Noether's Theorem Is Not So Simple

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Noether's theorem [1] is a "law of physics" that symmetries (invariance principles) are associated with conservation laws. In classical mechanics, invariance of the behavior of a system under translation in space, or in time, or under a rotation about an axis, is associated in "simple enough" systems with conservation of linear momentum, energy, and angular momentum, respectively. Clarification of whether a system is "simple enough" requires an understanding of the calculus of partial derivatives, so is beyond the typical level of introductory Newtonian mechanics.

A recent article [2] gave a discussion supposing the reader is only interested in "simple enough" systems, rather than the full power of Noether's theorem. For example, it was stated on p. 354 that "provided the laws of physics (especially those of the physical interactions) remain unchanged when a closed system anywhere in a given region is transported from one moment in time to another—thereby corresponding to temporal translational symmetry—the system's total energy must be conserved." Similarly on p. 355 it was stated that "provided the laws of physics (especially those of the physical interactions) are unchanged when a closed system is translated from any location in a region of space to another arbitrary location in that region—thereby corresponding to spatial translational symmetry—linear momentum must be conserved within that region."

The term "closed" was used 13 times in Ref. [2] without being defined. The author (private communication) indicated that its meaning is given by the statements [3]: "Let us now consider a system of particles which interact with one another but with no other bodies. This is called a closed system."

By this definition a "closed system" excludes all examples that involve the acceleration g due to the Earth.

If the "closed system" involves electrically charged particles, the mechanical momentum of these particles is not conserved, in general, because mechanical momentum can be exchanged with electromagnetic-field momentum. Hence, the discussion in [2] does not apply to charged particles.

The "closed systems" in Ref. [2] are largely limited to "billiard-ball" collisions in "empty" space". In such collisions the total force and torque on the system are always zero, according to Newton's third law, so total linear momentum and total angular momentum are conserved without need to consider "symmetry" arguments. It is possible that two balls stick together when they collide, so the "mechanical" kinetic energy of the system decreases, yet the "laws of physics" do not depend on time. Here, the "total" energy is conserved if we also consider the "internal" energy of the balls, but "symmetry" arguments have little to say about this.

The simplified discussion of Noether's theorem in Ref. [2] has so many qualifications that it applies to very few examples, which are typically so elementary that little insight is added. In contrast, Noether helped clarify the existence of conserved quantities in nontrivial cases provided a Hamiltonian/Lagrangian description of the system can be given.

Our understanding that the total energy of a system is conserved predates Noether's theorem.

Regarding time-translation "symmetry", Noether's actual statement is that if the Hamiltonian, $H = \sum_i p_{q_i}\dot{q}_i + T - V$, of a system with coordinates q_i (where $\dot{q}_i = dq_i/dt$), corresponding canonical momenta p_{q_i}, kinetic energy T and potential energy V, does not depend explicitly on time, *i.e.*, $\partial H/\partial t = 0$, then the Hamiltonian is a conserved quantity, *i.e.*, $dH/dt = 0$. This is not necessarily the energy of the system, as discussed in sec. 2.7 of the classic textbook of Goldstein [4]. Thus, Noether's theorem for time-translation "symmetry" is not about energy in general, but about other possible conserved quantities. However, the distinction here is not elementary.

Noether's theorem also relates to a consequence of Lagrange's analysis that if both the kinetic energy $T(q_i, \dot{q}_i, t)$ and the potential energy $V(q_i, \dot{q}_i, t)$ of a system described by a coordinates q_i with $\dot{q}_i = dq_i/dt$ do not depend explicitly on coordinate q_i, *i.e.*, $\partial T/\partial q_i = 0 = \partial V/\partial q_i$, then the canonical momentum, $p_{q_i} = \partial\,(T - V)/\partial\dot{q}_i$, associated with coordinate q_i is conserved.

The example of a (small) mass m that slides without friction on the inside of a cylinder that rolls without slipping on a horizontal plane has kinetic energy and potential energy that do not depend explicitly on the horizontal coordinate x of the mass.[1] According to Noether there is a conserved (canonical) momentum in this example, but it turns out not to be the linear (horizontal) momentum $m\,dx/dt$. This example was discussed in [5], with the unfortunate title "Breakdown of the connection between symmetries and conservation laws for semiholonomic systems". In fact, Noether's theorem holds for this example, as pointed out in Ref. [6].

Going beyond Newtonian physics, gravitational effects propagate with the speed of light, and the gravitational field can be regarded as containing energy, momentum and angular momentum that are globally conserved according to observers at "infinity" [7] (which includes the observation of black-hole mergers in the LIGO experiment). However, local conservation of energy, momentum and angular momentum does not hold in general relativity, as shown by the second theorem of Noether, also stated in the article of 1918 [1] that included her more well known first theorem (the topic of Ref. [2]). See, for example, Refs. [8, 9].

A symmetry not mentioned in [2] is invariance of quantum physics under local changes in the phase of the wave function. The corresponding conservation law is conservation of "charge", as in electromagnetism. While mass/energy can be called the "charge" of gravity, it is not conserved locally in general relativity according to Noether's second theorem [8, 9]. This suggests that a quantum theory of gravity should restore local conservation of mass/energy, which has been elusive.

References

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[2] E. Hecht, *Noether's Theorem and Newton's Laws*, Phys. Teach. **64**, 353-358 (2026).

[1] This example involves the acceleration g due to Earth's gravity, so the simplified discussion of Ref. [2] does not apply here.

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