between the lens and the screen, and the wave-length $\lambda$. By elementary diffraction theory the first darkness occurs when the difference of retardations of the various secondary rays issuing from the aperture ranges over one complete wave-length, i.e. when the projection of the aperture on the central secondary ray is equal to $\lambda$. The half-width ($\xi$) of the central band is therefore expressed by $\xi = f/\lambda/a$.

If a prism of relative index $\mu$, and of small angle $i$, be interposed near the lens, the geometrical focus of rays passing through the prism will be displaced through a distance $(\mu - 1)iF$. If we identify this with $\xi$ as expressed above, we have

$$(\mu - 1)i = \lambda/a, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)$$

as the condition that the half maximum brightness of the prism shall coincide with approximate extinction of the remainder of the field of view. If the linear aperture of the prism be $b$, supposed to be small in comparison with $a$, the maximum retardation due to it is

$$(\mu - 1)ib = \lambda \cdot b/a; \quad \ldots \ldots \ldots (2)$$

and we recognize that easy visibility of the prism on the darkened field is consistent with a maximum retardation which is a small fraction of $\lambda$.

In Cheshire's application of Foucault's method (for I think it should be named after him) the prism had an angle $i$ of $10^\circ$, and the aperture $a$ was 8 cms., although it would
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appear from the sketch that the whole of it was not used. Thus in (1) $\lambda/ia$ would be about $5 \times 10^{-5}$; and the accuracy with which $\mu$ was determined (about $\pm 0.0002$) is of the order that might be expected.

It is of interest to trace further and more generally what the wave theory has to tell us, still supposing that the source of light is from an infinitely narrow slit (or, what comes to the same, a slit of finite width at an infinite distance), and that the apertures are rectangular. The problem may then be supposed to be in two dimensions*, although in strictness this requires that the elementary sources distributed uniformly along the length of the slit should be all in one phase. The calculation makes the usual assumption, which cannot be strictly true, that the effect of a screen is merely to stop those parts of the wave which impinge upon it, without influencing the neighbouring parts. In fig. 1, $A$ represents

![Figure 1](image)

the lens with its rectangular aperture, which brings parallel rays to a focus. In the focal plane $B$ are two adjustable screens with vertical edges, and immediately behind is the eye or objective of a small telescope. The rays from the various points $Q$ of the second aperture, which unite at a point in the focal plane of the telescope, or of the retina, may be regarded as a parallel pencil inclined to the axis at a small angle $\phi$. $P$ is a point in the first aperture, $AP=x$, $BQ=\xi$, $AB=f$. Any additional linear retardation operative at $A$ may be denoted by $R$, a function of $x$. Thus if $V$ be the velocity of propagation and $\kappa=2\pi/\lambda$, the vibration at the

* Compare Wave Theory, Encyc. Brit. 1888; ‘Scientific Papers,’ vol. iii. p. 84.
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point $\xi$ of the second aperture will be represented by

$$\int dx \sin \kappa \left( Vt - f - R + \frac{x\xi}{f} \right),$$

or, if $x/f = \theta$, by

$$\int d\theta \sin \kappa (Vt - f - R + \theta\xi), \ldots \ldots \ldots \ldots \ldots (3)$$

the limits for $\theta$ corresponding to the angular aperture of the lens $A$. For shortness we shall omit $\kappa^*$, which can always be restored on considering "dimensions," and shall further suppose that $R$ is at most a linear function of $\theta$, say $\rho + \sigma \theta$, or, at any rate, that the whole aperture can be divided into parts for each of which $R$ is a linear function. In the former case the constant part $\rho$ may be associated with $Vt - f$, and if $T$ be written for $Vt - f - \rho$, (3) becomes

$$\sin T \int d\theta \cos (\xi - \sigma) \theta + \cos T \int d\theta \sin (\xi - \sigma) \theta. \ldots (4)$$

Since the same values of $\rho, \sigma$ apply over the whole aperture, the range of integration is between $\pm \theta$, where $\theta$ denotes the angular semi-aperture, and then the second term, involving $\cos T$, disappears, while the effect of $\sigma$ is represented by a shift in the origin of $\xi$, as was to be expected. There is now no real loss of generality in omitting $R$ altogether, so that (4) becomes simply

$$2 \sin T \frac{\sin \xi \theta}{\xi}, \ldots \ldots \ldots \ldots \ldots (5)$$

as in the usual theory. The borders of the central band correspond to $\xi \theta$, or rather $\kappa \xi \theta = \pm \pi$, or $\xi \theta = \pm \frac{1}{2} \lambda$, which agrees with the formula used above, since $2\theta = a/f$.

When we proceed to inquire what is to be observed at angle $\phi$ we have to consider the integral

$$2 \int d\xi \sin (T + \phi \xi) \frac{\sin \theta \xi}{\xi} = \sin T \left[ \frac{\sin (\theta + \phi) \xi + \sin (\theta - \phi) \xi}{\xi} d\xi \right.$$

$$+ \cos T \left[ \frac{\cos (\theta - \phi) \xi - \cos (\theta + \phi) \xi}{\xi} d\xi \right]. \ldots \ldots \ldots \ldots \ldots (6)$$

It will be observed that, whatever may be the limits for $\xi$, the first integral is an even and the second an odd function of $\phi$, so that the intensity (I), represented by the sum of the squares of the integrals, is an even function. The field of view is thus symmetrical with respect to the axis.

* Equivalent to supposing $\lambda = 2\pi$. 
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The integrals in (6) may be at once expressed in terms of the so-called sine-integral and cosine-integral defined by

$$ Si(x) = \int_{0}^{x} \frac{\sin t}{t} dt, \quad Ci(x) = \int_{x}^{\infty} \frac{\cos t}{t} dt. $$

If the limits of $\xi$ be $\xi_1$ and $\xi_2$ we get

\[
\sin T[Si\{(\theta + \phi)\xi_2\} - Si\{(\theta + \phi)\xi_1\} \\
+ Si\{(\theta - \phi)\xi_2\} - Si\{(\theta - \phi)\xi_1\}]
\]

\[
+ \cos T[Ci\{(\theta - \phi)\xi_2\} - Ci\{(\theta - \phi)\xi_1\} \\
- Ci\{(\theta + \phi)\xi_2\} + Ci\{(\theta + \phi)\xi_1\}] = (7)
\]

If $\xi_1 = -\xi_2 = -\xi$, so that the second aperture is symmetrical with respect to the axis, the Ci's, being even functions, disappear, and we have simply

\[
2 \sin T[Si\{(\theta + \phi)\xi\} + Si\{(\theta - \phi)\xi\}] = \cdot \cdot \cdot (8)
\]

If the aperture of the telescope be not purposely limited, the value of $\xi$, or rather of $k\xi$, is very great, and for most purposes the error will be small in supposing it infinite. Now $Si(\pm \infty) = \pm \frac{1}{4} \pi$, so that if $\phi$ is numerically less than $\theta$, $I = 4\pi^2$, but if $\phi$ is numerically greater than $\theta$, $I = 0$. The angular field of view $2\theta$ is thus uniformly illuminated and the transition to darkness at angles $\pm \theta$ is sudden—that is, the edges are seen with infinite sharpness. Of course, $\xi$ cannot really be infinite, nor consequently the resolving power of the telescope; but we may say that the edges are defined with full sharpness. The question here is the same as that formerly raised under the title "An Optical Paradox"*; the paradox consisting in the full definition of the edges of the first aperture, although nearly the whole of the light at the second aperture is concentrated in a very narrow band, which might appear to preclude more than a very feeble resolving power.

It may be well at this stage to examine more closely what is actually the distribution of light between the central and lateral bands in the diffraction pattern formed at the plane of the second aperture. By (5) the intensity of light at $\xi$ is proportional to $\xi^{-2} \sin^2 \theta \xi$ or, if we write $\eta$ for $\theta \xi$, to $\eta^{-2} \sin^2 \eta$. The whole light between 0 and $\eta$ is thus

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represented by

\[ J = \int_0^\pi \frac{\sin^2 \eta}{\eta} d\eta. \] (9)

\( J \) can be expressed by means of the Si-function. As may be verified by differentiation,

\[ J = \text{Si}(2\eta) - \frac{\sin^2 \eta}{\eta}, \] (10)

vanishing when \( \eta = 0 \). The places of zero illumination are defined by \( \eta = n\pi \), when \( n = 1, 2, 3, \&c. \); and, if \( \eta \) assume one of these values, we have simply

\[ J = \text{Si}(2\eta) = \text{Si}(2n\pi). \] (11)

Thus, setting \( n = 1 \), we find for half the light in the central band

\[ J = \text{Si}(2\pi) = \frac{1}{2} \pi - 15264. \]

On the same scale half the whole light is \( \text{Si}(\infty) \), or \( \frac{1}{2} \pi \), so that the fraction of the whole light to be found in the central band is

\[ 1 - \frac{2 \times 15264}{\pi} = 1 - 0.097174, \] (12)

or more than nine-tenths. About half the remainder is accounted for by the light in the two lateral bands immediately adjacent (on the two sides) to the central band.

We are now in a position to calculate the appearance of the field when the second aperture is actually limited by screens, so as to allow only the passage of the central band of the diffraction pattern. For this purpose we have merely to suppose in (8) that \( \theta \xi = \pi \). The intensity at angle \( \phi \) is thus

\[ 4 \left[ \text{Si} \left( \frac{\theta + \phi}{\pi} \right) + \text{Si} \left( \frac{\theta - \phi}{\pi} \right) \right]^2. \] (13)

The further calculation requires a knowledge of the function \( \text{Si} \), and a little later we shall need the second function \( \text{Ci} \). In ascending series

\[ \text{Si}(x) = x - \frac{1}{3} \frac{x^3}{1.2.3} + \frac{1}{5} \frac{x^5}{1.2.3.4.5} - \frac{1}{7} \frac{x^7}{1.2...7} + \ldots. \] (14)

\[ \text{Ci}(x) = \gamma + \frac{1}{2} \log(x^2) - \frac{1}{2} \frac{x^2}{1.2} + \frac{1}{4} \frac{x^4}{1.2.3.4} - \ldots; \] (15)

\( \gamma \) is Euler's constant -5772156, and the logarithm is to base \( e \).

These series are always convergent and are practically
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available when $x$ is moderate. When $x$ is great, we may use the semi-convergent series

$$
\text{Si}(x) = \frac{\pi}{2} - \cos x \left\{ \frac{1}{x} - \frac{1}{x^3} + \frac{1.2.3}{x^5} \right\} + \ldots
$$

$$
- \sin x \left\{ \frac{1}{x^2} - \frac{1.2.3}{x^4} + \frac{1.2.3.4.5}{x^6} \right\}
$$

$$
\text{Ci}(x) = \sin x \left\{ \frac{1}{x} - \frac{1.2}{x^3} + \frac{1.2.3.4}{x^5} \right\} - \ldots
$$

$$
- \cos x \left\{ \frac{1}{x^2} - \frac{1.2.3}{x^4} + \frac{1.2.3.4.5}{x^6} \right\}
$$

Tables of the functions have been calculated by Glaisher*. For our present purpose it would have been more convenient had the argument been $\pi x$, rather than $x$. Between $x=5$ and $x=15$, the values of $\text{Si}(x)$ are given for integers only, and interpolation is not effective. For this reason some values of $\phi/\theta$ are chosen which make $(1 + \phi/\theta)\pi$ integral. The calculations recorded in Table I. refer in the first instance to the values of

$$
\text{Si}(1 + \phi/\theta)\pi + \text{Si}(1 - \phi/\theta)\pi.
$$

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$\phi/\theta$ & (18) & (18)$^2$ \\
\hline
0.0000 & 3.704 & 13.72 \\
0.2732 & 3.475 & 12.08 \\
0.5000 & 2.979 & 8.87 \\
0.5915 & 2.721 & 7.40 \\
0.9099 & 1.707 & 2.91 \\
1.0000 & 1.418 & 2.01 \\
1.2282 & 0.758 & 0.57 \\
1.5465 & 0.115 & 0.01 \\
2.0000 & -0.177 & 0.03 \\
\hline
\end{tabular}
\caption{Table I.}
\end{table}

\begin{itemize}
\item $\kappa\theta\xi_1 = -\pi$, $\kappa\theta\xi_2 = +\pi$.
\end{itemize}

It will be seen that, in spite of the fact that nine-tenths of the whole light passes, the definition of what should be the edge of the field at $\phi=\theta$ is very bad. Also that the illumination at $\phi=0$ is greater than what it would be ($\pi^2$) if the second screening were abolished altogether ($\pm \xi = \infty$).

So far we have dealt only with cases where the second

* Phil. Trans. vol. clx. p. 367 (1870).
aperture is symmetrically situated with respect to the geometrical focus. This restriction we will now dispense with, considering first the case where $\xi_1 = 0$ and $\xi_2 (= \xi)$ is positive and of arbitrary value. The coefficient of $\sin T$ in (7) becomes simply

$$\text{Si}((\theta + \phi)\xi) + \text{Si}((\theta - \phi)\xi).$$  \hspace{1cm} (19)

In the coefficient of $\cos T$, $\text{Ci}((\theta + \phi)\xi)$, $\text{Ci}((\theta - \phi)\xi)$ assume infinite values, but by (15) we see that

$$\text{Ci}((\theta + \phi)\xi) - \text{Ci}((\theta - \phi)\xi) = \log \left|\frac{\theta + \phi}{\theta - \phi}\right|.$$  \hspace{1cm} (20)

so that the coefficient of $\cos T$ is

$$\text{Ci}((\theta - \phi)\xi) - \text{Ci}((\theta + \phi)\xi) + \log \left|\frac{\theta + \phi}{\theta - \phi}\right|.$$  \hspace{1cm} (21)

The intensity $I$ at angle $\phi$ is represented by the sum of the squares of (19) and (21). When $\phi = 0$ at the centre of the field of view, $I = 4(\text{Si}^2\theta\xi)$, but at the edges for which it suffices to suppose $\phi = + \theta$, a modification is called for, since $\text{Ci}((\theta - \phi)\xi)$ must then be replaced by $\gamma + \log |(\theta - \phi)\xi|$. Under these circumstances the coefficient of $\cos T$ becomes

$$\gamma + \log (2\theta\xi) - \text{Ci}(2\theta\xi),$$

and

$$I = \{\text{Si}(2\theta\xi)\}^2 + \{\gamma + \log (2\theta\xi) - \text{Ci}(2\theta\xi)\}^2.$$  \hspace{1cm} (22)

If in (22) $\xi$ be supposed to increase without limit, we find

$$I = \frac{1}{4}\pi^2 + \{\log \theta\xi\}^2,$$  \hspace{1cm} (23)

becoming logarithmically infinite.

Since in practice $\xi$, or rather $\kappa \xi$, is large, the edges of the field may be expected to appear very bright.

As may be anticipated, this conclusion does not depend upon our supposition that $\xi_1 = 0$. Reverting to (7) and supposing $\phi = \theta$, we have

$$\sin T[\text{Si}(2\theta\xi_2) - \text{Si}(2\theta\xi_1)]$$

$$+ \cos T[\text{Ci}(2\theta\xi_1) - \text{Ci}(2\theta\xi_2) + \log (\xi_2/\xi_1)],$$  \hspace{1cm} (24)

and $I = \infty$, when $\xi_2 = \infty$. If $\xi_1$ vanishes in (24), we have only to replace $\text{Ci}(2\theta\xi_1)$ by $\gamma + \log (2\theta\xi_1)$ in order to recover (22).

We may perhaps better understand the abnormal increase of illumination at the edges of the field by a comparison with the familiar action of a grating in forming diffraction
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spectra. Referring to (5) we see that if positive values of $\xi$ be alone regarded, the vibration in the place of the second aperture, represented by $\xi^{-1}\sin(\theta \xi)$, is the same in respect of phase as would be due to a theoretically simple grating receiving a parallel beam perpendicularly, and the directions $\phi = \pm \theta$ are those of the resulting lateral spectra of the first order. On account, however, of the factor $\xi^{-1}$, the case differs somewhat from that of the simple grating, but not enough to prevent the illumination becoming logarithmically infinite with infinite aperture. But the approximate resemblance to a simple grating fails when we include negative as well as positive values of $\xi$, since there is then a reversal of phase in passing zero. Compare fig. 2, where positive

values are represented by full lines and negative by dotted lines. If the aperture is symmetrically bounded, the parts at a distance from the centre tend to compensate one another, and the intensity at $\phi = \pm \theta$ does not become infinite with the aperture.

We now proceed to consider the actual calculation of $I = (19)^2 + (21)^2$ for various values of $\phi/\theta$, which we may suppose to be always positive, since $I$ is independent of the sign of $\phi$. When $\xi \theta$ is very great and $\phi/\theta$ is not nearly equal to unity, $\Si((\theta + \phi) \xi)$ in (19) may be replaced by $\frac{1}{2} \pi$ and $\Si((\theta - \phi) \xi)$ by $\pm \frac{1}{2} \pi$, according as $\phi/\theta$ is less or greater than unity. Under the same conditions the Ci's in (21) may be omitted, so that

$$I = \pi^2 (1, \text{or } 0) + \left\{ \log \left( \frac{\theta + \phi}{\theta - \phi} \right) \right\}^2.$$

But if we wish to avoid the infinity when $\phi = \theta$, we must make some supposition as to the actual value of $\theta \xi$, or rather of $2\pi \theta \xi/\lambda$. In some observations to be described later $\alpha = 1$ inch, $\xi = \frac{1}{2}$ inch, $1/\lambda = 40,000$, and $\theta = \frac{1}{2} \alpha / f$. Also $f$ was about 10 feet = 120 inches. For simplicity we may suppose $f = 40 \pi$, so that $2\pi \theta \xi/\lambda = 500$, or in our usual
(19) = \text{Si}\{500(1 + \phi/\theta)} + \text{Si}\{500(1 - \phi/\theta)\}, \quad (26)

and

(21) = \text{Ci}\{500(1 - \phi/\theta)} - \text{Ci}\{500(1 + \phi/\theta)}
\quad + \log (1 + \phi/\theta) - \log |1 - \phi/\theta| \quad \ldots \quad (27)

For the purposes of a somewhat rough estimate we may neglect the second \text{Ci} in \text{(27)} and identify the first \text{Si} in \text{(26)} with \text{Si} for all (positive) values of \phi/\theta. Thus when \phi = 0, I = \pi^2; and when \phi = \infty, I = 0.

When \phi/\theta = 1, we take

(26) = \frac{1}{2}\pi = 1.571, \quad (26)^2 = 2.467.

In \text{(27)}

\text{Ci}\{500(1 - \phi/\theta)} = \gamma + \log 500 + \log (1 - \phi/\theta),

so that

(27) = \gamma + \log 1000 = 7.485, \quad (27)^2 = 56.03;

and

I = 58.50.

For the values of \phi/\theta in the neighbourhood of unity we may make similar calculations with the aid of Glaisher's Tables. For example, if \phi/\theta = 1 \pm 0.02, we have

500(1 - \phi/\theta) = \pm 10.

From the Tables

\text{Si}(\pm 10) = \pm 1.6583, \quad \text{Ci}(\pm 10) = \mp 0.0455,

and thence

I(0.98) = 31.13, \quad I(1.02) = 20.89.

As regards values of the argument outside these units, we may remark that when \phi exceeds 10, \text{Si}(\phi) - \frac{1}{2}\pi and \text{Ci}(\phi) are approximately periodic in period 2\pi and of order \phi^{-1}. It is hardly worth while to include these fluctuations, which would manifest themselves as rather feeble and narrow bands, superposed upon the general ground, and we may thus content ourselves with \text{(25)}. If we apply this to \pm 10, we get

I(0.98) = 30.98, \quad I(1.02) = 21.30;

and the smoothed values differ but little from those calculated for \pm 10 more precisely. The Table (II.) annexed shows the values of I for various values of \phi/\theta. Those in the 2nd and 8th columns are smoothed values as explained, and they would remain undisturbed if the value of \theta^2 were increased. It will be seen that the maximum illumination near the edges is some 6 times that at the centre.
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TABLE II.
\( \kappa \theta \xi_1 = 0, \ k \theta \xi_2 = 500 \).

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<th>\phi/\theta</th>
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<td>2.00</td>
<td>1.21</td>
<td></td>
</tr>
<tr>
<td>0.950</td>
<td>23.27</td>
<td>59.36</td>
<td>1.020</td>
<td>26.14</td>
<td>( \infty )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1.000</td>
<td>58.50</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE III.
\( \kappa \theta \xi_1 = \pi, \ k \theta \xi_2 = 500 \).

<table>
<thead>
<tr>
<th>\phi/\theta</th>
<th>I</th>
<th>\phi/\theta</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.32</td>
<td>1.01</td>
<td>8.98</td>
</tr>
<tr>
<td>0.50</td>
<td>0.48</td>
<td>1.02</td>
<td>6.57</td>
</tr>
<tr>
<td>0.91</td>
<td>2.46</td>
<td>1.23</td>
<td>0.59</td>
</tr>
<tr>
<td>0.98</td>
<td>7.55</td>
<td>1.55</td>
<td>0.13</td>
</tr>
<tr>
<td>0.99</td>
<td>9.90</td>
<td>1.86</td>
<td>0.05</td>
</tr>
<tr>
<td>1.00</td>
<td>25.51</td>
<td>( \infty )</td>
<td>0.00</td>
</tr>
</tbody>
</table>

In the practical use of Foucault's method the general field would be darkened much more than has been supposed above where half the whole light passes. We may suppose that the screening just cuts off the central band, as well as all on one side of it, so that \( \theta \xi_1 = \pi \). In this case (7) becomes

\[
\sin T [\text{Si}(\theta + \phi) \xi + \text{Si}(\theta - \phi) \xi - \text{Si}(1 + \phi/\theta) \pi - \text{Si}(1 - \phi/\theta) \pi] \\
+ \cos T [\text{Ci}(\theta - \phi) \xi - \text{Ci}(\theta + \phi) \xi + \text{Ci}(1 + \phi/\theta) \pi - \text{Ci}(1 - \phi/\theta) \pi]. \quad (28)
\]

We will apply it to the case already considered, where \( \theta \xi = 500 \), as before omitting \( \text{Ci}(\theta + \phi) \xi \) and equating \( \text{Si}(\theta + \phi) \xi \) to \( \frac{1}{2} \pi \). Thus

\[
I = \left[ \frac{1}{2} \pi + \text{Si} 500(1 - \phi/\theta) - \text{Si}(1 + \phi/\theta) \pi - \text{Si}(1 - \phi/\theta) \pi \right]^2 \\
+ \left[ \text{Ci} 500(1 - \phi/\theta) + \text{Ci}(1 + \phi/\theta) \pi - \text{Ci}(1 - \phi/\theta) \pi \right]^2. \quad (29)
\]
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When \( \phi = \infty \), \( I = 0 \). When \( \phi = 0 \),

\[
I = [\pi - 2 \text{Si} \pi]^2 = 3.162.
\]

When \( \phi = \theta \),

\[
I = \left[ \frac{\pi}{2} - \text{Si}(2\pi) \right]^2 + \left[ \log(500/\pi) + \text{Ci}(2\pi) \right]^2 = 25.51;
\]

so that the brightness of the edges is now about 80 times that at the centre of the field. The remaining values of \( I \) in Table III. have been calculated as before with omission of the terms representing minor periodic fluctuations.

Hitherto we have treated various kinds of screening, but without additional retardation at the plane of the first aperture. The introduction of such retardation is, of course, a complication, but in principle it gives rise to no difficulty, provided the retardation be linear in \( \theta \) over the various parts of the aperture. The final illumination as a function of \( \phi \) can always be expressed by means of the Si- and Ci-functions.

As the simplest case which presents something essentially novel, we may suppose that an otherwise constant retardation \( R \) changes sign when \( \theta = 0 \), is equal (say) to \( +\rho \) when \( \theta \) is positive and to \( -\rho \) when \( \theta \) is negative. Then (3) becomes

\[
\int_{-\theta}^{0} \sin(T + \rho + \theta \xi) d\theta + \int_{0}^{\theta} \sin(T - \rho + \theta \xi) d\theta
\]

\[=
2 \sin T \left[ \cos \rho \frac{\sin \theta \xi}{\xi} + \sin \rho \frac{1 - \cos \theta \xi}{\xi} \right], \quad (30)
\]

reducing to (5) when \( \rho = 0 \). This gives the vibration at the point \( \xi \) of the second aperture. If \( \xi = 0 \), (30) becomes

\[
2\theta \cos \rho \sin T,
\]

and vanishes when \( \cos \rho = 0 \), for instance, when the whole difference of retardation \( 2\rho = \pi \), or (reckoned in wave-lengths) \( \frac{1}{2} \lambda \).

The vibration in direction \( \phi \) behind the second aperture is to be obtained by writing \( T + \phi \xi \) for \( T \) in (30) and integrating with respect to \( \xi \). This gives

\[
2 \sin T \int d\xi \cos \phi \xi \left\{ \cos \rho \frac{\sin \theta \xi}{\xi} + \sin \rho \frac{1 - \cos \theta \xi}{\xi} \right\}
\]

\[+
2 \cos T \int d\xi \sin \phi \xi \left\{ \cos \rho \frac{\sin \theta \xi}{\xi} + \sin \rho \frac{1 - \cos \theta \xi}{\xi} \right\}, \quad (31)
\]

and the illumination (I) is independent of the sign of \( \phi \), which we may henceforward suppose to be positive.

If the second aperture be symmetrically placed, we may
Retardations, and on the Theory of Foucault's Test.

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take the limits to be expressed as ±ξ, and (31) becomes

\[
2 \sin T \cos \delta \int_{0}^{\xi} \frac{\sin ((\theta + \phi)\xi + \sin ((\theta - \phi)\xi) d\xi}{\xi} + 2 \cos T \sin \delta \int_{0}^{\xi} \frac{\sin \phi\xi - \sin ((\theta + \phi)\xi + \sin ((\theta - \phi)\xi) d\xi}. \tag{32}
\]

If we apply this to \(\xi = \infty\) to find what occurs when there is no screening, we fall upon ambiguities, for (32) becomes

\[
2 \sin T \cos \delta \{\frac{1}{2} \pi \pm \frac{1}{2} \pi\} + 2 \cos T \sin \delta \{2 \text{Si}(\phi\xi) - \frac{1}{2} \pi \pm \frac{1}{2} \pi\},
\]

the alternatives following the sign of \(\theta - \phi\), with exclusion of the case \(\phi = \theta\). If \(\phi\) is finite, \(2 \text{Si}(\phi\xi)\) may be equated to \(\pi\), and we get

\[
I = 4 \pi^2 (1 \text{ or } 0),
\]

according as \(\theta - \phi\) is positive or negative. But if \(\phi = 0\) absolutely, \(\text{Si}(\phi\xi)\) disappears, however great \(\xi\) may be; and when \(\phi\) is small,

\[
I = 4 \pi^2 \cos^2 \rho + 4 \sin^2 \rho \{2 \text{Si}(\phi\xi)\}^2,
\]

in which the value of the second term is uncertain, unless indeed \(\sin \rho = 0\).

It would seem that the difficulty depends upon the assumed discontinuity of \(R\) when \(\theta = 0\). If the limits for \(\theta\) be \(\pm \alpha\) (up to the present written as \(\pm \theta\)), what we have to consider is

\[
\int_{-\infty}^{+\infty} d\xi \left[ \int_{-\alpha}^{+\alpha} d\theta \sin \{T - R + (\theta + \phi)\xi\} \right],
\]

in which hitherto we have taken first the integration with respect to \(\theta\). We propose now to take first the integration with respect to \(\xi\), introducing the factor \(e^{\pm \mu \xi}\) to ensure convergency. We get

\[
2 \sin (T - R) \int_{0}^{\infty} e^{-\mu \xi} \cos (\theta + \phi)\xi \cdot d\xi = \frac{2 \mu \sin (T - R)}{\mu^2 + (\theta + \phi)^2}. \tag{33}
\]

There remains the integration with respect to \(\theta\), of which \(R\) is supposed to be a continuous function. As \(\mu\) tends to vanish, the only values of \(\theta\) which contribute are confined more and more to the neighbourhood of \(-\phi\), so that ultimately we may suppose \(\theta\) to have this value in \(R\). And

\[
\int_{-\alpha}^{+\alpha} \frac{\mu d\theta}{\mu^2 + (\theta + \phi)^2} = \tan^{-1} \frac{\phi + \alpha}{\mu} - \tan^{-1} \frac{\phi - \alpha}{\mu},
\]

which is \(\pi\), if \(\phi\) lies between \(\pm \alpha\), and 0 if \(\phi\) lies outside these limits, when \(\mu\) is made vanishing small. The intensity
in any direction \( \phi \) is thus independent of \( R \) altogether. This procedure would fail if \( R \) were discontinuous for any values of \( \theta \).

Resuming the suppositions of equation (31), let us now further suppose that the aperture extends from \( \xi_1 \) to \( \xi_2 \), where both \( \xi_1 \) and \( \xi_2 \) are positive and \( \xi_2 > \xi_1 \). Our expression for the vibration in direction \( \phi \) becomes

\[
\begin{align*}
\sin T \left[ \cos \rho \left\{ \text{Si}(\theta + \phi) \xi + \text{Si}(\theta - \phi) \xi \right\} \right. \\
+ \sin \rho \left\{ 2\text{Ci}(\phi \xi) - \text{Ci}(\theta + \phi) \xi - \text{Ci}(\theta - \phi) \xi \right\} \xi_1 \\
+ \cos T \left[ \cos \rho \left\{ \text{Ci}(\theta - \phi) \xi - \text{Ci}(\theta + \phi) \xi \right\} \right. \\
+ \sin \rho \left\{ 2\text{Si}(\phi \xi) - \text{Si}(\theta + \phi) \xi + \text{Si}(\theta - \phi) \xi \right\} \xi_1 \\
\end{align*}
\]

(34)

We will apply this to the case already considered where \( \xi_2 \theta = 500 \), \( \xi_1 \theta = \pi \); and since we are now concerned mainly with what occurs in the neighbourhood of \( \phi = 0 \), we may confine \( \phi \) to lie between the limits 0 and \( \frac{1}{2} \theta \). Under these circumstances, and putting minor rapid fluctuations out of account, we may neglect \( \text{Ci}(\theta \pm \phi) \xi_2 \) and equate \( \text{Si}(\theta \pm \phi) \xi_2 \) to \( \frac{1}{2} \pi \). A similar simplification is admissible for \( \text{Si}(\phi \xi_2) \), \( \text{Ci}(\phi \xi_2) \), unless \( \phi / \theta \) is very small.

When \( \phi = 0 \), (34) gives

\[
\begin{align*}
\sin T \left[ \cos \rho \left\{ \pi - 2\text{Si}(\pi) \right\} \right. \\
+ \sin \rho \left\{ 2 \log(500/\pi) + 2\text{Ci}(\pi) \right\} \xi_1
\end{align*}
\]

in which

\[
\pi - 2\text{Si}(\pi) = -0.5623, \quad \text{Ci}(\pi) = 0.0738, \quad \log(500/\pi) = 5.0699.
\]

Thus for the intensity

\[
I(0) = \left[ -0.5623 \cos \rho + 10.2874 \sin \rho \right]^2.
\]

(35)

If \( \rho = 0 \), we fall back upon a former result (‘3162). If \( \rho = \frac{1}{4} \pi \), \( I(0) = 47.3 \).

Interest attaches mainly to small values of \( \rho \), and we see that the effect depends upon the sign of \( \rho \). A positive \( \rho \) means that the retardation at the first aperture takes place on the side opposite to that covered by the screen at the second aperture. As regards magnitude, we must remember that \( \rho \) stands for an angular retardation \( \kappa \rho \), or \( 2\pi \rho / \lambda \); so that, for example, \( \rho = \frac{1}{4} \pi \) above represents a linear retardation \( \lambda / 8 \), and a total relative retardation between the two halves of the first aperture equal to \( \lambda / 4 \).

The second column of Table IV. gives the general expression for the vibration in terms of \( \rho \) for various values of \( \phi / \theta \), followed by the values of the intensity (I) for \( \sin \rho = \pm 1/10 \) and \( \sin \rho = \pm 1/\sqrt{2} \).
Retardations, and on the Theory of Foucault's Test. 177

TABLE IV.

\[ \kappa \theta \xi_1 = \pi, \quad \kappa \theta \xi_2 = 500. \]

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>Formula for Vibration.</th>
<th>I. ( \sin \rho )</th>
<th>I. ( \sin \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[ \sin T \left{ -0.56 \cos \rho + 10.29 \sin \rho \right} ]</td>
<td>+1.</td>
<td>-1.</td>
</tr>
<tr>
<td>-001</td>
<td>[ \sin T \left{ -0.56 \cos \rho + 10.16 \sin \rho \right} + \cos T \times 53 \sin \rho ]</td>
<td>22</td>
<td>250</td>
</tr>
<tr>
<td>-010</td>
<td>[ \sin T \left{ -0.56 \cos \rho + 5.53 \sin \rho \right} + \cos T \times 3.10 \sin \rho ]</td>
<td>10</td>
<td>134</td>
</tr>
<tr>
<td>-050</td>
<td>[ \sin T \left{ -0.55 \cos \rho + 2.71 \sin \rho \right} + \cos T \left{ -10 \cos \rho + 2.83 \sin \rho \right} ]</td>
<td>11</td>
<td>83</td>
</tr>
<tr>
<td>-100</td>
<td>[ \sin T \left{ -0.53 \cos \rho + 1.57 \sin \rho \right} + \cos T \left{ -20 \cos \rho + 2.52 \sin \rho \right} ]</td>
<td>16</td>
<td>66</td>
</tr>
<tr>
<td>-250</td>
<td>[ \sin T \left{ -0.37 \cos \rho - 17 \sin \rho \right} + \cos T \left{ -0.46 \cos \rho + 1.66 \sin \rho \right} ]</td>
<td>23</td>
<td>52</td>
</tr>
<tr>
<td>-500</td>
<td>[ \sin T \left{ -0.16 \cos \rho - 67 \sin \rho \right} + \cos T \left{ -0.67 \cos \rho + 64 \sin \rho \right} ]</td>
<td>38</td>
<td>59</td>
</tr>
</tbody>
</table>

It will be seen that the direction of the discontinuity \((\phi = 0)\) is strongly marked by excess of brightness, and that especially when \(\rho\) is small there is a large variation with the sign of \(\rho\).

Perhaps the next case in order of simplicity of a variable \(R\) is to suppose \(R = 0\) from \(\theta = -\theta\) to \(\theta = 0\), and \(R = \sigma \theta\) from \(\theta = 0\) to \(\theta = + \theta\), corresponding to the introduction of a prism of small angle, whose edge divides equally the field of view. For the vibration in the focal plane we get

\[
\sin T \left[ \frac{\sin \theta \xi}{\xi} + \frac{\sin (\xi - \sigma) \theta}{\xi - \sigma} \right] + \cos T \left[ \frac{1 - \cos (\xi - \sigma) \theta}{\xi - \sigma} - \frac{1 - \cos \xi \theta}{\xi} \right]. \quad (36)
\]

In order to find what would be seen in direction \(\phi\), we should have next to write \((T + \phi \xi)\) for \(T\) and integrate again with respect to \(\xi\) between the appropriate limits. As to this there is no difficulty, but the expressions are rather long. It may suffice to notice that whatever the limits may be, no infinity enters at \(\phi = 0\), in which case we have merely to integrate (36) as it stands. For although the denominators...
become zero when $\xi = 0$ or $\xi = \sigma$, the four fractions themselves always remain finite. The line of transition between the two halves of the field is not so marked as when there was an actual discontinuity in the retardation itself.

In connexion with these calculations I have made for my own satisfaction a few observations, mainly to examine the enhanced brightness at the edges of the field of view. The luminous border is shown in Draper's drawing, and is described by Töpler as due to diffraction. The slit and focussing lens were those of an ordinary spectroscope, the slit being drawn back from the "collimating" lens. The telescope was from the same instrument, now mounted independently at a distance so as to receive an image of the slit and itself focussed upon the first lens. The rectangular aperture at the first lens was originally cut out of the black card. The principal dimensions have already been given. A flat paraffin-flame afforded sufficient illumination. The screens used in front of the telescope were razor-blades (Gillettes), and were adjusted in position with the aid of an eyepiece, the telescope being temporarily removed. It is not pretended that the arrangements used corresponded fully to the suppositions of theory.

The brightness of the vertical edge of the field of view is very conspicuous when the light is partly cut off by the advancing screen. A question may arise as to how much of it may be due to light ordinarily reflected at the edges of the first aperture. With the aperture cut in cardboard, I think this part was appreciable, but the substitution of a razor-edge at the first aperture made no important difference. The strongly illuminated border must often have been seen in repetitions of Foucault's experiment, but I am not aware that it has been explained.

To examine the sudden transition from one uniform retardation to another, I used a piece of plate glass which had been etched in alternate strips with hydrofluoric acid to a depth of about $\frac{1}{4} \lambda$. When this was set up in front of the first aperture with strips vertical, the division-lines shone out brightly, when the intervening areas were uniformly dark or nearly so. No marked difference was seen between the alternate division-lines corresponding to opposite signs of $\rho$. Perhaps this could hardly be expected. The whole relative retardation, reckoned as a distance, is $\frac{1}{8} \lambda$, and is thus intermediate between the values specified in Table IV. It would be of interest to make a similar experiment with a shallower etching.

Terling Place, Witham.
Jan. 5, 1917.