Fraunhofer diffraction patterns from apertures illuminated with nonparallel light

Paul E. Klingsporn  
Bendix, Kansas City Division, P. O. Box 1159, Kansas City, Missouri 64141  
(Received 27 March 1978; accepted 11 September 1978)

The size of the Fraunhofer diffraction pattern produced when a circular aperture is illuminated with a diverging beam of light, in contrast with a parallel beam, is derived on the basis of simple lens theory by introducing the concept of a virtual diffraction pattern. The results of the simple analysis agree with the diffraction pattern size determined by a formal analysis of the light wave amplitude. The formal analysis reveals a general scale factor relating the Fraunhofer diffraction pattern size for an aperture illuminated with parallel light to the size for nonparallel illumination. The diffraction pattern intensity is independent of the position of the aperture in its plane, based on a property of the Fourier transform. The invariance in intensity for nonparallel illumination of the aperture is also shown by a simple argument based on the Abbe theory of image formation by diffraction. Finally, some experimental results are given.

I. INTRODUCTION

The subject of Fraunhofer diffraction is treated in many textbooks1-5 and the examples given most frequently deal with the classical diffraction phenomena from circular and rectangular apertures. The texts discuss characteristics of Fraunhofer diffraction from rectangular and circular apertures only for the case of parallel incident light. Some excellent articles on phenomena related to diffraction have appeared in this journal,6-8 but these treat only apertures or objects illuminated by plane waves.

A typical textbook illustration for Fraunhofer diffraction is that shown in Fig. 1. Coherent light from a point source $P_s$ is rendered parallel by a lens $L_c$ and allowed to fall normally on a screen containing an aperture. Beyond the aperture is placed a second lens $L$ of focal length $F$, and the Fraunhofer diffraction effects are observed on a screen in the focal plane at distance $F$ beyond the convergent lens $L$. In the Fraunhofer approximation the light wave field $u$ at a point $(x, y)$ on the screen in the focal plane is given by

$$u(x, y) = C' \int \int_{aperture} e^{-i(2\pi/\lambda)[(x/F)x_0 + (y/F)y_0]} dx_0 dy_0,$$

(1)

where $(x_0, y_0)$ are the coordinates of a point in the aperture, $\lambda$ is the wavelength of the light, and $C'$ is a constant. For purposes of discussion, consider the classical case of diffraction by a circular aperture. The intensity distribution $I(x, y)$ observed on the screen consists of a central bright spot (the Airy disc) centered at 0 and surrounded by alternately dark and bright concentric circular rings. The intensity distribution is correctly described by the function $I(x, y) = u(x, y)u^*(x, y)$ calculated from the light wave amplitude in Eq. (1). The first three minima or dark rings and the two included relative maxima or bright rings have radii given by the well known expressions9

$$r_{1\text{min}} = 0.61(\lambda/r_0)F, \quad r_{2\text{min}} = 1.16(\lambda/r_0)F,$$

$$r_{3\text{min}} = 1.61(\lambda/r_0)F, \quad r_{1\text{max}} = 0.817(\lambda/r_0)F,$$

$$r_{2\text{max}} = 1.339(\lambda/r_0)F.$$

Because of symmetry about the axis $zz'$, the intensity distribution in the diffraction pattern on the screen can be represented schematically in the cross-sectional view of Fig. 2.

What are the radial positions of the minima in the Fraunhofer diffraction pattern when lens $L_c$ in Fig. 2 is removed and a diverging beam of light strikes the aperture? The above references do not discuss this case. Born and Wolf10 state that the light waves incident on the aperture need not be plane for Fraunhofer diffraction, but no elaboration is given. Longhurst11 states that if the light incident on the aperture is not parallel, the Fraunhofer diffraction is formed not in the focal plane of the lens but in the image plane conjugate to the plane in which the point source lies; however, Longhurst does not investigate the size of the Fraunhofer pattern for this arrangement. In the following sections several aspects of Fraunhofer diffraction from apertures illuminated by diverging light are considered.

First, the size of the Fraunhofer diffraction pattern produced by illuminating a circular aperture with nonparallel light is deduced from simple lens theory in which the concept of a virtual diffraction pattern is introduced. Second, it is shown that a formal analysis leads to the same result for the diffraction pattern size as obtained from the simple lens theory. A generalization to apertures of arbitrary shape is made which shows that the sizes of the Fraunhofer diffraction patterns with and without the collimating lens in Fig. 1 are related by a simple scale factor. The form of the scale factor indicates a unique position of the aperture relative to the lens $L$ and source $P_s$ for which the Fraunhofer diffraction pattern size is the same with and without the lens $L_c$. For both parallel and nonparallel illumination the diffraction pattern intensity is independent of the position of the aperture in its plane, based on a property of the Fourier transform. The invariance of the intensity with respect to movement of the aperture is also shown by a simple but interesting application of the Abbe theory of image formation by diffraction, for an aperture illuminated with nonparallel light. Finally, some experimental results are given.

II. THEORY

A. Simple lens theory analysis of the diffraction pattern size from a circular aperture illuminated with nonparallel light

In the arrangement in Fig. 2, a real image of the aperture is formed beyond the focal plane of lens $L$ if the magnitude
Fig. 1. Arrangement for observing the Fraunhofer diffraction pattern from an aperture illuminated with parallel light. Light from a point source $P_s$ is rendered parallel by a convergent lens $L_s$ with focal length $f$ and falls normally on a screen containing the aperture. The Fraunhofer pattern is observed in the focal plane of the lens $L$ of focal length $F$.

of $S_0$ exceeds $F$. If $S_0$ is less than $F$ a virtual image of the aperture is formed to the left of lens $L$. In either case the Fraunhofer diffraction pattern of the aperture is formed in the focal plane of lens $L$ and its dimensions and intensity distribution are independent of the aperture position along the axis $zz'$. This is true even for relatively large distances $S_0$ provided the diameter of lens $L$ is sufficiently large to collect virtually all the light diffracted by the aperture. In the absence of the screen containing the aperture, the lens $L$ forms an image of the point source $P_s$ in its focal plane at $F$, and it is in this plane that the Fraunhofer diffraction pattern is formed when the aperture is present. Consider the case in which the collimating lens $L_c$ is not present, as shown in Fig. 3. For $S_0$ greater than $F$ the lens $L$ forms a real image of the aperture in the transverse plane at distance $S_2$ from the lens. An image of the point source $P_s$ is formed in the plane at distance $S_1$ from the lens. This plane is analogous to the focal plane at $F$ in Figs. 1 and 2 in which the image of the point source is formed. The dimensions of the Fraunhofer diffraction pattern formed in the plane at $S_1$ in Fig. 3 can be determined by the following reasoning. Consider a cone of light diffracted by the circular aperture at half-angle $\theta$ relative to the axis as indicated in Fig. 3. The diffracted rays are shown as dashed lines extending to the left of the aperture to indicate that they represent virtual diffracted rays in the region to the left of the screen containing the aperture. Thus, in the plane through the point source $P_s$ the virtual diffracted rays can be considered to form a virtual diffraction pattern, and in the plane through $S_1$ the lens $L$ forms a real Fraunhofer diffraction pattern which is the real image of the virtual diffraction pattern in the plane through $P_s$. The radial distance $r_{\min}$ from the axis to the first intensity minimum in the virtual Fraunhofer diffraction pattern in the plane through $P_s$ is

$$r_{\min} = R \tan \theta_{\min} \approx R \sin \theta_{\min} = 0.61 \left(\frac{\lambda R}{r_0}\right).$$

where $r_0$ is the radius of the aperture. The planes through $P_s$ and $S_1$ are conjugate planes and therefore the magnitude of the distance $r_{\min}$ to the first minimum intensity of the diffraction pattern in the plane at $S_1$ is

$$r_{\min} = \frac{S_1}{R + S_0} = \left(\frac{S_1}{R + S_0}\right) \frac{0.61 \lambda R}{r_0}.$$  

where $S_1/(R + S_0)$ is the factor by which the lens $L$ magnifies the virtual pattern. Correspondingly, the distances to the second and third intensity minima and the first and second relative intensity maxima in the pattern in the plane at $S_1$ are

$$r_{2\min} = 1.116 \frac{\lambda R S_1}{r_0(R + S_0)}, \quad r_{3\min} = 1.619 \frac{\lambda R S_1}{r_0(R + S_0)},$$

$$r_{1\max} = 0.817 \frac{\lambda R S_1}{r_0(R + S_0)}, \quad r_{2\max} = 1.339 \frac{\lambda R S_1}{r_0(R + S_0)}.$$  

Comparison of Eqs. (4) and (5) with (2) shows that the positions of the Fraunhofer diffraction pattern minima and maxima, when the circular aperture is illuminated with nonparallel light, differ from the minima and maxima positions for the case of parallel light illumination of the aperture. The minima and maxima positions in the pattern vary with the position of the aperture relative to the lens $L$. This is in contrast to the case of illumination with parallel light in Fig. 2 for which the diffraction pattern size is independent of the separation between the lens $L$ and the aperture, provided the lens diameter is sufficiently large as pointed out earlier. The results contained in Eqs. (4) and (5) are obtained on a formal basis below.

**B. Light wave amplitude analysis of the diffraction pattern intensity for nonparallel illumination**

If the point source $P_s$ in Fig. 3 emits spherical waves of wavelength $\lambda$, then it can be shown that the light wave amplitude $u(x_1, y_1, z_1)$ at a point $(x_1, y_1)$ in the transverse plane at $S_1$ is given by

![Image of Fig. 2. Fraunhofer diffraction from a circular aperture centered on the axis and illuminated by parallel light. The pattern consists of a circular symmetric bright spot surrounded by alternately dark and bright concentric rings.](image-url)

![Image of Fig. 3. Fraunhofer diffraction from a circular aperture illuminated by nonparallel light. The Fraunhofer pattern is observed in the transverse plane at $S_1$ which is the plane conjugate to the plane containing the point source $P_s$. The plane at $S_2$ is the conjugate plane containing the image of the aperture.](image-url)
\[ u(x_1, y_1, S_1) \]
\[ = \frac{i u_0 M}{\lambda R(S_1 - S_2)} e^{-i(2\pi/\lambda)(x_1 + y_1)(S_1 - S_2)/2(S_1 - S_2)} \times \int \int e^{-i(2\pi/\lambda)M[(x_1 x_0 + y_1 y_0)/(S_1 - S_2)]} dx_0 dy_0. \]  
(6)

where \( u_0 \) is a constant amplitude factor, \( x_0 \) and \( y_0 \) are the coordinates of a point in the plane of the aperture, and \( M \equiv -S_2/S_0 \) is the factor by which the image of the aperture is magnified. The latter holds for a plane aperture of arbitrary shape. The light intensity \( I(x_1, y_1, S_1) \) at a point \( (x_1, y_1) \) in the plane at \( S_1 \) is
\[ I(x_1, y_1, S_1) = u(x_1, y_1, S_1) \ast u^*(x_1, y_1, S_1), \]  
(7)

where \( u^* \) is the complex conjugate of \( u \). In applying Eqs. (6) and (7) to calculate the light intensity at a point \( (x_1, y_1) \) in the plane at \( S_1 \) caused by a circular aperture of radius \( r_0 \) in the aperture plane at distance \( R \) from the source \( P_0 \), the integrals over \( x_0 \) and \( y_0 \) must be evaluated over the area of the circular aperture. The evaluation of the double integral can be performed by transforming to polar coordinates \((\rho, \theta)\), so that \( x_0 = \rho \cos \theta \) and \( y_0 = \rho \sin \theta \), where it is assumed that the circular aperture is centered on the axis \( zz' \) in Fig. 3. If the variables \( Mx_1/(S_1 - S_2) \) and \( My_1/(S_1 - S_2) \) are defined as
\[ Mx_1/(S_1 - S_2) = V \cos \phi, \]
\[ My_1/(S_1 - S_2) = V \sin \phi, \]  
(8)

then the double integral \( J \) in Eq. (6) can be written
\[ J = \int_0^{2\pi} \int_0^{r_0} 2\pi e^{-i(2\pi/\lambda)\rho \cos(\theta - \phi)} d\rho d\theta. \]  
(9)

From properties of Bessel functions the double integral in Eq. (9) becomes
\[ J = 2\pi r_0 J_1(2\pi r_0 V/\lambda)/(2\pi r_0 V/\lambda), \]  
(10)

and hence the intensity from Eq. (7) can be expressed as
\[ I(x_1, y_1, S_1) = \frac{4\pi^2 u_0^2 M^2 r_0^2}{\lambda^2 R^2(S_1 - S_2)^2} \left[ \frac{J_1(2\pi r_0 V/\lambda)}{(2\pi r_0 V/\lambda)} \right]^2. \]  
(11)

The first minimum in the diffraction pattern intensity distribution given by Eq. (11) occurs for the value of \( 2\pi r_0 V/\lambda \) corresponding to the first zero of the Bessel function \( J_1 \), which from tables is
\[ 2\pi r_0 (V/\lambda) = 1.2197\pi. \]  
(12)

From Eq. (8) \( V \) can be written
\[ V = [M/(S_1 - S_2)](x_1^2 + y_1^2)^{1/2} = Mr/(S_1 - S_2), \]  
(13)

where \( r \) is the radial distance from the axis \( zz' \) to the point \((x_1, y_1)\) in the plane at \( S_1 \) in Fig. 3. Therefore, from Eqs. (12) and (13) the distance \( r_{1\text{min}} \) from the center of the diffraction pattern to the first minimum intensity is
\[ r_{1\text{min}} = [(S_1 - S_2)/M](0.61\pi/r_0). \]  
(14)

The plane at \( S_2 \) in Fig. 3 contains the image of the aperture, and the plane at \( S_1 \) is conjugate to the plane in which \( P_0 \) lies. The distances \( S_0, R, S_1, \) and \( S_2 \) are therefore related to the lens focal length \( F \) by the relations
\[ \frac{1}{R} + \frac{1}{S_0} = \frac{1}{F}, \quad \frac{1}{S_1} + \frac{1}{S_2} = \frac{1}{F}, \quad M = -\frac{S_2}{S_0}. \]  
(15)

When the expressions in Eq. (15) are combined, it can be shown after lengthy algebraic manipulations that Eq. (14) can be written
\[ r_{1\text{min}} = [S_1/(S_0 + R)](0.61\pi/r_0), \]  
(16)

which is identical with the result obtained earlier by the simple analysis leading to Eq. (4). It follows that the positions of the relative maxima in intensities and the higher order minima are also identical with the earlier results in Eq. (5).

It is interesting to note that the simple analysis leading to Eq. (4) involves the magnification factor between the conjugate planes at the point source \( P_0 \) and at \( S_1 \), whereas the analysis leading to Eq. (14) involves the magnification \( M \) between the conjugate planes at \( S_0 \) and \( S_2 \). However, the variables \( S_1, S_2 \), and \( M = -S_2/S_0 \) are present in Eq. (14) in such a way that the latter yields an expression identical to that of Eq. (4) for the location of the first minimum intensity in the Fraunhofer diffraction pattern in the plane at \( S_1 \).

C. Relative sizes of the diffraction pattern for parallel and nonparallel illumination of the aperture

When an aperture of arbitrary shape is illuminated with nonparallel light as in Fig. 3 the light wave amplitude at a point \((x_1, y_1)\) in the plane at \( S_1 \) is given by Eq. (6). If a quantity \( Q \) is defined as
\[ Q = (S_1 - S_2)/MF, \]  
(17)

then the double integral in the expression for the light amplitude in Eq. (6) can be written
\[ \int \int e^{-i(2\pi/\lambda)(x_1/f + (y_1/f)\rho \cos(\theta - \phi)} dx_0 dy_0. \]  
(18)

where \( x_1 = x_1/Q \), \( y_1 = y_1/Q \). The latter integral is formally identical to the integral in Eq. (1) for the light amplitude at a point in the focal plane of the lens \( L \) for parallel illumination of the aperture in Fig. 1. [The exponential factor outside the integral in Eq. (6) is immaterial because it cancels when the intensity is calculated from Eq. (7).] The scaling between the two cases can be determined by comparing Eqs. (1) and (18) which show that
\[ x = x_1/Q, \quad y = y_1/Q. \]  
(19)

That is, a given characteristic of the intensity distribution in the diffraction pattern at the point \((x, y)\) in the focal plane at \( F \) for parallel illumination will occur at the point \((xQ, yQ)\) in the plane at \( S_1 \) for nonparallel illumination. The patterns are the same in their symmetries and relative intensities but their sizes differ by the scale factor \( Q \).

The way in which the scaling factor \( Q \) varies with position of the aperture relative to the lens \( L \) can be determined by first expressing \((S_1 - S_2)/M \) in terms of \( S_0, F \), and \( R \) with the aid of Eq. (15). If the distance \( R + S_0 \) between the point source \( P_0 \) and lens \( L \) is fixed and equal to \( D \), then the distance \( S_1 \) is fixed. \( Q \) can be expressed as
\[ Q = (D - S_0)/(D - F), \]  
(20)

where \( 0 \leq S_0 \leq D \). A schematic plot of \( Q \) as a function of the separation \( S_0 \) between lens \( L \) and the aperture is shown in Fig. 4. It is interesting that the scale factor \( Q \) is unity when the aperture is at a distance \( S_0 \) equal to the lens focal length \( F \). That is, for a given lens of focal length \( F \) placed at a fixed distance \( R + S_0 \) from a point source, there exists a unique position of the aperture between the lens and
source for which the Fraunhofer diffraction pattern size in the plane at $S_1$ is the same as that in the plane at $F$ when lens $L'$ is present. When the aperture is at distance $S_0 = F$ to the left of lens $L$, there should be no difference in the intensity distribution in the diffraction pattern observed in the plane at $S_1$, compared with that observed in the focal plane at $F$. For $S_0 < F$ the diffraction pattern at $S_1$ is larger in size than the pattern at $F$ for parallel illumination, and for $S_0 > F$ the pattern at $S_1$ is smaller than that at $F$.

D. Invariance of the diffraction pattern intensity with position of the aperture in its plane

It can be seen from Eq. (1) that the intensity distribution in the focal plane at $F$ in Fig. 1 is proportional to the Fourier transform of the object which in this case is the diffracting aperture. If the aperture is displaced to a point $(L, N)$ in its plane the only effect is to introduce a phase factor $\exp[-i2\pi (L + N)/\lambda]$ in Eq. (1), according to a property of the Fourier transform known as the “shift theorem.” However, when the intensity is calculated from the product $uu^*$ the phase factor vanishes and therefore the intensity distribution in the diffraction pattern is independent of the position of the aperture in its plane, provided the lens $L$ is sufficiently large to collect virtually all the diffracted light.

When the aperture is illuminated with nonparallel light as in Fig. 3, the Fraunhofer diffraction pattern observed in the plane at $S_1$ enjoys the same property that the intensity is independent of the position of the aperture in its plane at $S_0$. This is true again because the double integral in Eq. (6) is proportional to the Fourier transform of the disturbance represented by the diffracting aperture. The invariance of intensity with aperture position can also be shown by the following simple but revealing geometric analysis based on the Abbe theory of image formation by diffraction.

According to the Abbe theory of image formation by diffraction in a microscope, applied to the case in Fig. 3, light from a point source $P_1$ is diffracted by an object (in this case the aperture), and the lens $L$ brings the diffracted rays to a focus in the plane at $S_1$, which is the plane in which the image of $P_1$ is formed in the absence of the screen and aperture. The diffracted rays proceed beyond this plane and combine with the undiffracted rays from $P_1$ to form an image of the object (the aperture in Fig. 3) in the plane at $S_2$, which is conjugate to the aperture plane. To demonstrate the invariance of the diffraction pattern intensity relative to movement of the aperture in its plane, first let the aperture be centered at $a$ on the axis $zz'$ and consider those rays diffracted at an arbitrary angle $\theta$ relative to the axis as shown in Fig. 5. The rays diffracted at angle $\theta$ will strike the thin lens at a height $y$ from the axis and will be brought to a focus at a distance $y_1'$ from the axis in the plane at $S_1$. These rays proceed beyond $S_1$ and combine with...
other rays to form an image of the aperture. Let \( y_2 \) be the
distance from the axis at which the rays under consideration
contribute to the image in the plane at \( S_2 \). From the
gemometry the distances \( y_1, y, \) and \( y_2 \) can be related through
the equation
\[
y_1 = (1 - S_1/S_2)y + (S_1/S_2)y_2.
\] (21)

Next, let the diffracting aperture be moved off axis a dis-
tance \( \epsilon \) to the position \( a' \) in its plane and consider the rays
diffracted at the same angle \( \theta \) relative to the direction de-
\[\text{fined by points } P_1 \text{ and the aperture center. Let the dis-

tances of the diffracted ray from the axis } a'z' \text{ at the lens position,}
\]
\( S_1 \) plane, and \( S_2 \) plane be \( y', y_1, \) and \( y_2 \), respectively.

Inspection of Fig. 5 shows that these distances are related
functionally in the same way as \( y, y_1, \) and \( y_2 \) and therefore
\[
y'_1 = (1 - S_1/S_2)y' + (S_1/S_2)y_2'.
\] (22)

The distances from the axis of a given point on the image
of the aperture for the shifted and unshifted aperture po-
sitions are given by \( y_2 \) and \( y_2' \), respectively, and therefore
\[
y'_2 - y_2 = M\epsilon = -(S_2/S_0)\epsilon,
\] (23)

because movement of the aperture in its plane by a distance
\( \epsilon \) results in movement of the image in the plane \( S_2 \) by the
magnified amount \( M\epsilon \).

If \( \delta \) is the angle between the axis \( a'z' \) and the line joining
the source point \( P_2 \) with the center of the aperture in its
off-axis position, then \( \delta = \epsilon/R \) for \( \epsilon \ll R \). For angles small
enough that the approximation \( \tan \delta \approx u \) is valid, \( y \) and \( y' \)
can be related through the expression
\[
y' = \epsilon + y + S_0\epsilon/\epsilon R).
\] (24)

If \( y', y_2, \) and \( y \) from Eqs. (21), (23), and (24) are sub-
stituted into Eq. (22), then the distances \( y_1 \) and \( y_1 \) in the plane
at \( S_1 \) in Fig. 5 are related by
\[
y'_1 = y_1 + \epsilon((1 - S_1/S_2) + (S_0 + R) - S_1/S_0).
\] (25)

The lens \( L \) of focal length \( F \) in Fig. 5 forms an image of
the point \( P_2 \) in the plane at \( S_1 \) and an image of the aperture in
the plane at \( S_2 \). For a thin lens the distances \( R, S_0, S_1, \) and
\( S_2 \) are related to the lens focal length \( \ell \) by the relations in
Eq. (15). When the latter are solved for \( S_1 \) and \( S_2 \) and sub-
stituted into Eq. (25) it is found that the term in square
brackets multiplying the aperture movement \( \epsilon \) vanishes and thus
\[
y'_1 = y_1.
\] (26)

The result in Eq. (26) indicates that although the image of
the aperture shifts in the plane at \( S_2 \) when the aperture is
moved in its plane, the diffracted rays, which according to
Abbe are focused in the plane at \( S_1 \), do not shift for aperture
movement \( \epsilon \) small compared with \( R \). That is, the diffrac-
tion pattern intensity distribution in the plane at \( S_2 \) does not shift
position as the aperture is moved in its plane.

III. EXPERIMENTAL

A somewhat novel experiment can be used to illustrate
the concept of the virtual diffraction pattern and to provide
an experimental check of the Fraunhofer diffraction pattern
size in the plane at \( S_1 \) for an optical arrangement of the type
in Fig. 3. Instead of an aperture in the conventional sense
of a small hole in an opaque screen, the diffracting "aper-
ture" was produced by using the tip of a precision microdrill
to make a small circular defect on the surface of a highly
reflective metal sphere of diameter 15.6 mm. The diameter
of one "aperture" used, as determined with a microscope,
was 0.0356 mm, which corresponds to \( r_0 = 0.0178 \) mm.

The optical arrangement for observing the diffraction
pattern is shown in Fig. 6. Light from a helium-neon laser
is expanded and stray or off-axis light components are re-
moved by the action of the lens and pinhole of a spatial
filter. A converging lens \( L_1 \) of focal length \( f_1 \) is placed in the
expanding beam from the pinhole at a distance from the
pinhole equal to the lens \( L_1 \) focal length, to provide a collimi-
ated beam with diameter approximately three times that
of the original laser beam. A lens \( L_2 \) is used to focus the
collimated beam to a virtual point at the center of the
sphere. Each ray is incident normally on the sphere and
therefore the light is retroreflected from the sphere and is
diverted by a thin membrane beamsplitter (pellicle), and a
lens \( L \) is used to focus the light to a point on the face of a
photodetector array. The lens \( L \) in effect forms a real image
of the virtual point at the center of the sphere.

The theory presented above deals with the case of a small
aperture in an opaque screen, whereas the small "aperture"
or diffracting object in the illuminated spot on the surface
of the sphere represents the complementary case of a small
opaque spot in the interior of a large aperture. According
to Babinet’s principle the Fraunhofer diffraction patterns
from complementary screens are the same and therefore
the above theory applies to the diffraction patterns observed
in the plane of the photodetector array in Fig. 6.

The similarity between Figs. 3 and 6 can be seen. The
combined distance \( d_1 + d_2 \) in Fig. 6 corresponds to the
distance \( S_0 \) in Fig. 3. When the sphere is positioned so that
the "aperture" on its surface is illuminated, the point at the
center of the sphere can be thought of as a virtual source \( P_2 \),
or, the light incident on the sphere can be thought to form
a virtual diffraction pattern in the plane through the sphere
center perpendicular to the common axis of lenses \( L_1, L_2 \),
and the sphere center. The sphere is mounted in a fixture
which provides for accurate rotation of the sphere about its
center. When the sphere is positioned so that the aperture
is not illuminated, the spot of light at the center of the de-
tector array is very intense. When the sphere is positioned so that the “aperture” on its surface is illuminated, there is still a very bright spot at the detector which results from the fact that the “aperture” is very small in relation to the illuminated spot of light on the sphere. However, if a small card with a hole in it is placed in the plane of the detector so that the bright spot passes through the hole, the concentric circular ring diffraction pattern from the “aperture” can be seen very well.

In one arrangement of the system, the diameter of the sphere was 15.6 mm, corresponding to a value of $R = 7.8$ mm in Fig. 3. The focal length $F$ of lens $L$ was 200 mm and the distance of lens $L$ from the photodetector array was 929 cm, corresponding to distance $S_1$ in Fig. 3. The corresponding distance $S_0$ was 204 mm. The distance from the center of the observed Fraunhofer diffraction pattern to the first minimum intensity, as measured with the photodetector array, was 7.6 mm. When the helium-neon laser wavelength $\lambda = 6328 \text{Å}$ and the above values of $R, S_0, S_1, r_0$ are substituted into either Eq. (4) or (16), the theoretical radius of the first minimum intensity ring in the Fraunhofer diffraction pattern is 7.4 mm, which is within 2.7% of the experimentally observed value. The diameter of the sphere, from which the radius $R$ was obtained, was measured with an accuracy of 0.2%, and the estimated accuracy of the measurement of $S_1$ was 0.5%. However, the measured distance 7.6 mm to the first minimum intensity was accurate to within 0.3 mm or approximately 4%. The accuracy of the measurement for $S_0$ was 2% and that of the “aperture” radius $r_0$ was 3.5%. Therefore, the observed distance to the first minimum intensity is in agreement with the theory within the uncertainty of the experimental measurements.

A larger radius sphere was also used, along with a 100-mm focal length lens $L$ and other “apertures” as small as 0.0102-mm diam. In all cases the size of the observed diffraction pattern measured with the photodetector array agreed well with the size calculated from Secs. II A and II B of the theory.

The fixture holding the sphere was aligned such that the sphere could be rotated accurately about its center, with the axis perpendicular to the plane of the page in Fig. 6. It was found that the concentric ring diffraction pattern remained centered on the detector as the “aperture” was moved through the illuminated spot on the sphere by rotating the sphere slowly about its axis. The overall intensity in the diffraction pattern varied, of course, because the $TEM_{00}$ mode of the laser beam is a maximum at its center and decreases monotonically with radial distance outward from its center, in a plane perpendicular to its axis. This means that the same type of intensity variation exists in the illuminated spot on the sphere in Fig. 6. In accordance, the overall intensity in the Fraunhofer diffraction pattern in the plane of the detector was observed to be a maximum when the “aperture” was at the center of the illuminated spot on the sphere, and the overall pattern intensity gradually decreased as the “aperture” was moved from the center outward to the edge of the illuminated spot. However, the positions of the minima and relative maxima intensity rings in the diffraction pattern remained fixed in space as the “aperture” was moved in its plane by rotating the sphere through the small angle subtended on the circumference by the illuminated spot. This experimental observation is in agreement with Sec. II D of the theory.

ACKNOWLEDGMENT

The author wishes to thank R. T. Stagner for assistance in making some of the experimental measurements.

9Reference 2, p. 397.
10Reference 2, p. 384.
13Reference 2, pp. 395–396.
14Reference 2, pp. 420–424.
Comment on "Fraunhofer diffraction patterns from apertures illuminated with nonparallel light"

Frank S. Crawford
Department of Physics and Lawrence Berkeley Laboratory, University of California, Berkeley.
California 94720
(Received 21 May 1979; accepted 10 September 1979)

In a recent paper, Klingsporn\(^1\) uses simple lens theory to relate the Fraunhofer diffraction pattern for nonparallel illumination of an aperture to the well-known results for parallel illumination. We can restate his result in a simpler and yet more general form which is also useful in practical optical design.

For simplicity we consider only slits (rather than circles) for apertures. We also take all angles to be small. First, recall the familiar case of a parallel light beam of wavelength \(\lambda\) incident on a slit of width \(D\), undergoing Fraunhofer diffraction, and then being focused to a real image by a positive lens of focal length \(f\). (See Image 1 of Fig. 1.) Let \(\Delta x\) denote the image width measured from the central maximum to the first zero. Then we have the familiar result

\[
\Delta x = f \lambda / D. \tag{1}
\]

Next reexpress this result. The ratio \(f/D\) is commonly called the \(f\) number of the lens. Its inverse, \(D/f = \Delta \theta\), is the angular full width of the beam as it converges on the image. We rewrite Eq. (1) in the form

\[
\Delta x = \lambda / \Delta \theta \tag{2}
\]

so as to express the image width as the wavelength times the \(f\) number.

Now consider a general case where we start with a point source, go through any number of converging and diverging lenses, pass through any number of real or virtual images, scrape on any number of aperture stops located wherever you please (but not too close to a real image), and are finally brought to focus at a particular real image of interest. (See Image 2 of Fig. 1.) What is the diffraction width \(\Delta x\) of the image? Consider the beam as it converges to the image; let it have angular full width \(\Delta \theta\) as determined by geometrical optics and the aperture stops. Then the general result is that Eq. (2) still holds. We can also define \(1/\Delta \theta\) to be the generalized \(f\) number, i.e., the distance from the last lens to the image, divided by the beam diameter at the last lens. Then the generalized interpretation of Eq. (2) is that the image width due to diffraction is the wavelength times the generalized \(f\) number.

To prove this result we may use geometrical optics à la Klingsporn\(^1\) (See Fig. 1 caption.) Alternatively, we may rewrite Eq. (2) so that its truth becomes "well known" to the physicist (although less useful to the designer). Multiply both sides of Eq. (2) by \(k = 2 \pi / \lambda\) and by \(\Delta \theta\), and let \(k \Delta \theta = \Delta k_x\). Then Eq. (2) becomes

\[
\Delta x \Delta k_x = 2 \pi. \tag{3}
\]

The left-hand side of Eq. (3) is the "volume in phase space" at the image. Its numerical value of \(2 \pi\) is due to our particular choices for \(\Delta \lambda\) (central maximum to first zero) and for \(\Delta k_x\) (corresponding to \(\Delta \theta\) being the angular full width). Then the fact that the general interpretation of Eq. (2) is simply obtained from the result (1) for parallel light follows from the well-known result of geometrical optics that volume in phase space is conserved as the beam progresses from one image to the next\(^2\); and because we can imagine that just before one of these images the beam was parallel and of diameter \(D\) before being brought to a focus a distance \(f\) from a lens, thus establishing the numerical value of \(2 \pi\) for the conserved phase space.

As an extended application of Eq. (2) consider the interference pattern formed by a system of identical slits having separation \(d\) between neighboring slits. We know that for parallel light incident on the slits the separation \(\Delta x\) between adjacent principal maximae at an image formed by a lens is given by \(\Delta x = \lambda f d / \Delta \theta\). Now define \(\Delta \theta\) to be the angle between adjacent "rays" converging towards the image from adjacent slits. For parallel light \(\Delta \theta = df / f\) and we can write \(\Delta x = \lambda / \Delta \theta\). For a slit system located anywhere in the system, in converging, diverging, or parallel light, that same formula for \(\Delta x\) gives the interference pattern at the image, provided \(\Delta \theta\) (which is generally not \(df / f\)) is still taken to be the angle given by geometrical optics for adjacent rays converging to the image.

ACKNOWLEDGMENT

This work was supported by the High Energy Physics Research Division of the U.S. Department of Energy under Contract No. W-7405-ENG-48.

2See, for example, M. Born and E. Wolf, Principles of Optics, 3rd ed. (Pergamon, New York, 1965), p. 165, where what we have called "volume in phase space" is called the "Smith–Helmholtz invariant."

Fig. 1. Diffraction from apertures illuminated by parallel or nonparallel light. Any one of the three aperture stops shown can be taken to be the limiting aperture. The image width at Image 2 equals that at Image 1 multiplied by the magnification \(q/p\): \(\Delta x_2 = (q/p)\Delta x_1 = (q/p)(f/D)\lambda = (q/D')\lambda = \lambda / \Delta \theta_2\).