Rainbow dust
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Beautiful "rainbows" produced by tiny glass spheres are described.

I. INTRODUCTION
Small glass spheres (diam 0.1–0.3 mm) are sometimes deposited on newly painted white roadway lines so as to partially "retroreflect" the headlight beams of cars. They are called "highway safety spheres" by the manufacturers. Some of the spheres may escape and become distributed loosely over the neighboring roadway. The roadway gives a relatively dark background against which one can easily view a complete (360°) circular "rainbow" encircling the shadow of your head produced by the overhead Sun. Charles Wohl discovered such "glass rainbows" or "glass-bows" on the roadway at Lawrence Berkeley Laboratory and pointed them out to me. Using a board with pins stuck in it, I measured an angle of about 22° for the bow, which had red on the outside and blue on the inside. (I could see no second bow.) I call the stuff rainbow dust. It has some lovely properties.

II. RAINBOW OF ORDER k
Rainbows have long been admired and studied. For the most lucid diagrams, explanations, and derivations, the best source I know is an article by Jarle Walker. We now paraphrase his results. In a rainbow of order k the ray has suffered k internal reflections inside the sphere, with k = 1 for the primary bow, k = 2 for the secondary bow, etc. A ray that impinges with angle of incidence i on a sphere having index of refraction n suffers deviation i - r as it enters the sphere, where r is the angle of refraction given by Snell's law,

\[ n \sin r = \sin i. \]  (1)

The ray emerges after k internal reflections, having suffered deviation 180° - 2r at each reflection, and a second deviation i - r as it refracts out of the sphere. The total deviation \( \delta \) is thus given by \( 2(i - r) + k(180° - 2r) \), i.e.,

\[ \delta = k \cdot 180° + 2i - 2(k + 1)r. \]  (2)

The bow occurs at the angle where \( \delta \) goes through a minimum with respect to i, found by differentiating Eq. (2) with respect to i and setting the result to zero. This occurs at \( i = i_k \) given by

\[ \cos i_k = \left( \frac{n^2 - 1}{k(k + 2)} \right)^{1/2}. \]  (3)

For water spheres (raindrops) having \( n = 1.331 \) for red light and \( n = 1.343 \) for blue light, Eqs. (1)–(3) give, for the primary bow (\( k = 1 \)), \( \delta \text{(red)} = 137.6° \), and \( \delta \text{(blue)} = 139.4° \). The observer, with her back to the Sun and measuring the bow angle from the shadow of her head, sees the red at viewing angle \( \theta = 180° - \delta \text{(red)} = 42.4° \) and the blue at viewing angle \( \theta = 180° - \delta \text{(blue)} = 40.6° \). Thus red is on the outside (larger viewing angle) of the primary bow. Similarly, for \( k = 2 \), the observer sees red of the secondary rainbow at deviation \( \delta = 230.4° \) or viewing angle \( \theta = 53.5° \). Thus red is on the inside of the secondary bow. Of course, the angular width of the Sun is 0.5°, which gives some overlap to these angles.

For glass spheres having index 1.51 for green light, Eqs. (1)–(3) give for the primary bow \( \delta \text{(green)} = 158.1° \), i.e., viewing angle 180° - \( \delta = 21.9° \). That agrees very well with my crude measurement of 22°. For the secondary bow, Eqs. (1) and (2) give \( \delta \text{(green)} = 268.6° \), i.e., viewing angle 88.6°. That is too close to 90° to be seen by simply looking down at the roadway, and explains why I could see only one bow.

III. THINGS TO DO WITH RAINBOW DUST
I have by now acquired several pounds of rainbow dust, at first by sweeping it up from a roadway (thus generating considerable interest from a police officer), and later as a gift from an acquaintance in the local Streets and Highways Department. My front sidewalk now exhibits a gorgeous complete rainbow on sunny days, and an eerie soft yellowish bow every night from the high-pressure sodium street light overhead.

For a striking indoor demonstration, spectacular at night or in another dimly lit room, sprinkle a thin layer of rainbow dust over a flat dark horizontal surface and hold above the layer a bright point source of light. (A lighted match will do, but a much better point source is the flashlight MINI-MAGLITE™ with its reflector easily removed by twisting.) Looking down from directly above the source, a person with normal depth perception will "fuse" the rainbow seen by the left eye with that seen by the right eye and thereby perceiv a beautiful "3-D" (three-dimensional) rainbow suspended in space, completely encircling the light source at a height slightly above the source, and having apparent diameter about 80% of the source height above the surface. For a derivation of its perceived 3-D location in space see the Appendix.

If the layer of rainbow dust just described is submerged under water, the 3-D rainbow from the point source disappears. That is because the relative index of glass and water for green light, 1.51/1.334 = 1.13, gives a primary rainbow with \( \delta = 94.7° \). Thus a ray incident vertically on a sphere reemerges and impinges on the underside of the water–air interface at a glancing angle of only 4.7°, and is totally internally reflected, never getting back into the air.

IV. HIGHER-ORDER BOWS AND HALOS FROM GLASS SPHERES
For \( k = 1 \) and 2 we get the results given above for the primary and secondary bows. For these and for higher values of \( k \) we get the results shown in Table I. When the deviation \( \delta \) lies in the backward hemisphere (quadrants II and III), we have a bow, with viewing angle \( \theta \) measured from the shadow of the observer's head (or eye). When \( \delta \)
Table I. Bows and halos from glass spheres of index 1.51. Here, $k$ is the number of internal reflections. The minimum deviation $\delta$ lies in quadrant "quad." The viewing angle $\theta$ is measured from the Sun for halos and from the shadow of the observer's head for bows. The last column tells where on the glass sphere an observer with sufficient angular resolution will see the colored spot that corresponds to the bow.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\delta$</th>
<th>quad</th>
<th>$\theta$</th>
<th>$\theta$ is a kind</th>
<th>red is on spot at sphere's</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>158.1°</td>
<td>II</td>
<td>21.9°</td>
<td>max bow outside</td>
<td>inner edge</td>
</tr>
<tr>
<td>2</td>
<td>268.6°</td>
<td>III</td>
<td>88.6°</td>
<td>min bow inside</td>
<td>outer edge</td>
</tr>
<tr>
<td>3</td>
<td>371.6°</td>
<td>I</td>
<td>11.6°</td>
<td>min halo inside</td>
<td>outer edge</td>
</tr>
<tr>
<td>4</td>
<td>472.1°</td>
<td>II</td>
<td>67.9°</td>
<td>max bow inside</td>
<td>inner edge</td>
</tr>
<tr>
<td>5</td>
<td>571.4°</td>
<td>III</td>
<td>31.4°</td>
<td>min bow inside</td>
<td>outer edge</td>
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<td>6</td>
<td>670.1°</td>
<td>IV</td>
<td>49.9°</td>
<td>max halo outside</td>
<td>inner edge</td>
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<tr>
<td>7</td>
<td>768.3°</td>
<td>I</td>
<td>48.3°</td>
<td>max halo inside</td>
<td>outer edge</td>
</tr>
<tr>
<td>8</td>
<td>866.3°</td>
<td>II</td>
<td>33.7°</td>
<td>max bow outside</td>
<td>inner edge</td>
</tr>
</tbody>
</table>

lies in the forward hemisphere (quadrants I and IV), we have a “halo” around the Sun, with viewing angle $\theta$ measured from the Sun. Deviation $\delta$ is always at a minimum, but viewing angle $\theta$ is at a maximum for $\delta$ in quadrants II and IV and at a minimum for $\delta$ in quadrants I and III. All of the information in columns 5 through 8 of Table I follow from the quadrant, given in Column 3, and the fact that larger $n$ corresponds to larger $\delta$ and shorter wavelength (bluer color).

According to Table I, the secondary bow ($k = 2$) should be seen at viewing angle about 89°. I have not succeeded in seeing it using rainbow dust. However, using a single 2-in.-diameter glass sphere, and the Sun, I can see the corresponding ray. Glass spheres should give a halo for $k = 3$ at a viewing angle of about 12°. To search for this halo, I sprinkled a thin layer of rainbow dust on top of a horizontal pane of window glass, placed a point source below the pane, and looked down at the source through the dust. I saw no halo. However, using the single 2-in. sphere and the Sun I can see it! Similarly, I cannot see the bows at $k = 4$ and 5 using the rainbow dust, but I can see them with the single glass sphere and the Sun.

After making these observations, it was obvious why I could succeed in finding the rays corresponding to higher-order bows and halos using the single large glass sphere but fail with rainbow dust. Success is due to the advantage of good angular resolution! For example, when I search for the halo for $k = 3$ using the single large glass sphere, I immediately identify two very bright white spots "on the sphere" due to forward external reflection (a single external reflection with no refraction) and forward refraction (two refractions and no reflections). These two white spots emerge toward my eye from two very different points on the sphere. Forward external reflection gives a spot near the inner edge (closest to the Sun). Forward refraction gives a spot slightly beyond the center (nearer to the outer edge than the center). Once I have identified these brilliantly bright "background" rays, I can easily shield them out with my hand, and ignore them. In searching for the halo for $k = 3$ I know it should appear on the outer edge of the sphere, farthest from the Sun, when I have located the sphere correctly. As I vary the location of the sphere, starting at larger angles than 12°, I see two faint white spots near the outer edge. As I move to smaller angles, these two spots finally coalesce and become colored, and I have found the halo. Instead, when I search using rainbow dust, I am overwhelmed by forward refraction and reflection background that I cannot resolve and discard.

Similarly, in searching for bows, say the bow for $k = 1$, with my back to the Sun, I can separately identify three white spots "on the sphere," if I am holding the sphere at less than the bow angle of 22°. One of these is due to back reflection (single external reflection) and is a true background. The other two are due to singly internally reflected rays, but are not really background, since they do not obscure the bow but are part of the light "inside" (at smaller viewing angles than) the primary bow. Suppose, for example, I am holding the 2-in. sphere at arms length (24 in.) at about 11° viewing angle. Then ray tracing (back to Descartes!) shows that a white ray with impact parameter 0.97$R$ (where $R$ is the radius of the sphere) is deviated through 174° and enters my eye, appearing to come from the inner edge of the sphere. Another white ray at impact parameter 0.26$R$ is deviated through 170° and enters my eye, appearing to come from about 90° of the way from the inner edge to the center of the sphere. As I move the sphere outward, these two white spots move closer together finally becoming colored and then coalescing into a single colored "bow" ray, at the bow 1 minimum deviation of 158°, appearing to come from near the inner edge of the sphere. Of course, these two white spots are not an obscuring background. But the backscattered light is. When I search for bows and halos with $k > 2$ using either a garden hose for water drops or using glass rainbow dust, the spheres are so small that I cannot resolve in angle the rainbow rays from the background rays, which are then overwhelming, for $k > 2$.

For an excellent description of measurements on higher-order rainbows and halos, see Ref. 5.

V. GLASS SPHERES AS RETROREFLECTORS: SCOTchlITE AND GLORY DUST

A glass sphere with index $n = 2$ would have its "primary rainbow" ($k = 1$) at $i_s = 0°$, $r = 0°$, and deviation $\delta = 180°$, according to Eqs. (2) and (3). Such spheres should therefore make excellent "retroreflectors" of light. Glass with $n = 2$ is not easy to manufacture, but $n = 1.9$ is common. A tape with tiny glass spheres of index about 1.9 embedded on a sticky silvery surface can be obtained at some hardware stores. It is called Scotchlite and is a good retroreflector. Tiny (1-mm) spheres with $n = 1.92$ are made by Potters Industries and are deposited on the white lines of airplane runways so as to retroreflect airplane landing lights.

The beads with $n = 1.92$ do not give an observable 3-D rainbow of the kind described above for rainbow dust ($n = 1.51$). But, if you submerge them under water, they give a beauty! That is because the relative index 1.92/1.334 = 1.439 of these glass spheres and water is almost that of glass and air, 1.5, and gives a primary rainbow at $\delta = 151.2°$, i.e., a viewing angle $\theta = 28.8°$. Using a MINI-MAGLITE® we see a fine 3-D rainbow similar (but with different length ratios) to that for rainbow dust.

With the Sun at 45° zenith angle the shadow of my head on a horizontal sidewalk subtends an angular diameter of about 4.5° at my eyes, obscuring the 2° (diameter) bow or Glory® produced by the beads of index 1.92 that now cover a portion of my sidewalk. In order to see the Glory, I must
then use a small (2-in. diam) mirror held at arm’s length transverse to the sunbeam, with the shadow of the mirror falling on the beads and with the mirror turned so I can see its shadow, which is too small to obscure the bow. Because the 0.5° angular diameter of the Sun smears out the bow (which has a radial width of about 0.5° from violet to red), I see not a well-defined circular bow, but rather a blazing white fireball with a beautiful red outer edge and a less easily seen blue inner region superposed on white background. If I do not use the mirror, I see encircling the shadow of my head a brilliant white halo from the retroreflection with δ greater than 182°. So I call this stuff Glory Dust.  

For more about rainbows and halos, see the magnificent article by Jearl Walker, the beautiful book by Minnaert, the fine articles by Nussenzveig and by Bryant and Jarmie, the magnificent historical and mathematical study by Boyer, the delightful book by Jearl Walker, and the incredibly beautiful book by Greenler.

APPENDIX: THE 3-D BOW SEEN WITH A POINT SOURCE ABOVE RAINBOW DUST. WHERE IS IT?

The rainbow dust is sprinkled over the x-y plane. The point light source S is located at x = y = 0, at height z = h above the x-y plane. See Fig. 1. The observer’s two eyes (R and L) have separation 2D, and are both located at the same height z = H (with H > h). The eyes are aligned along the x axis, and are centered over the source, so that they both have y = 0, and eyes R and L have x = +D and −D, respectively. The observer can look straight down to see the source, or swivel his eyes to the right (without turning his head, so as to maintain z = H for both eyes) to see the part of the rainbow B that crosses the positive x axis. The line of sight from the R eye through the rainbow intercepts the x-y plane at z = 0, y = 0, x = x(R). The line of sight from the left eye L through the rainbow intercepts the plane at z = 0, y = 0, x = x(L). These two lines [R to x(R) and L to x(L)] intersect at the point B common to both lines of sight (after the observer “fuses” the two images). Point B has coordinates x = x_B and z = z_B. By inspection of the sketch, the ray Sx(R) from the source S to the rainbow-dust grain at x = x(R) makes an angle to the vertical equal to \( \tan^{-1}\left(\frac{x(R)}{h}\right) \). The reflected ray x(R), R from that grain to the R eye located at x = D, z = H, makes an angle to the vertical equal, by inspection, to \( \tan^{-1}\left(\frac{x(R) - D}{H}\right) \). This difference between these two angles is what we have called the “viewing angle” \( \theta \) (180° minus the deviation), which equals 21.9° for \( n = 1.51 \), for the primary bow with \( k = 1 \). Thus for the path S to x(R) to R, we have

\[
21.9° = \theta = \tan^{-1}\left(\frac{x(R) - D}{H}\right) - \tan^{-1}\left(\frac{x(R)}{h}\right)
\]

(A1)

Given numerical values of h, H, D, and \( \theta \) we easily solve Eq. (A1) for x(R) by using the “solve” button on our HP15C. Similarly, for the left eye, located at x = −D, height H, we have

\[
21.9° = \theta = \tan^{-1}\left(\frac{x(L) - D}{H}\right) - \tan^{-1}\left(\frac{x(L)}{h}\right)
\]

(A2)

which can be similarly solved for x(L).

Points x, z on the R, x(R) line of sight satisfy (by similar triangles and inspection of the sketch)

\[
z/H = (x(R) - x)/x(R) - D
\]

(A3)

Points x, z on the L, x(L) line of sight similarly satisfy

\[
z/H = (x(L) - x)/x(L) + D
\]

(A4)

The values of x, z on the line R, x(R) differ from those on the line L, x(L), except at their common point of intersection at the perceived bow B. The intersection at the perceived bow B is thus found by equating the right-hand sides of Eqs. (A3) and (A4), solving for x = x_B, and then inserting this x_B into either Eqs. (A3) or (A4) to get z_B. We thus find

\[
x_B/D = (x(L) + x(R))/(x(L) - x(R) + 2D)
\]

(A5)

and

\[
z_B/H = (x(L) - x(R))/(x(L) - x(R) + 2D)
\]

(A6)

Experimentally, for h = 16.5 cm and H = 80 cm, and using my depth perception to locate the bow, I measured a bow radius x_B = 6.2 cm at a height z_B = 20.5 cm. For my eyes, I measure 2D = 6.4 cm. For \( \theta_B = 21.9° \), h = 16.5 cm, and H = 80 cm, Eqs. (A1) and (A2) give x(R) = 7.748 cm and x(L) = 10.028 cm. Then Eq. (A5) gives x_B = 6.567 cm, which agrees fairly well with my experimental value of 6.2 cm, and Eq. (A6) gives z_B = 20.77 cm, which agrees fairly well with my measured value of 20.5 cm.

Equations (A5) and (A6) may give the false impression that x_B is proportional to D and z_B proportional to H, but, of course, x(L) and x(R) also depend on D and H, through Eqs. (A1) and (A2). For our example, if we keep h = 16.5 cm but double H from 80 cm to 160 cm, the bow radius x_B only changes from 6.57 to 6.63 cm, and z_B only from 20.8 to 19.6 cm. Experimentally, the 3-D bow position in space is not noticed to change at all as one moves one’s head up and down so as to vary H. Going back to h = 16.5, H = 80, if we could double our eye separation from 6.4 to 12.8 cm (!) that would only change the perceived bow radius from 6.57 to 6.58 cm, but would greatly change the perceived bow height, from 21.0 to 33.1 cm. So if you disagree with your friends as to the vertical location of the bow, you had better measure the separation 2D between their eyes and compare it with yours.

Fig. 1. Geometry of the 3-D bow from rainbow dust. The angles S, x(R), and S, x(L), L are each equal to the rainbow angle 21.9°. When the left (L) and right (R) eyes "fuse" their images to get depth perception, the R eye looks along the line R, x(R) and the L eye looks along the line L, x(L). The bow is perceived to be suspended in space at the intersect point B of these two lines.

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For the numbers chosen, \( h = 16.5 \), \( H = 80 \), and \( 2D = 6.4 \text{ cm} \), the bow looks, experimentally, like a circle. But all that we actually calculated above was the intersection \( x_B \) of the bow with the \( x \) axis, which we then called the radius. If we direct our eyes along the \( y \) axis, instead of along the \( x \) axis, and calculate the intersection of the perceived bow with the \( y \) axis at \( y = y_B \) for the same height \( x_B = 20.77 \text{ cm} \) as found above, then I calculate \( y_B = 6.566 \text{ cm} \), which is essentially the same as the result \( x_B = 6.567 \text{ cm} \) found above. (I leave the formulas to the reader.) So the bow is indeed very close to being a circle. However, that is partly an accident of my eye separation. If I take the "wide eyed" \( 2D = 12.8 \text{ cm} \), then, for \( h = 16.5 \) and \( H = 80 \), I find \( x_B = 6.58 \text{ cm} \) but \( y_B = 5.07 \text{ cm} \). Thus the perceived bow is ellipselike, with major axis along the line connecting the two eyes.

If your eyes are not directly above the light source, the bow is strongly ellipselike rather than circelike, and is "tilted" relative to the \( x-y \) plane. It is very pretty. But I have not tried to find the equations.

ACKNOWLEDGMENTS

I thank Charles Wohl and Richard Muller for helpful comments and suggestions, Cynthia Broderson and Larry Webster for providing me with my first sample of Rainbow Dust, and Jack Navone, of Potters Industries, Inc., for a sample of Glory Dust.

1Highway Safety Spheres, Type 11 M.P. 0587 VG RG, index \( n = 1.51 \), manufactured by Potters Industries Inc. Anaheim, CA. Sold retail at about 
25 per 50 lbs at traffic and roadway suppliers all over the country, for example, at Hawkins Safety Products, 1255 E. Bayshore, Berkeley, CA. Potters also makes airport runway safety spheres with index \( n = 1.9 \).
4M. Minnaert, The Nature of Light and Colour in the Open Air (Dover, New York, 1954), Chap. X.
6Manufactured by MAG INSTRUMENT, Ontario, California.

The division of the Martian eccentricity from Hipparchos to Kepler: A history of the approximations to Kepler motion

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Any planetary theory with pretensions of quantitative validity must grapple with the inequality of movement due to Kepler motion. This was so in antiquity no less than today. The technical details of six historically important planetary models are discussed. The relationships of the models to one another and to the real motions of the planets are examined with simple geometrical techniques. The discussion centers on the history of the attempts to provide a model for the motion of Mars.

I. INTRODUCTION

In roughest approximation, the planets may be considered to travel at constant speed upon circular orbits centered on the Sun. This model of planetary motion may be designated the "zero-eccentricity model," as it corresponds to Kepler motion in the limit of an ellipse of zero eccentricity. The uniform circular motion of the zero-eccentricity model lends itself to elementary instruction in physics and astronomy; it serves well for many order-of-magnitude calculations. But it does not agree with the motions of the planets. Already in the fifth century B.C., the departure of the Sun from the zero-eccentricity model became apparent in the unequal lengths of the four seasons. In the case of Mars, Kepler motion produces a striking inequality in the widths and spacings of the planet's retrograde arcs. This inequality was not only known in antiquity, but appears to have played an important part in Ptolemy's discovery of the equant. No planetary theory with pretensions of quantitative validity can fail to deal with the inequalities due to Kepler motion. Hipparchos' eccentric circle model for the Sun, Ptolemy's equant point, Copernicus' minor epicycle, and the "vicarious hypothesis" that Kepler employed before his discovery of the law of areas and the ellipticity of the orbits—all these represent historically important efforts to grapple with the departures of the planets from the simple motions of the zero-eccentricity model.

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