Noether's Theorem Is Not So Simple

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Noether's theorem is a "law of physics" that symmetries (invariance principles) are associated with conservation laws. In classical mechanics, invariance of the behavior of a system under translation in space, or in time, or under a rotation about an axis, is associated in "simple enough" systems with conservation of linear momentum, energy, and angular momentum, respectively. Clarification of whether a system is "simple enough" requires an understanding of the calculus of partial derivatives, so is beyond the typical level of introductory Newtonian mechanics.

A recent article [1] gave a simplified discussion of Noether's theorem for classical mechanics, and in particular stated on p. 355 that "provided the laws of physics (especially those of the physical interactions) are unchanged when a closed system is translated from any location in a region of space to another arbitrary location in that region—thereby corresponding to spatial translational symmetry—linear momentum must be conserved within that region".

This qualification is satisfied by the motion of a particle in free fall in a uniform gravitational field (without any other interaction). While the linear momentum of the particle is constant in any horizontal direction, it changes with time in the vertical direction.

The simplified version of Noether's theorem stated in [1] does not hold, but the actual version stated by Noether does hold. Her version relates to a consequence of Lagrange's analysis that if both the kinetic energy $T(q, \dot{q}, t)$ and the potential energy $V(q, \dot{q}, t)$ of a system described by a coordinate q with $\dot{q} = dq/dt$ do not depend explicitly on coordinate q, *i.e.*, $\partial T/\partial q = 0 = \partial V/\partial q$, then the so-called canonical momentum, $p_q = \partial(T - V)/\partial\dot{q}$, associated with coordinate q is conserved (constant in time).

Since the potential energy of a particle in a uniform gravitational field depends explicitly on the vertical coordinate, the vertical (linear) momentum of the particle is not conserved.[1]

The example of a (small) mass m that slides without friction on the inside of a cylinder that rolls without slipping on a horizontal plane has kinetic energy and potential energy that do not depend explicitly on the horizontal coordinate x of the mass. According to Noether there is a conserved (canonical) momentum in this example, but it turns out not to be the linear (horizontal) momentum $m\, dx/dt$. This example was discussed in [5], with the unfortunate title "Breakdown of the connection between symmetries and conservation laws for semiholonomic systems". In fact, Noether's theorem holds for this example, as pointed out in [6].

It was also stated on p. 354 of [1] that "provided the laws of physics (especially those of the physical interactions) remain unchanged when a closed system anywhere in a given re-

[1]Going beyond Newtonian physics, gravitational effects propagate with the speed of light, and the gravitational field can be regarded as containing energy, momentum and angular momentum that are globally conserved according to observers at "infinity" [2] (which includes the observation of black-hole mergers in the LIGO experiment). However, local conservation of energy, momentum and angular momentum does not hold in general relativity, as shown by the second theorem of Noether, also stated in the article of 1918 that included her more well known first theorem (the topic of [1]). See, for example, [3, 4].

gion is transported from one moment in time to another—thereby corresponding to temporal translational symmetry—the system's total energy must be conserved". However, Noether's actual statement is that if the Hamiltonian, $H = \sum_i p_{q_i}\dot{q}_i + T - V$, of a system with coordinates q_i does not depend explicitly on time, $\partial H/\partial t = 0$, then the Hamiltonian is a conserved quantity, $dH/dt = 0$. This is not necessarily the energy of the system, as discussed in sec. 2.7 of the classic textbook of Goldstein [7].

A symmetry not mentioned in [1] is invariance of quantum physics under local changes in the phase of the wave function. The corresponding conservation law is conservation of "charge", as in electromagnetism. While mass/energy can be called the "charge" of gravity, it is not conserved locally in general relativity according to Noether's second theorem [3, 4]. This suggests that a quantum theory of gravity should restore local conservation of mass/energy, which has been elusive.

References

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