Heaviside and the "Murder" of the Potentials

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(November 3, 2026)

Oliver Heaviside was well known for his acerbic charactierzation of ideas he did not like. Here we review his antipathy towards the electromagnetic vector and scalar potentials $\mathbf{A}$ and V.

The concept of a scalar potential for gravity was introduced in 1747 by Daniel Bernoulli [1], and Poisson [2, 3] (1812 and 1824) considered electric and magnetic scalar potentials (assuming electricity is due to electric charges and magnetism is due to magnetich charges/poles). These potentials were generally regarded as useful mathematical constructs, rather than "physical" entities.

In 1831, Faraday [4] intuited aspects of the electromagnetic vector potential $\mathbf{A}$, which he called the electronic state, and since then attitudes as to the significance of the vector potential have varied considerably.[1]

Maxwell (1864) discussed the "electromagnetic momentum of a current" in sec. 22 of [9], and identified this in sec. 59 with what he later called the vector potential.[2,3] That is, Maxwell regarded the electromagnetic vector potential as more "physical" than the electric scalar potential.

In secs. 98-99 of [9], Maxwell gave the first discussion of what is now called a gauge transformation, which he noted can change the form of the vector potential. To maintain the interpretation of the vector potential as an "electromagnetic momentum" (per unit charge), Maxwell argued that the vector potential should obey $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, which is now called the Coulomb-gauge condition.

Maxwell (1873) returned to this theme in Arts. 616-617 of [8] where he claimed that the vector potential has the form $\mathbf{A} = \int \mathbf{J}\, d\mathrm{Vol}/cr$ (in Gaussian units as will be used in this Note) where c is the speed of light in vacuum. We now recognize this as a possible form of the vector potential only if the electric-current density $\mathbf{J}$ is static (*i.e.*, $\mathbf{J}/\partial t = 0$). Maxwell did not live long enough to reflect on the conundrum that his vision of the vector potential as an "electromagnetic momentum" applied only to static examples, in which the total momentum should be zero, so if there is a nonzero "electromagnetic momentum" there must also be an equal and opposite "hidden" mechanical momemtum.[4]

Heaviside, one of Maxwell's immediate followers, was skeptical of the significance of the vector potential, and stated in 1886 [16], "the very artificial nature of $\mathbf{A}$ and P (*the electric*

[1] Gauss, p. 609 of [5] (published posthumously in 1867), claimed to have invented a version of the vector potential in 1835. The vector potential is often attributed to Neumann (1845) [6], but he only considered a scalar magnetic-field (potential) energy rather than a vector potential. The earliest published example of a vector potential is due to Thomson (1846), p. 63 of [7]. The first published use of the term "vector potential" is in Art. 405 of Maxwell's *Treatise* (1873) [8].

[2] In contrast, Maxwell stated (1881) [10] "The electrical potential ... is a mere scientific concept. We have no reason to regard it as denoting a physical state."

[3] The development of Maxwell's thinking on electromagnetism is discussed in, for example, Refs. [11, 12, 13], and in Appendix A.28 of Ref. [14].

[4] It is now recognized that the "hidden" mechanical momentum is a relativistic effect associated with the current density $\mathbf{J}$. See, for example [15].

scalar potential) greatly obscures and complicates many investigations". In 1889 [17] he added that "the fact that $\mathbf{A}$ has often a scalar potential parasite ... causes sometimes great mathematical complexity and indistinctness; and it is, for practical reasons, best to murder the whole lot, or at any rate merely employ them as subsidiary functions."

In 1893, p. 293 of [18], Heaviside added some scathing remarks about potential energy: "Potential energy, when regarded merely as expressive of the work that can be done by forces depending upon configuration, does not admit of much argument. It is little more than a mathematical idea, for there is scarcely any physics in it. It explains nothing."

In 1890, Hertz [20] characterized the electromagnetic potentials as "scaffolding ... which serves for calculation only".[5]

In 1900 Poincaré [22] argued that the volume density $\mathbf{p}_{\rm field}$ of electromagnetic-field momentum is the Poynting vector $\mathbf{S} = (c/4\pi)\,\mathbf{E}\times\mathbf{B}$ divided by c^2, where $\mathbf{E}$ and $\mathbf{B}$ are the electromagnetic fields.[6] Then, the electromagnetic-field momentum is related by $\mathbf{P}_{\rm field} = \int \mathbf{p}_{\rm field}\, d{\rm Vol} = \int \mathbf{E}\times\mathbf{B}\, d{\rm Vol}/4\pi c$.

The electromagnetic-field momentum $\mathbf{P}_{\rm field}$ is a global property of the electromagnetic fields and depends on the details of their distant sources.

In general, electromagnetic-field momentum is not associated with the vector potential, although in static examples the electromagnetic-field momentum can also be computed (following Maxwell, and in Gaussian units) as $\mathbf{P}_{\rm field} = \int \rho\mathbf{A}^{\rm (C)}\, d{\rm Vol}/c$ where $\mathbf{A}^{\rm (C)}(\mathbf{x}) = \int \mathbf{J}(\mathbf{x}')\, d{\rm Vol}/cr$ is the vector potential in the Coulomb gauge supposing that $\mathbf{A}^{\rm (C)} = 0$ at infinity, ρ is the electric-charge density, $\mathbf{J}$ is the electric-current density and $r = |\mathbf{r}| = |\mathbf{x}-\mathbf{x}'|$. This reinforces that the vector potential is "useful" and "valid", but it does not imply that the vector potential is "physical", as suggested in [24]-[33].[7] In science, the term "physical" should be reserved for entities that exist in Nature, and not applied to concepts such as words, numbers, mathematical symbols, and potentials, however useful they may be in discussions of "physical" objects.

In static examples the electromagnetic-field momentum can be computed in other ways as well. The form $\mathbf{P}_{\rm field} = \int V^{\rm (C)}\mathbf{J}\, d{\rm Vol}/c^2$, where $V^{\rm (C)} = \int \rho\, d{\rm Vol}/r$ is the electric scalar potential in the Coulomb gauge was first advocated by Furry [35]. And, the form $\mathbf{P}_{\rm field} = \int (\mathbf{J}\cdot\mathbf{E})\,\mathbf{r}\, d{\rm Vol}/c^2$ where $\mathbf{E} = \int \rho\,\hat{\mathbf{r}}\, d{\rm Vol}/r^2$ was introduced by Aharonov, Pearle and Vaidman [36].

The density of momentum in the electromagnetic field is $\mathbf{p}_{\rm field} = \mathbf{S}/c^2 = \mathbf{E}\times\mathbf{B}/4\pi c$, and not $\rho\mathbf{A}^{\rm (C)}/c$ nor $V^{\rm (C)}\mathbf{J}/c^2$ nor $(\mathbf{J}\cdot\mathbf{E})\,\mathbf{r}/c^2$.

Recall that the canonical momentum of an electric charge q with (rest) mass m and velocity $\mathbf{v}$ in an electromagnetic field can be written as $\mathbf{p}_{\rm canon} = \mathbf{p}_{\rm mech} + q\mathbf{A}/c$ for the vector potential $\mathbf{A}$ in any gauge, with $\mathbf{p}_{\rm mech} = m\mathbf{v}/\sqrt{1-v^2/c^2}$. Then, the Hamiltonian, $H = c[m^2c^2 + (\mathbf{p}_{\rm canon} - q\mathbf{A}/c)^2]^{1/2} + qV$, of the electric charge in an electromagnetic field appears not to be gauge invariant, but the equation of motion, obtained by taking derivatives

[5] A recent review of the history of attitudes about the vector potential [21] notes that its role as "scaffolding" was changed in quantum theory, where it is more fundamental than in classical theory, as mentioned at the end of this note.

[6] For a brief history of this insight, see p. 246 of [23].

[7] These references could be characterized as being about four topics: 1. Canonical momentum and mechanics, Refs. [24] and [32]; 2. Pedagogic understanding, Refs.[25], [28] and [31]; 3. Gauge conditions and physical interpretation, Refs. [27], [30] and [33]; 4. Applications emphasizing potentials, Refs.[26] and [29].

of the Hamiltonian, is gauge invariant, being $m\,\mathbf{a} = q(\mathbf{E} + \mathbf{v}/c \times \mathbf{B})$, *i.e.*, the Lorentz force law. This is perhaps the most significant descendant of Maxwell's notion that the vector potential is a kind of "electromagnetic momentum".[8]

Maxwell's partial understanding of the gauge invariance of the electromagnetic potentials was made more explicit by Lorentz in 1905, p. 157 of [37]. See also p. 673 of [38]. That is, the physical fields $\mathbf{E}$ and $\mathbf{B}$ can be deduced from potentials $\mathbf{A}$ and V according to $\mathbf{E} = -\boldsymbol{\nabla} V - \partial\mathbf{A}/\partial ct$ and $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, while the nonphysical potentials can be changed arbitrarily, with no effect on $\mathbf{E}$ or $\mathbf{B}$, according to $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi$ and $V \to V + \partial\chi/\partial ct$ for any (doubly) differentiable function χ.

In quantum theory, where the Hamiltonian plays a key role in Schrödinger's equation, the appearance of the electromagnetic potentials in the Hamiltonian of a charged particle in an electromagnetic field suggests that the unphysical potentials are nonetheless "fundamental".[9]

In 1926, Fock [42] noted that Schrödinger's equation for an electric charge in an electromagnetic field is gauge invariant only if the gauge transformation of the potentials, $A_\mu(x_\nu) \to A_\mu + \partial_\mu\chi(x_\nu)$, is accompanied by a phase change of the quantum wavefunction, $\psi(x_\nu) \to e^{-ie\chi(x_\nu)/\hbar c}\,\psi$. See also [43, 44]. Yang and Mills (1954) [45, 46] were the first to point out that Fock's argument can be inverted such that a requirement of local phase invariance of the form $\psi(x_\nu) \to e^{-ie\chi(x_\nu)/\hbar c}\,\psi$ implies the existence of an interaction described by a 4-potential A_μ which satisfies gauge invariance, namely the electromagnetic interaction.[10] This emphasizes the significance of potentials in the quantum realm, and has led to the understanding of the electroweak and strong interactions as further examples of "gauge theories".[11]

It can remain unsettling that nonphysical potentials play a key role in determining what type of physical fields are possible in Nature.

We close by noting that while Heaviside argued against interpreteting the electromagnetic vector potential too physically, he remained commited to the concept of a mechanical æther, although he did say (p. 352 of [48]): "...mechanical speculations in general should be received with much caution, and regarded rather as illustrations or analogies than expressions of fact. We do not know enough yet about the æether for dogmatising."[12] However, it was left to Einstein to "murder" the æther.

[8]The canonical momentum of a charge in an electromagnetic field may have first appeared in quantum theory on p. 611 of Dirac's great paper [34] that deduced his famous equation.

[9]Because the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be deduced from knowledge of the electromagnetic potentials V and $\mathbf{A}$, the latter are often considered to be more "fundamental" than the former. However, a version of the potentials V and $\mathbf{A}$ can be deduced from the fields $\mathbf{E}$ and $\mathbf{B}$ via an argument attributed to Poincaré [39], so it is not self evident which is more "fundamental" than the other. Consideration of the vector potential in the Poincaré gauge [40] for the Aharanov-Bohm effect [41] indicates that one should not suppose the vector potential always provides a "local" explanation of this effect.

[10]Note that an implication of local phase invariance of the quantum wave function ψ is that the wave function ψ (like the electromagnetic potentials) is not physical.

[11]See, for example, sec. I of [38], and [47].

[12]In 1911, §534 of [49], Heaviside reconsidered electrodynamics for electrons moving with velocity close to the speed of light, but still presumed the existence of an electromagnetic æther. Nonetheless, his subsection (17), p. 444 of [49] gave an insightful way to think about what is now called synchroton radiation, as popularized by Feynman in Chap. 34 of [50].

References

[1] D. Bernoulli, *Commentationes de immutatione et extensione principii conservationis virium viuarum, quae pro motu corporum coelestium requiritur*, Comment. Acad. Sci. Imp. Petropol **10**, 116 (1747). [Written in 1738.]
https://www.biodiversitylibrary.org/item/113649#page/128/mode/1up
https://kirkmcd.princeton.edu/examples/mechanics/bernoulli_casip_10_116_47.pdf

[2] S.D. Poisson, *Sur la Distribution de l'Électricité a la Surface des Corps Conducteurs*, Mém. Inst. Imp., 1 (1812), 163 (1813).
https://digital.deutsches-museum.de/en/digital-catalogue/library-object/BV022482627/
kirkmcd.princeton.edu/examples/EM/poisson_12.pdf

[3] S.D. Poisson, *Sur la Théorie du Magnétisme*, Mém. Acad. Roy. Sci. **5**, 247, 486 (1826).
https://catalog.hathitrust.org/Record/011822778
https://kirkmcd.princeton.edu/examples/EM/poisson_24.pdf

[4] M. Faraday, *Experimental Researches in Electricity*, Phil. Trans. Roy. Soc. London **122**, 125-162 (1832). See Art. 60. https://www.jstor.org/stable/107956
kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_122_163_32.pdf

[5] C.F. Gauss, *Werke*, Vol. 5 (Göttingen, 1867).
https://archive.org/details/Werkecarlf05gausrich/page/n620/mode/1up
kirkmcd.princeton.edu/examples/EM/gauss_werke_v5.pdf

[6] F.E. Neumann, *Allgemeine Gesetze der inducirten elektrischen Ströme*, Abh. König. Akad. Wiss. Berlin, pp. 1-87 (1845).
https://archive.org/details/diemathematische00neumuoft
kirkmcd.princeton.edu/examples/EM/neumann_akawb_1_45.pdf

[7] W. Thomson, *On the Mathematical Theory of Electricity in Equilibrium*, Cam. Dublin Math. J. **1**, 75 (1846). https://babel.hathitrust.org/cgi/pt?id=cub.u183019465063&seq=83
kirkmcd.princeton.edu/examples/EM/thomson_cdmj_1_75_46.pdf

[8] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2 (Clarendon Press, Cambridge, UK, 1873). https://archive.org/details/electricandmagne02maxwrich
http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_73.pdf

[9] J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459-512 (1865). doi/10.1098/rstl.1865.0008
kirkmcd.princeton.edu/examples/EM/maxwell_ptrsl_155_459_65.pdf

[10] J.C. Maxwell, *An Elementary Treatise on Electricity* (Clarendon Press, 1881). See p. 53.
https://archive.org/details/elementarytreati00maxwrich/page/56/mode/1up
http://kirkmcd.princeton.edu/examples/EM/maxwell_elementary_treatise_81.pdf

[11] A.M. Bork, *Maxwell and the Vector Potential*, Isis **58**, 210-222 (1967).
https://www.jstor.org/stable/228226
http://kirkmcd.princeton.edu/examples/EM/bork_isis_58_210_67.pdf

[12] J.C. Maxwell, *Origins of Clerk Maxwell's Electric Ideas: As Described in Familiar Letters to William Thomson*, Ed. by J. Larmor (Cambridge University Press, 1937). See p. 56, *etc.*

[13] E.T. Whittaker, *A History of the Theories of Aether and Electricity* (Longmans, Green, 1910). https://archive.org/details/historyoftheorie00whitrich/page/n5/mode/2up
kirkmcd.princeton.edu/examples/EM/whittaker_history.pdf

[14] K.T. McDonald, *Is Faraday's Disk Dynamo a Flux Rule Exception?* (July 27, 2019).
http://kirkmcd.princeton.edu/examples/faradaydisk.pdf

[15] D. Babson *et al.*, *Hidden momentum, field momentum, and electromagnetic impulse*, Am. J. Phys. **77**, 826-833 (2009). See p. 828. https://doi.org/10.1119/1.3152712
http://kirkmcd.princeton.edu/examples/EM/babson_ajp_77_826_09.pdf

[16] O. Heaviside, *On the Self-induction of Wires*, Phil. Mag. **22**, 118-138 (1886). See p. 124.
https://doi.org/10.1080/14786448608627908
https://kirkmcd.princeton.edu/examples/EM/heaviside_pm_22_118_86.pdf

[17] O. Heaviside, *The general solution of Maxwell's electromagnetic equations in a homogeneous isotropic medium, especially in regard to the derivation of special solutions, and the formulæ for plane waves*, Phil. Mag. **27**, 29-50 (1889). See p. 45.
https://doi.org/10.1080/14786448908628314
https://kirkmcd.princeton.edu/examples/EM/heaviside_pm_22_118_86.pdf

[18] O. Heaviside, *Gravitational and Electromagnetic Analogy*, Electrician **31**, 281 (1893).
https://kirkmcd.princeton.edu/examples/EM/heaviside_electrician_31_281_93.pdf
Also p. 462 of [19].

[19] O. Heaviside, *Electromagnetic Theory*, Vol. 1 (Macmillan, 1893).
https://archive.org/details/bub_gb_9ukEAAAAYAAJ/page/n3/mode/2up
https://kirkmcd.princeton.edu/examples/EM/heaviside_electromagnetic_theory_1.pdf

[20] H. Hertz, *Ueber die Grundgleichungen der Electrodynamik für ruhende Körper*, Ann. d. Phys. **40**, 577-624 (1890). See p. 578. https://doi.org/10.1002/andp.18902760803
https://kirkmcd.princeton.edu/examples/EM/hertz_ap_40_577_90.pdf
Translation in H. Hertz, *Electric Waves* (Macmillan, 1900). See p. 107.
https://archive.org/details/electricwavesbe00jonegoog/page/n219/mode/1up
http://kirkmcd.princeton.edu/examples/EM/hertz_electric_waves.pdf

[21] M.F.M. Urán, *The Price of Removing the Scaffolding* (Sept. 22, 2026).
https://arxiv.org/pdf/2609.28525

[22] H. Poincaré, *La Théorie de Lorentz et la Principe de Réaction*, Arch. Neer. **5**, 252 (1900). https://henripoincarepapers.univ-lorraine.fr/chp/hp-pdf/hp1900an.pdf
http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00.pdf
English translation: http://physicsinsights.org/poincare-1900.pdf
http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00_english.pdf

Poincaré's paper deals with macroscopic electrodynamics, in which $\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{B}$ with $\mathbf{E} = (f, g, h)$, $\mathbf{B} = (\alpha, \beta, \gamma)$. The equations on p. 6 before eqs. (4) state that the electromagnetic-field-energy density u $(= K_0 J)$ times the energy flow velocity $\mathbf{U}$ equals the Poynting vector $\mathbf{S}$, *i.e.*, $u\mathbf{U} = \mathbf{S}$. [Poincaré's J is not related to the electric-current density denoted by $\mathbf{J}$.] Just after his eq. (7) Poincaré stated that the field-energy density u equals the effective mass density $\rho_{\rm eff}$ times c^2, *i.e.*, $u = \rho_{\rm eff}\, c^2$. Then, we have $\rho_{\rm eff}\mathbf{U}c^2 = \mathbf{g}\, c^2 = \mathbf{S}$, where $\mathbf{g} = \rho_{\rm eff}\mathbf{U}$ is the field-momentum density.

[23] K.T. McDonald, *The Poynting vector should be* $\mathbf{E}\times\mathbf{B}/\mu_0$ *not* $\mathbf{E}\times\mathbf{H}$, Am. J. Phys. **94**, 245-247 (2026). https://doi.org/10.1119/5.0297408
http://kirkmcd.princeton.edu/examples/EM/mcdonald_ajp_94_245_26.pdf

[24] E.J. Konopinski, *What the electromagnetic vector potential describes*, Am. J. Phys. **46**, 499-502 (1978). https://doi.org/10.1119/1.11298
http://kirkmcd.princeton.edu/examples/EM/konopinski_ajp_46_499_78.pdf

[25] M.D. Semon and J.R. Taylor, *Thoughts on the magnetic vector potential*, Am. J. Phys. **64**, 1361-1369 (1996). https://doi.org/10.1119/1.18400
http://kirkmcd.princeton.edu/examples/EM/semon_ajp_64_1361_96.pdf

[26] N.K. Georgieva and H.W. Tam, *Potential Formalisms in Electromagnetic-Field Analysis*, IEEE Trans. Micro. Tech. Meas. **51**, 1330-1338 (2003).
https://doi.org/10.1109/TMTT.2003.809188
https://kirkmcd.princeton.edu/examples/EM/georgieva_ieeemtt_51_1330_03.pdf

[27] G. Rousseaux, *On the physical meaning of the gauge conditions of Classical Electromagnetism: the hydrodynamics analogue viewpoint*, Ann. Fond. Louis de Broglie **28**, 261-269 (2003). https://hal.science/hal-00013234
https://kirkmcd.princeton.edu/examples/EM/rousseaux_aflb_28_261_03.pdf

[28] D. Iencinella and G. Matteucci, *An introduction to the vector potential*, Eur. J. Phys. **25**, 249-256 (2004). https://doi.org/10.1088/0143-0807/25/2/011
https://kirkmcd.princeton.edu/examples/EM/iencinella_ejp_25_249_04

[29] G. Rousseaux, R. Kofman and O. Minazzoli, *The Maxwell-Lodge effect: Significance of electromagnetic potentials in the classical theory*, Eur. Phys. J. D **49**, 249-256 (2008).
https://doi.org/10.1140/epjd/e2008-00142-y
https://kirkmcd.princeton.edu/examples/EM/rousseaux_epjd_49_249_08.pdf

[30] G. Giuliani, *Vector potential, electromagnetic induction and 'physical meaning'*, Eur. J. Phys. **31**, 871-880 (2010). http://dx.doi.org/10.1088/0143-0807/31/4/017
https://kirkmcd.princeton.edu/examples/EM/giuliani_ejp_31_871_10.pdf

[31] S. Barbieri, M. Cavinato and M. Giliberti, *An educational path for the magnetic vector potential and its physical implications*, Eur. J. Phys. **34**, 1209-1219 (2013).
http://dx.doi.org/10.1088/0143-0807/34/5/1209
https://kirkmcd.princeton.edu/examples/EM/barbieri_ejp_34_1209_13.pdf

[32] J. Adams and C. Dunston, *Numerical modeling of the magnetic vector potential and the associated electromagnetic momenta*, Am. J. Phys. **94**, 113-119 (2026).
https://doi.org/10.1119/5.0246553
http://kirkmcd.princeton.edu/examples/EM/adams_ajp_94_113_26.pdf

[33] A.K. Singal, *Electromagnetic momentum in the Aharonov-Bohm quantum interference experiment from a physical perspective*, to appear in Can. J. Phys. (2026).
https://arxiv.org/html/2301.06502v2
http://kirkmcd.princeton.edu/examples/EM/singal_cjp_26.pdf

[34] P.A.M. Dirac, *The Quantum Theory of the Electron*, Proc. Roy. Soc. (London) **117** 610-624 (1928). https://doi.org/10.1098/rspa.1928.0023
https://kirkmcd.princeton.edu/examples/QED/dirac_prsla_117_610_28.pdf

[35] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621-636 (1969).
https://doi.org/10.1119/1.1975729
http://kirkmcd.princeton.edu/examples/EM/furry_ajp_37_621_69.pdf

[36] Y. Aharonov, P. Pearle and L. Vaidman, *Comment on "Proposed Aharonov-Casher effect: Another example of an Aharonov-Bohm effect arising from a classical lag,"* Phys. Rev. A **37**, 4052-4055 (1988). https://doi.org/10.1103/PhysRevA.37.4052
http://kirkmcd.princeton.edu/examples/QM/aharonov_pra_37_4052_88.pdf

[37] H.A. Lorentz, *Weiterbildung der Maxwellischen Theorie. Elektronentheorie*, in *Encyklopädie der Mathematischen Wissenschaften*, Band 5, Teil 2 (1904), pp. 145-280.
https://kirkmcd.princeton.edu/examples/EM/lorentz_emw_5_2_145_04.pdf

[38] J.D. Jackson and L.B. Okun, *Historical roots of gauge invariance*, Rev. Mod. Phys. **73**, 663-680 (2001). https://doi.org/10.1103/RevModPhys.73.663
https://kirkmcd.princeton.edu/examples/EM/jackson_rmp_73

[39] W.E. Brittin, W.R. Smythe and W. Wyss, *Poincaré gauge in electrodynamics*, Am. J. Phys. **50**, 693-696 (1982). https://doi.org/10.1119/1.12731
https://kirkmcd.princeton.edu/examples/EM/brittin_ajp_50_693_82.pdf

[40] K.T. McDonald, *Vector Potential of a Long Solenoid in the Poincaré Gauge* (January 15, 2017). https://kirkmcd.princeton.edu/examples/poincare.pdf

[41] Y. Aharonov and D. Bohm, *Significance of Electromagnetic Potentials in Quantum Theory*, Phys. Rev. **115**, 485-491 (1959). https://doi.org/10.1103/PhysRev.115.485
http://kirkmcd.princeton.edu/examples/QM/aharonov_pr_115_485_59.pdf

[42] V. Fock, *Über die invariante Form der Wellen- und der Bewegungsgleichungen für einen geladenen Massenpunkt*, Z. Phys. **39**, 226-232 (1926).
https://doi.org/10.1007/BF01321989
http://kirkmcd.princeton.edu/examples/QM/fock_zp_39_226_26.pdf
http://kirkmcd.princeton.edu/examples/QM/fock_zp_39_226_26_english.pdf

[43] F. London, *Quantenmechanische Deutung der Theorie von Weyl*, Z. Phys. **42**, 375-389 (1927). https://doi.org/10.1007/BF01397316
http://kirkmcd.princeton.edu/examples/QM/london_zp_42_375_27.pdf

[44] W. Pauli, *Relativistic Field Theories of Elementary Particles*, Rev. Mod. Phys. **13**, 203-232 (1941). See p. 206. https://doi.org/10.1103/RevModPhys.13.203
http://kirkmcd.princeton.edu/examples/EP/pauli_rmp_13_203_41.pdf

[45] C.N. Yang and R.L. Mills, *Conservation of Isotopic Spin and Isotopic Gauge Invariance*, Phys. Rev. **96**, 191-195 (1954). https://doi.org/10.1103/PhysRev.96.191
http://kirkmcd.princeton.edu/examples/EP/yang_pr_96_191_54.pdf

[46] See also R. Utiyama, *Invariant Theoretical Interpretation of Interaction*, Phys. Rev. **101**, 1597-1607 (1956). https://doi.org/10.1103/PhysRev.101.1597
http://kirkmcd.princeton.edu/examples/EP/utiyama_pr_101_1597_56.pdf

[47] H. Gomes and B.W. Roberts, *Gauge Matters More Than Curvature* (Sept. 14, 2026). https://arxiv.org/abs/2609.13017

[48] O. Heaviside, *Electromagnetic Theory—XLIII*, Electrician **30**, 351 (1893).
https://kirkmcd.princeton.edu/examples/EM/heaviside_electrician_30_351_93.pdf
Also p. 327 of [19].

[49] O. Heaviside, *Electromagnetic Theory*, Vol. 3 (Macmillan, 1912).
https://archive.org/details/electromagnetict03heavuoft/page/n5/mode/2up
https://kirkmcd.princeton.edu/examples/EM/heaviside_electromagnetic_theory_3.pdf

[50] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics*, Vol. 1, Chap. 34, (Addison-Wesley, 1964), https://www.feynmanlectures.caltech.edu/I_34.html