Volume and Surface Area of an $N$-Sphere

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(February 4, 2003)

1 Problem

Deduce expressions for the volume and surface area of a (Euclidean) $N$-sphere.

2 Solution

This solution follows http://db.uwaterloo.ca/~alopez-o/math-faq/node75.html by A. Lopez-Ortiz. Who first gave this solution?

We expect that the volume $V_N$ of an $N$-sphere varies with its radius $r$ as,

\[ V_N = C_N r^N, \tag{1} \]

where the $C_N$ are constants to be determined.

If we consider the $N$-sphere to be made up of a set of concentric shells, then the volume $dV_N$ of a shell of radius $r$ and thickness $dr$ is related the surface area $A_N$ of the shell by,

\[ dV_N = A_N dr. \tag{2} \]

Thus,

\[ A_N = \frac{dV_N}{dr} = N C_N r^{N-1}. \tag{3} \]

A clever method to evaluate the $C_N$ is to consider the integral of $e^{-r^2}$ in both rectangular and “spherical” coordinates,

\[
\begin{align*}
\int e^{-r^2} \, dV_N &= \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N \, e^{-x_1^2 - \cdots - x_N^2} = \left( \int_{-\infty}^{\infty} dx \, e^{-x^2} \right)^N = \pi^{N/2} \\
&= \int_{0}^{\infty} e^{-r^2} \, A_N \, dr = N C_N \int_{0}^{\infty} e^{-r^2} \, r^{N-1} \, dr = \frac{N C_N}{2} \int_{0}^{\infty} e^{-s} \, s^{N/2-1} \, ds \\
&= \frac{N C_N}{2} \Gamma(N/2) = \frac{N C_N}{2} (N/2 - 1)! = \frac{N C_N (N/2)!}{N/2} = C_N (N/2)! \tag{4}
\end{align*}
\]

Thus,

\[ C_N = \frac{\pi^{N/2}}{(N/2)!}, \tag{5} \]

so the volume and surface area of an $N$-sphere are,

\[ V_N = \frac{\pi^{N/2}}{(N/2)!} r^N, \quad A_N = N \frac{\pi^{N/2}}{(N/2)!} r^{N-1}. \tag{6} \]
An expression for \((N/2)!\) for odd integer \(N\) can be deduced from the fact that,

\[
\Gamma(1/2) = \int_0^\infty e^{-s}s^{-1/2} \, ds = 2 \int_0^\infty e^{-r^2} \, dr = \sqrt{\pi}
\]  

(7)

and the recurrence relation,

\[
\Gamma(x + 1) = x\Gamma(x).
\]  

(8)

Thus,

\[
(N/2)! = \Gamma(N/2 + 1) = \sqrt{\pi} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{N}{2} = \sqrt{\pi} \cdot \frac{1}{2} \cdot \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{2} \cdot \cdots \frac{N}{2} \cdot \frac{N+1}{2} = \sqrt{\pi} \cdot \frac{(N + 1)!}{(N/2)! \cdot 2^{N+1}} \quad \text{(odd \(N\)).}
\]  

(9)

With this, we find the first few \(C_N\) to be,

\[
C_1 = 2, \quad C_2 = \pi, \quad C_3 = \frac{4}{3}\pi, \quad C_4 = \frac{\pi^2}{2}, \quad C_5 = \frac{8\pi^2}{15}, \quad C_6 = \frac{\pi^3}{6}, \quad C_7 = \frac{16\pi^3}{105}, \quad C_8 = \frac{\pi^4}{24}.
\]  

(10)

For large \(N\), Stirling’s approximation yields,

\[
C_N \approx \frac{1}{\sqrt{N\pi}} \left(\frac{2e\pi}{N}\right)^{N/2} \quad (N \gg 1).
\]  

(11)