Babinet’s principle in the Fresnel regime studied using ultrasound

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The diffraction of ultrasound by a circular disk and an aperture of the same size has been investigated as a demonstration of Babinet’s principle in the Fresnel regime. The amplitude and the phase of the diffracted ultrasonic waves are measured and a graphical treatment of the results is performed by drawing vectors in the complex plane. The results verify Babinet’s principle. It is also found that the incident wave is $\pi/2$ behind the phase of the wave passing through on the central axis of a circular aperture. Because both waves travel the same path and the same distance, they should be in phase. This paradox has previously been regarded as a defect of Fresnel’s theory.

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I. INTRODUCTION

The high intensity spot just behind a circular opaque obstacle, known as the Poisson’s bright spot (the spot of Arago),

arouses curiosity and can be used to gain the attention and interest not only of students of physics but also of those of general science classes. The additional observation of the complementary circular aperture provides materials for quantitative discussions on the mathematical theory of waves.

A general description of Babinet’s principle states that the sum of the diffraction fields $U_1$ and $U_2$ behind two complementary objects (for example, an aperture and a disk of the same shape and size) is the field $U_0$ observed without the diffracting objects.

$$U_0 = U_1 + U_2.$$  (1)

This principle is generally fulfilled within the framework of the scalar theory of diffraction.

Verification of Babinet’s principle has been almost exclusively performed with light. The diffracted intensity patterns $I = UU^*$, where $U^*$ is the complex conjugate) are compared in the Fraunhofer regime,

where the source and observation plane are very far from the diffracting object. In the Fresnel regime, where the source and observation plane are a finite distance from the diffracting object, the intensity patterns behind the complementary objects are different from each other and are compared using calculations.

The difficulty of direct verification is mainly because the wavelength of light is too short and the frequency is too high to observe the phase. In the microwave region, the wavelength becomes a few centimeters. It is possible to make use of this advantage. For example, the phase dependence of the dielectric transient absorption allows the observation of a phenomenon several orders of magnitude faster than the time resolution, which is determined by the $Q$ value of the cavity.

The wavelength of ultrasound is several millimeters and the period is a few tens of microseconds, and hence it is possible to manipulate and observe the phase as well as the amplitude, enabling Babinet’s principle to be studied directly by comparing the amplitude and the phase. The essence of the diffraction theory can be seen by simply drawing vectors (phasors) in the complex plane without complicated calculations.

We will observe diffraction by a circular aperture, having a diffracted field $U_d$ and a complementary circular disk $U_d$ and study Babinet’s principle, $U_0 = U_a + U_d$, by using ultrasound. The apparatus is inexpensive and can be assembled by hand. The measurement is as simple as learning how to use an oscilloscope. A graphical treatment, the vibration spiral, based on Huygens–Fresnel diffraction theory, is used to analyze the observed amplitude and phase.

A curious feature of both the graphical approaches and mathematical analysis is a $\pi/2$ advance of the phase for the wave passing through on the central axis of a circular aperture with respect to the incident wave $U_0$. This phase difference has been regarded as a defect of Huygens–Fresnel diffraction theory. However, the phase difference has never been actually measured as far as we know. We measure the phase due to a small aperture with a diameter close to the wavelength being used. The vibration spiral for the circular aperture will be investigated in detail.

II. THEORETICAL BACKGROUND

Huygens’ principle explains the propagation of a wave as follows: every point of a wave front may be regarded as a source of secondary spherical wavelets, and the wave front at any later instant is constructed as the envelope of these wavelets, as shown in Fig. 1. Fresnel constructed a diffraction theory based on Huygens’ principle together with the principle of interference. We will discuss the theoretical basis using graphical and mathematical treatments.

A. Graphical treatment of diffraction by a circular object

A graphical treatment based on Huygens–Fresnel diffraction theory gives a clear physical picture of the origin of the diffraction pattern.

Consider a point source $S$ located at a distance $z_0$ from a circular aperture and the diffracted field $U_d$ at point $P$ on the axis at a distance $z$ from the aperture. Imagine a set of spheres centered at $P$ of radii $z + \lambda/2, z + 2\lambda/2, z + 3\lambda/2, \ldots$, differing by $\lambda/2$, as shown in Fig. 1. Their intersections at the aperture with a spherical wave front emanating from the source $S$ form circular zones, $F_1, F_2, F_3, \ldots$. The zones are called Fresnel or half-period zones. There is a $\pi$ phase difference between consecutive zones, and therefore, the contributions of the successive zones to the diffracted field are alternately positive and negative. Figure 2(a) shows the amplitude diagram in the complex plane when the first half-period zone is divided into eight subzones. The phasor $a_1$ represents the contribution...
from the first subzone. The phasors $a_2$ and $a_3$ due to the second and third subzones are added. The phase difference $\delta$ between each successive subzone is $\pi/8$. Adding all eight subzones gives the phasor $OG$ as the resultant amplitude from the first half-period zone. Similarly, the contribution of each of the subzones gives the phasor $UG$ representing the unobstructed field $U_0$. This argument also applies for plane wave geometry.

According to Babinet’s principle, $U_0=U_d+U_d'$, where $U_d$ is $DC$. The spiral is practically circular for many turns. Huygens–Fresnel diffraction theory predicts that the Poisson’s bright spot appears behind a circular disk, its amplitude being almost equal to $U_0$.

The spiral also allows one to determine directly the resultant amplitude and the phase due to any fractional number of zones. The phase can be estimated by simply calculating $\Delta \beta$, the difference between the path length along the axis and that via the edge of the circular object. The angle $\phi_0$ ($\angle OCD$) on the spiral (see Fig. 2) is given by $2\pi \Delta \beta/\lambda$. Because $\Delta \beta$ can be larger than $\lambda$, $\phi_0$ can exceed $2\pi$.

Figure 2 applies for the time $t=0$. The phasors rotate clockwise with the same angular velocity $\omega$. The triangle representing $U_0$, $U_a$, and $U_d$ rotates as a rigid frame. The simple harmonic wave is presented as the projection on the coordinate of the real part. We adopt here the convention in optics of representing an advance of time by a clockwise rotation of the amplitude phasor.

A problem arises. The resultant phasor $U_0$ turns out to be $\pi/2$ behind the wave from the center of the zone system, whereas according to Huygen’s principle, a wavelet emitted at $O$ (Fig. 1) in phase with the incident wave $U_0$ should arrive at $P$ still in phase with $U_0$. This paradox has been regarded as a defect of Fresnel’s theory resulting from the approximations made therein.

B. Mathematical treatment in relation to the vibration spiral

We discuss now the scalar theory of diffraction in relation to the vibration spiral. Kirchhoff gave a mathematical expression for Fresnel’s wave theory on the basis of Green’s theorem. Fresnel–Kirchhoff diffraction theory provides an accurate solution in many instances and is widely used. The diffraction patterns for the circular aperture and disk are given by Lommel functions, which are given in terms of the Bessel functions. We consider for simplicity diffraction on the axis.

The Fresnel–Kirchhoff diffraction integral for a point source and a circular aperture can be expressed in terms of the incident wave $U_0(P)$. For point $P$ on the axis, one obtains in polar coordinates:  

$$U_0(P) = -iU_0(P) \int_0^{\phi_0} e^{i\phi Q(\phi)} d\phi, \quad (2)$$

$$U_0(P) = A_0 \exp[-i(\omega t - kr - \theta)]/r, \quad (3)$$

where $k=2\pi/\lambda$, $A_0$ is the amplitude of the incident wave, and $\phi_0$ is the value of the phase difference at $P$ between a wave that comes from the edge of the aperture and one that follows the $z$-axis. The inclination factor $Q(\phi)=1/2(\cos \alpha + \cos \beta)$ is a smooth monotonically decreasing function. The quantities $\alpha$ and $\beta$ are angles that the diffracted and incident waves, respectively, make with a normal unit vector $n$ perpendicular to the diffraction plane (see Fig. 1). The factor $-i$ appearing in Eq. (2) accounts for the $\pi/2$ phase shift of the diffracted wave relative to the incident wave. The phasor integral

$$I(\phi_0) = \int_0^{\phi_0} Q(\phi)e^{i\phi} d\phi \quad (4)$$

gives the vibration spiral. For a few turns, $Q(\phi)$ is close to 1. Therefore, the spiral may be regarded as a circle. $I(\phi_0)$ $= 1 + i$, $2i$, $-1 + i$, and 0 for the phase difference $\phi_0 = \pi/2$, $\pi$, $3\pi/2$, and $2\pi$, respectively, and the points for $I(\phi_0)$, multiplied by $A_0$, lie on the circle in Fig. 2(b). In the limit of infinite aperture radius, $I(\phi_0)$ becomes $i$, and $U_a(P)$ approaches $U_0(P)$, the incident wave. The resultant phase from Eq. (2) is $-\pi/2$ for a wave that follows the $z$ axis and 0 for no aperture. The vibration spiral in Fig. 2 should be turned $\pi/2$ clockwise so that $U_0$ is on the real axis.

However, Fresnel–Kirchhoff diffraction theory has inconsistencies in the boundary conditions and is valid when the
The data for the vibration spiral (Fig. 4) were taken on the axis (r=0) by changing z.

diffraction object is large compared with the wavelength (that is, the diameter of the circular object D ≫ λ) but small compared with z₀ and z (z₀, z ≫ D).\textsuperscript{3,5}

Rayleigh–Sommerfeld diffraction theory gives a rigorous expression for the scalar field.\textsuperscript{11} This theory has not been discussed in relation to the vibration spiral. Closed solutions for the diffracted field for the spherical wave by a circular aperture and a disk along the axis are given as\textsuperscript{12}

\[
U_d(z) = A_0 \left[ \frac{\exp(ik(z_0 + z))}{(z_0 + z)} + \exp(i\varphi) \frac{\exp(ik(Z_0 + Z))}{Z} \left( Z_0 + Z \right) \right],
\]

(5)

where Z₀ is the distance between the edge and the point source on the axis and Z is the distance between the edge and the observation point on the axis (see Fig. 3). These simple equations are valid anywhere along the axis behind the object. The first term in Eq. (5) is the incident spherical wave U₀(z) and the second term is \( -U_d(z) \exp(i\varphi) \). Babinet’s principle is satisfied as \( U_0 = U_0 + U_d \) along the axis.\textsuperscript{12}

Now we move into the Fresnel regime (z₀, z > D) and look at \( U_d(z) \) as a function of D at a fixed z on the axis. We rewrite Eq. (6) as

\[
U_d(z) = A_0 \frac{z_0 + z}{Z} \frac{\exp(ik(Z_0 + Z))}{(z_0 + z)}.
\]

(7)

The amplitude \( A_d \) for a disk is given by \( (z_0 + z)/(Z_0 + Z) \cdot A_0 \) and is almost equal to \( A_0 \) when D is not too large. The phase difference from the incident wave is given by \( 2\pi(Z_0 + Z)/(z_0 + z) - 2\pi \Delta_\phi/\lambda \) and is equal to \( \varphi = 2\pi \Delta_\phi/\lambda \), which is given by the vibration spiral, as discussed in Sec. II A. Let us put the end point of phasor \( U_d \) onto that of \( U_0 \) and see the trace of the starting point of \( U_d \), which is also the end point of \( U_0 \), on the complex plane by increasing D. In this way, we obtain the vibration spiral, as shown in Fig. 2(b), except \( U_0 \) is on the real axis. Fresnel zones can be obtained with the condition \( (z_0 + z) + \Delta_\phi = (z_0 + z) + n\lambda/2 \), where \( n = 1, 2, 3, \ldots \), as we discussed in Sec. II A. For a very small aperture, \( \Delta_\phi \) is small and \( U_d \) becomes close to \( U_0 \). Therefore, the resultant phasor \( U_0 - U_d \) has a phase \( \pi/2 \) ahead of \( U_0 \).

The diffracted field for the plane wave by a circular aperture along the axis is expressed simply as\textsuperscript{13,14}

\[
U_d(z) = A_0 \left[ \frac{\exp(ikz)}{Z} \exp(ikZ) \right].
\]

(8)

A one-dimensional integral over the rim of the aperture\textsuperscript{13} and a conventional two-dimensional integration over the area of the aperture\textsuperscript{14} yield the same result.

As we have seen, the theories all show a \( \pi/2 \) phase difference between the incident wave and the wave passing through on the central axis of a circular aperture.

### III. Experimental Method

The experimental setup is shown schematically in Fig. 3. The transmitter \( T \) (Nicera T4016) was driven at 40 kHz by an oscillator and gives a wavelength \( \lambda \) of 8.58 mm at room temperature. The frequency was monitored by a frequency counter. The transmitter and the center of the diffracting objects were placed on the axis so that the path of the ultrasound has a cylindrical geometry about the normal to the object through its center. The receiver \( R \) (Nicera R-4010A1) was chosen because it has a wide signal accepting angle (95° FWHM). It was set on a rail placed perpendicularly to the axis and scanned the radial distribution of the signal beyond the circular aperture or disk. The distances \( z_0 \) and \( z \) are from the diffracting object (the aperture or the disk) to the source and the scanning line, respectively. A transmitter with a narrow emitting angle (50° FWHM) was chosen to avoid undesirable reflections mainly from the table. The distance \( z_0 \) was chosen to be larger than \( z \) to have uniform sound pressure at the aperture plane. The signal from the receiver was put into channel 1 of the oscilloscope and the voltage to the transmitter was monitored in channel 2. The voltage used was typically 2–4 Vp-p. The amplitude of the received signal and the phase difference with respect to the signal in channel 2 were observed. The amplitude and the phase were registered as a function of the radial distance \( r \). The diameters of the transducers were 7 and 6 mm for T4016 and R4010A1, respectively.

A. Circular aperture and disk for the study of Babinet’s principle

A circular aperture and a complementary disk of 10A in diameter were prepared to study Babinet’s principle. A circular aperture of 86 mm in diameter cut into 0.6 mm thick A4 size hard paper was placed on a cork board of 900 mm wide, 600 mm high, and 6 mm thick with a circular hole of 118 mm in diameter. The cork board acts as a frame supporting the aperture and also absorbs the sound. A circular disk of the same diameter was also prepared. A circular disk taken from the same paper was backed by a circular cork disk of 69 mm in diameter and 2 mm thick. The disk was supported by a carbon rod of 3 mm in diameter. The apparatus was set up on a 5 cm-square-ruled mat which covered the desk to guide alignment. The heights of the centers of the transmitter, the receiver, the aperture, and the disk were set to be 340 mm.

The receiver was put on a rail placed 858 mm (100λ) away from the transmitter so that the radial distribution of the diffracted wave could be obtained. The position on the vibration spiral was selected by altering the position \( z \) of the aperture, as shown in Table I.
Table I. Experimental parameters for the study of Babinet’s principle. The frequency of the ultrasound is 40 kHz and the wavelength \( \lambda \) is 8.6 mm. The transmitter-receiver distance \( z_0 + z \) is 100\( \lambda \), and \( z \) is the distance between the diffracting object to the receiver on the axis. \( \Delta \phi \) is the difference between the path length along the axis and that via the edge of the circular object. The angle \( \phi \) on the spiral is \( 2\pi \Delta \phi / \lambda \). The ratio of the amplitude expected for the disk to the incident wave \( A_1/A_0 \) is obtained by closed solutions [Eqs. (5) and (6)] for Rayleigh-Sommerfeld diffraction theory (Ref. 12).

| \( z (\lambda) \) | 17 | 26.3 | 38 |
| \( \Delta \phi (\lambda) \) | 0.87 | 0.64 | 0.53 |
| \( \phi (\degree) \) | 313 | 230 | 190 |
| \( A_1/A_0 \) | 0.95 | 0.98 | 0.99 |

Table II. Experimental parameters for the study of the vibration spiral. \( \Delta \theta \) is the phase difference from the incident wave. The rest of the symbols are the same as those in Table I.

| Sublabels | sa1 | sa2 | sa3 |
| \( z (\lambda) \) | 17 | 26.3 | 38 |
| \( \Delta \phi (\lambda) \) | 0.017 | 0.013 | 0.010 |
| \( \phi (\degree) \) | 6.3 | 4.6 | 3.7 |
| \( \Delta \theta (-(90+\phi/2)) (\degree) \) | \(-87\) | \(-88\) | \(-88\) |
| \( A_1/A_0 \) | 1.00 | 1.00 | 1.00 |

B. Small aperture for phase study of vibration spiral

According to the vibration spiral, the phase difference \( \Delta \theta \) observed on the axis for a circular aperture field \( U_a \) relative to the unobstructed field \( U_0 \) is \(-\pi/2\) in the zero diameter limit, and \(-\pi/4, 0, \) and \( \pi/2\), respectively, for an aperture containing a quarter-period zone, a half-period zone, and one period zone (see Fig. 2). Given the relatively long wavelength of the ultrasound, the diameter of the first Fresnel half-period zone can be several centimeters. Therefore, it is possible to observe effects within a small fraction of the half-period zone. A small aperture, which will contain only a tiny fraction of half-period zones, was prepared. The phase differences with and without the aperture were measured.

We used a small circular aperture of 1.4\( \lambda \) in diameter and performed a graphical treatment. A4 size hard paper of 0.6 mm thick with a 12 mm diameter hole at the center was placed on the aperture system used in the experiment discussed in Sec. III A. The distance \( x_0 + z \) was also set to be 100\( \lambda \). The fraction of the half-period zone covered was changed by altering the position of the small aperture set between \( z = 17\lambda \) to 38\( \lambda \). The fraction can be as small as 1/30 to 1/50 of the half-zone. The phase and the amplitude of the unobstructed wave \( U_0 \) on the axis were compared with the amplitude and phase of the wave \( U_{sa} \) with the small aperture. The phase was also observed off the axis as a function of the radial distance \( r \) at \( z = 17\lambda \) and 38\( \lambda \). The position \( \phi \) on the vibration spiral and the expected phase difference \( \Delta \theta \) \((-90+\phi/2\)\) on the axis are shown in Table II. Sublabels 1, 2, and 3 stand for the distances \( z \) of 17\( \lambda \), 26.3\( \lambda \), and 38\( \lambda \), respectively.

Fig. 4. The amplitude and phase relation obtained for unobstructed wave \( U_0 \), a circular aperture \( U_a \), and a circular disk \( U_d \). The receiver was set on the axis. The phase \( \theta_0 \) for \( U_0 \) was taken to be \( \pi/2 \). The end points for phasors \( U_a \) and \( U_a - U_d \) are plotted as open and closed symbols, respectively. The \( \phi \) values calculated for each setup are shown with bars on the circle.

IV. RESULTS

A. Babinet’s principle

The relation between \( U_0 \), \( U_{sa} \), and \( U_d \) for the 10\( \lambda \) circular aperture and the disk placed 17\( \lambda \), 26.3\( \lambda \), and 38\( \lambda \) from the receiver set on the axis is shown in Fig. 4. The distance between the transmitter and the receiver was 100\( \lambda \). The experimental parameters are shown in Table I. Babinet’s principle is rewritten as \( U_a = U_0 - U_d \), and the end points for the phasors \( U_a \) and \( U_0 - U_d \) are plotted as open and closed symbols. Measurements of \( U_a \) and \( U_d \) can be compared with theory independent of each other in this plot (see Table I). The phase \( \theta_0 \) for no diffracting object was taken to be \( \pi/2 \), and following the convention of representing the Fresnel vibration spiral. If the graphical treatment is correct, the measured points lie on the spiral. The amplitude values for the aperture \( A_a \) and the disk \( A_d \) are normalized to \( A_0 \). The \( \phi \) values calculated for the phase difference \( \Delta \phi \) for each setup are shown with bars on the circle. The aperture contains less than two half-period zones with these setups. The scattered points are results for different measurements. The arrows show the average values for \( z = 17\lambda \). The main contribution to the error comes from errors in the phase. The amplitude for \( U_d \) is observed to be larger than that for \( U_0 \) and the points are mostly outside the circle except for \( z = 17\lambda \). In contrast, most points for \( U_a \) lie inside the circle. The reason may be due to the finite thickness of the edges, the finite sizes of the transducers, and the limited radial width of the incident ultrasound. Although there still are small discrepancies in the positions for \( U_d \) and \( U_{sa} \), there is good agreement between the graphical treatment and the experimental results. It has also been shown that the point moves counterclockwise on the spiral as \( \Delta \phi \) increases when the object distance is altered from the center to the receiver (see Table I and Fig. 4).

The \( A_a \) values on the axis can be between 0 and twice \( A_0 \), depending on the position \( z \) of the aperture. In the present setup, two \( A_a \) maxima were observed at \( z = 44\lambda \) and 56\( \lambda \), where \( \Delta \phi \) is the same. \( A_a \) was observed to be \( \approx 1.9A_0 \) and \( \theta_0 \) was almost equal to \( \theta_0 \) at the maximum. An \( A_a \) minimum was observed at \( z = 14\lambda \); it is also observed as a dark spot in

the direct beam. The phase $\theta_a$ changed abruptly from being behind to ahead of $\theta_0$ as the aperture approached the receiver in the vicinity of the minimum.

The amplitude and the phase measurements were also performed for off-axis positions. The typical radial diffraction patterns, the amplitude $A$, and the phase $\theta$ obtained for $U_0$, $U_a$, and $U_d$ are shown in Fig. 5. There are five characteristic positions shown in Fig. 5. One position is the axis, and two are the edge of geometrical shadows (vertical broken lines) which lie symmetrically about the axis. The positions denoted by $M_1$, where all phases can come close, correspond to positions for the eclipse of the first half-period zone (dot-dash lines). When the observation point $P$ moves out from the center and goes into the geometrical shadow of the aperture, the Fresnel zone system collapses. As $P$ moves further, the total eclipse of a half-period zone occurs.

Phases also come close to each other at the geometrical edges. The position of the geometrical edge and the position where three phases coincide differ a little from each other. The difference may be due to the finite sizes of the transducers.

The measured $A_0$ and $\theta_0$ change smoothly as expected over the whole range observed. The values for $A_0$ are almost constant. $\theta_0$ lags as the receiver moves out from the axis and the distance from the source increases. The parabolic curve shows the phase shift of $U_0$, $2\pi(TR' - (z_0 + z))/\lambda$, calculated from the difference in the transmitter-receiver distance between the receiver on the axis $(z_0 + z)$ and on a radial position $(TR')$ (see Fig. 3). The curve represents the phase of a spherical wave originated at the transmitter observed on a line perpendicular to the axis as a function of the radius $r$. The observed $\theta_0$ agrees well with the calculated curve as expected.

The amplitude $A_a$ for the aperture increases slowly when the receiver approaches the geometrical edge from outside. Two $A_a$ maxima lie symmetrically about the axis in the direct beam. $A_a$ on the axis depends on $z$. The amplitude $A_d$ for the disk is almost constant and is the same as $A_0$, outside the shadow except for a small minimum near $M_1$, until the receiver approaches the geometrical edge. Then it begins to decrease as the receiver comes close to the geometrical edge. Structures are seen in the shadow. The Poisson’s bright spot appears on the axis surrounded by small maxima. The amplitude of the spot is almost equal to $A_0$, as predicted by the graphical treatment.

The phase $\theta_a$ for the aperture changes slowly inside the edge, except in the vicinity of the center. However, the phase $\theta_d$ changes rapidly outside the edge (in the geometrical shadow). Outside the edge, $\theta_d$ goes backward as $r$ increases until it approaches the position $M_1$; in the vicinity of $M_1$, all phases, $\theta_0$, $\theta_a$, and $\theta_d$ coincide. After passing through $M_1$, $\theta_d$ again goes backward as $r$ increases. However, $A_a$ is small and the changes in $A_a$ are not evident in this region. The same behavior is observed at the boundaries of the successive outer zones.

The phase $\theta_d$ for the disk is almost the same as $\theta_0$ outside the geometrical shadow. Inside the shadow, $\theta_d$ goes backward as the receiver goes inside. The phase $\theta_a$ changes rapidly in the vicinity of the minima of $A_d$. $\theta_d$ on the axis differs by about $2\pi$ from $\theta_0$ with this setup. One cannot tell the difference by only observing phases for $\theta_d$ and $\theta_0$ near the axis. However, the difference is clear by observing over a wide range of $r$. The difference is also seen in the vibration spiral, as shown in Fig. 4, where $U_d$ has rotated about $2\pi$.

The amplitude patterns for aperture $A_a$ and disk $A_d$ are very different from each other, as shown in Fig. 5, and one cannot find a relation for $U_a$ and $U_d$ by just looking at those patterns. However, Babinet’s principle is satisfied when one compares the amplitudes and phase off the axis. The phasors representing $U_0$, $U_a$, and $U_d$ off the axis do not lie on the circle. Still they make a triangle as expected by Babinet’s principle. It was found that the phases tend to coincide or differ by $\pi$ at characteristic positions. Three phases coincide just outside the geometrical edge and $A_0$ is given by $A_a + A_d$. The same relation is also observed at $M_1$, although the error is large because $A_a$ is very small. At the maxima of $A_a$, $\theta_d$ differs by $\pi$ from $\theta_0$ and $\theta_a$, and the patterns show $A_0 = A_a - A_d$. Therefore, Babinet’s principle is also satisfied at these positions.

B. Phase difference

The amplitudes and phases obtained for the 12 mm circular aperture are plotted on a vibration spiral, as shown in Fig. 6. The end points representing phasor $U_a$ for the small aperture are plotted with different symbols according to $z$. In Fig. 6, the arrow $U_a$ shows the average value for all $z$ positions and $U_{a3}$ shows that for $z=38\lambda$. The phase difference $\phi$ between the path via the center of the small aperture and via the edge is very small, as shown in Table II. Only a tiny fraction of a half-period zone appears in the small aperture. The $\phi$ values calculated by the path difference $\Delta_\theta$ for $z=17\lambda$ and $38\lambda$ are shown with dot-dash and broken bars on the arc. The observed $\phi$ agrees well with the calculated values. However, the observed end points for $U_{a1}$ and $U_{a2}$ lie inside the circle as observed for $U_a$ in the experiment discussed in Sec. III A; consequently, the values of $|\Delta\theta_{a1}|$ are observed a little smaller than the theoretical values. The reason may be due to the finite size of the receiver and errors in alignment. Influences of these are expected to be larger for smaller $z$. In contrast, the points for $U_{a3}$ lie mostly on the arc and the $\Delta\theta$ value obtained was $-85^\circ$, close to the theoretical...
The broken parabolic curves represent the phase shift of aperture to the receiver for a broad peak with a small structure at the center was observed. The value of $-88^\circ$. It is clearly seen that the wave from the center of the zone system is $\pi/2$ ahead of the incident wave as expected from the Huygens–Fresnel diffraction theory.

The radial diffraction patterns for the amplitudes $A_{s1}$ and $A_{s3}$ obtained for a small aperture at $z = 17\lambda$ and $38\lambda$, respectively, are shown in Fig. 7 together with $A_0$. The vertical lines represent the edges of the geometrical shadows. A broad peak with a small structure at the center was observed for $A_{s1}$. The amplitude on the axis is almost 1/10 that observed for the unobstructed wave. The pattern for $A_{s3}$ is broader and the structure is not clear due to the smallness of the signal. The phases $\theta_{s1}$ and $\theta_{s3}$ observed at $z = 17\lambda$ and $38\lambda$, respectively, are also shown in Fig. 7. The dot-dash and broken parabolic curves represent the phase shift of $U_{s1}$ and $U_{s3}$, respectively. The origin of the spherical wave is chosen as $74^\circ$ or $90^\circ$. The physical meaning of the origin of the $\pi/2$ phase difference is not fully understood. The presence of the factor $-i$ in Eq. (2) gives the necessary $\pi/2$ phase difference. However, it indicates that the secondary sources oscillate a quarter of period ahead of the primary disturbance.5,7 Huygens’ principle states that the wavelets oscillate in phase with the incident wave and the boundary diffraction wave. We observe the phase relations at the edge of geometrical shadows and at an eclipse of a half-period zone shows another aspect of diffraction.

The phase and amplitude observed on the axis are simply predicted by the path difference between the wave along the axis and the wave via the edge of the aperture, independent of the position of the aperture or the disk. The wave observed for a small aperture, which contains only a fraction of a half-period zone, has a phase which is always $\pi/2$ ahead of the wave without the aperture, provided that the aperture is small enough and is not too close to the source or the observation point.

The physical meaning of the origin of the $\pi/2$ phase difference is not fully understood. The presence of the factor $-i$ in Eq. (2) gives the necessary $\pi/2$ phase difference. However, it indicates that the secondary sources oscillate a quarter of period ahead of the primary disturbance.5,7 Huygens’ principle states that the wavelets oscillate in phase with the primary wave. The parabolic curve $\theta_0$ in Fig. 7 represents a spherical wave centered at the transmitter, while $\theta_{s1}$ or $\theta_{s3}$, after passing through the small aperture, represents spherical waves centered at the small aperture. The latter can be regarded as an envelope of Huygens’ spherical wavelet originating at the small aperture but with the phase shifted ahead by $\pi/2$.

A secondary wavelet approach by Huygens–Fresnel and an edge diffraction approach by Young are two fundamental physical models of diffraction which can provide insight into the $\pi/2$ phase shift. The Huygens–Fresnel approach expresses the disturbance at $P$ as the sum of all the contributions from every point in the aperture, as shown by the vibration spiral. The edge gives no contributions. An edge diffraction approach, the Maggi–Rubinowicz theory,3,5 describes the field behind the object as the superposition of the incident wave and the boundary diffraction wave $U_B$ (wave originated at the edge) in the direct beam and, in the geometrical shadow, just as the boundary diffraction wave. We obtain

$$U_d(r) = A_0 \frac{\exp(ikr)}{r} + U_B(r) \quad (P \text{ in the direct beam}),$$

(9)

Fig. 6. The amplitude and phase relation obtained for a small aperture of 1.4$\lambda$ in diameter. The receiver was set on the axis. The area enclosed by a square is displayed on an expanded scale on the right. The end points for phasors $U_{s1}$, $U_{s2}$, and $U_{s3}$ at distance $z$ equal to 17$\lambda$, 26$\lambda$, and 38$\lambda$ are plotted with triangles, squares, and circles, respectively. The arrow $U_{s1}$ shows the average value for all $z$ positions and $U_{s3}$ shows that for $z = 38\lambda$. The $\varphi$ values calculated for $z = 17\lambda$ and 38$\lambda$ are indicated with bars on the arc.

Fig. 7. The diffraction patterns for the amplitude $A$ (bottom curves) and the phase $\theta$ (top curves) as a function of the radial distance observed for a small aperture of 1.4$\lambda$ in diameter. Sublabels 1 and 3 stand for the distances of 17$\lambda$ and 38$\lambda$, respectively. The dot-dash and broken parabolic curves show phase differences calculated for the geometrical path difference between the center of the small aperture to the point on the axis ($z$) and the center of the aperture to the receiver ($OR'$) with an added phase difference of $-74^\circ$ or $-90^\circ$ (see text). The vertical lines ($E_{Gal}$) show the edges of geometrical shadows.
The amplitude observed for a circular aperture on the axis has a maximum when the aperture contains an odd number of half-period zones, as shown in Fig. 2(b). $U_0$ and $U_B$ interact constructively, while the path difference for the two waves is $(2n-1)\lambda/2$, as shown in Fig. 1. This fact requires that the phase for $U_B$ originated at the edge of a circular aperture is shifted by $\pi$.

Rubinowicz showed that the Kirchhoff diffraction integral (the mathematical expression of the Huygens–Fresnel approach) can be divided into the incident wave and the boundary wave, which is a line integral over the rim of an aperture, according to Young’s approach. The two approaches are considered to be mathematically equivalent treatments. However, the edge diffraction theory has not been proven beyond the limits of validity of the Kirchhoff theory (\(\pi, z \gg D, D \gg \lambda\)). The physical reality of the boundary diffraction wave has been demonstrated by Ganci at the straight edge using Young’s double slit.

We have tried to detect the boundary diffraction wave by observing the phase of the wave diffracted by a small aperture. The first terms in Eqs. (5) and (8) are the incident wave and the second terms \((-U_d)\) can be regarded as the boundary diffraction wave $U_B$ [compare Eqs. (5) and (8) with Eq. (9)]. On the axis, we have $U_B = -U_d$ in Eqs. (5) and (8). As discussed in Sec. II B, $U_0$ and $U_B$ have almost the same amplitude and a phase difference of $\pi$ for a small aperture. Then, $U_a$ is $\pi/2$ ahead of $U_0$. This phase relation is also shown in the present measurements. When $P$ moves deep into the shadow, $U_0$ vanishes, and only $U_B$ remains. Then $U_d$ should have a phase difference of $\pi$ in the shadow. However, the phase difference for $U_d$ remains $\pi/2$ both on the axis and in the geometrical shadow, as shown by the parabolic curves $\theta_{sa1}$ and $\theta_{sa2}$ in Fig. 7. The reason may be that the “shadow” is not defined clearly when $D$ becomes small. The present experimental setup is beyond the validity of the Kirchhoff theory (\(\pi, z \gg D \gg \lambda\)). Sophisticated measurements such as Ganci’s experiment may be needed to observe the physical reality of the boundary diffraction wave.

The apparatus we used was intended as a tabletop instrument for the student laboratory and is far from ideal. However, it is still capable of showing fundamental aspects of the theory of diffraction. Ideally, a uniform wave and a large room with no reflections are needed to conduct a deeper investigation of wave diffraction theory including the phase. In the classroom, satisfactory results can be obtained by setting the transmitter and the receiver close to each other with proper alignment. Because Babinet’s principle can be studied directly, the diffracting object need not be circular. An annular shape, an elliptical form, or squares, or any shape can be used.

VI. CONCLUSIONS

Babinet’s principle in the extreme Fresnel regime has been directly verified by observing ultrasound waves diffracted by a circular aperture and a complementary disk. The phase as well as the amplitude were measured and analyzed by a graphical treatment. It was found that the wave without the diffracting objects is $\pi/2$ behind the wave from the center of the zone system, which has been regarded as a defect of Fresnel’s theory. The $\pi/2$ phase difference appears also in Fresnel–Kirchhoff diffraction theory, Rayleigh–Sommerfeld diffraction theory, and in the edge-diffracted approach by Young.

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