Is There a Mass Shift of a Permanent Magnetic Moment in an External, Static Magnetic Field?

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1 Introduction

The existence of a "mass shift",

$$m = m_0 - \frac{\boldsymbol{\mu} \cdot \mathbf{B}_{\rm ext}}{c^2}\,, \tag{1}$$

(where m is the rest mass of a permanent magnetic moment $\boldsymbol{\mu}$ such as that of an electron, proton or neutron, whose rest mass in zero magnetic field is m_0, c is the speed of light in vacuum, and $\mathbf{B}_{\rm ext}$ is an external, static magnetic field) has been argued from time to time, perhaps starting with a brief remark by Frenkel in the final paragraph of [1]. See also eq. (18·9) of [2], eq. (8′), p. 1837 of [3], eq. (3.17), p. 1621 of [4], Appendix B of [5],[1] eq. (22), p. 15 of [7], p. 4 of [8],[2] and p. 64 of [9].[3]

Here, we consider whether the magnetic-field interaction energy,

$$U_{\boldsymbol{\mu},\mathbf{B}_{\rm ext}} = \boldsymbol{\mu} \cdot \mathbf{B}_{\rm ext}, \tag{2}$$

which is "tied" to the magnetic moment $\boldsymbol{\mu}$, should be "renormalized" into the rest mass of the moment (as is the self magnetic-field energy of the moment), which would restore the value of the rest mass m of the moment to m_0.

2 An Argument for the "Mass Shift"

An argument, as in sec. 4.3 of [9], is that when, a permanent magnetic moment $\boldsymbol{\mu}$ is drawn into an external, static magnetic field $\mathbf{B}_{\rm ext}$ with $\boldsymbol{\mu}$ parallel to $\mathbf{B}_{\rm ext}$, then the moment takes on positive kinetic energy. To conserve energy, some other energy must have decreased.

The force $\mathbf{F}$ on a magnetic moment $\boldsymbol{\mu}$ in an external magnetic field $\mathbf{B}_{\rm ext}$ can be written as,

$$\mathbf{F} = (\boldsymbol{\mu} \cdot \boldsymbol{\nabla})\mathbf{B}_{\rm ext}, \tag{3}$$

whether the moment is due to equal and opposite magnetic "poles" (Gilbertian [13, 14]), or to electric currents (Ampèrian, [15, 16]). Clarification that permanent magnetism, due to

[1] For comments by the author on this paper, see [6].

[2] This paper notes on p. 8 that the "mass shift" is not predicted in either the classical Foldy-Wouthuysen model or in Dirac's quantum theory of the electron.

[3] A "mass shift" associated with potential energy has been discussed by Brillouin [10, 11]. See also [12].

the magnetic moments of electrons, is Ampèrian (rather than Gilbertian) came only after detailed studies of positronium (e^+e^- "atoms") in the 1940's [17, 18].

For a static, external magnetic field, $\boldsymbol{\nabla}\times\mathbf{B}_{\rm ext} = 4\pi\mathbf{J}_{\rm ext}/c$, where $\mathbf{J}_{\rm ext}$ is the source-current density, and we use Gaussian units in this note. At the location of the magnetic moment, $\boldsymbol{\nabla}\times\mathbf{B}_{\rm ext} = 0$, so eq. (3) can be rewritten as,[4,5,6]

$$\mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}) = -\boldsymbol{\nabla}(-\boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}) = -\boldsymbol{\nabla}U_{\rm potential}, \tag{4}$$

where $U_{\rm potential} = -\boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}$ can be called the magnetic potential energy of the system.[7]

We suppose the system starts "at rest" with moment $\boldsymbol{\mu}$ far from the sources of the external magnetic field $\mathbf{B}_{\rm ext}$, such that the initial, total energy of the system is,

$$U_0 = U_{\boldsymbol{\mu},{\rm internal},0} + U_{\mathbf{B}_{\rm ext},{\rm internal},0}, \tag{5}$$

where we suppose that the energies of the self magnetic fields of $\boldsymbol{\mu}$ and $\mathbf{B}_{\rm ext}$ are part of their internal energies.

There is no classical model of a permanent magnetic moment as due to electric currents, although we say that permanent moment is Ampèrian, as noted, for example, in [17, 18]. We are left with the view that the internal energy of a permanent magnetic moment is just its rest energy mc^2, where m is the rest mass of the moment.

The final energy $U_{\rm f}$ of the system includes the kinetic energy of the moment $\boldsymbol{\mu}$ and of the sources of $\mathbf{B}_{\rm ext}$, although we will neglect the latter, and suppose that the external magnetic field (and its sources) remains at rest, and is independent of time (static).

The final kinetic energy of the magnetic moment is related by,

$$\mathrm{KE}_{\boldsymbol{\mu},{\rm f}} = \int_0^{\rm f} \mathbf{F}\cdot d\mathbf{x} = \int_0^{\rm f} \boldsymbol{\nabla}(\boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext})\cdot d\mathbf{x} = \boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}. \tag{6}$$

Conservation of energy can then be written as,

$$U_0 = U_{\rm f} = \mathrm{KE}_{\boldsymbol{\mu},{\rm f}} + U_{\boldsymbol{\mu},{\rm internal},{\rm f}} + U_{\mathbf{B}_{\rm ext},{\rm internal},{\rm f}} + U_{\boldsymbol{\mu},\mathbf{B}_{\rm ext}}, \tag{7}$$

where $U_{\boldsymbol{\mu},\mathbf{B}_{\rm ext}}$ is the magnetic-field interaction energy,

$$U_{\boldsymbol{\mu},\mathbf{B}_{\rm ext}} = \boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}, \tag{8}$$

as computed in sec. 4.3 of [9], and in Appendix A below by another method.

Combining eqs. (5)-(8), we have,

$$U_{\boldsymbol{\mu},{\rm internal},{\rm f}} + U_{\mathbf{B}_{\rm ext},{\rm internal},{\rm f}} = U_{\boldsymbol{\mu},{\rm internal},0} + U_{\mathbf{B}_{\rm ext},{\rm internal},0} - \boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext} - U_{\boldsymbol{\mu},\mathbf{B}_{\rm ext}}. \tag{9}$$

If the external magnetic field $\mathbf{B}_{\rm ext}$ is due to currents driven by "batteries", it could be that the internal energy of the permanent moment $\boldsymbol{\mu}$ does not change as the moment enters the

[4]Equation (4) also holds for a magnetic moment with steady currents held constant by a "battery".

[5]For a different derivation of eq. (4), see Sec. 2.18 of [19].

[6]The force (4) is called the Stern-Gerlach force in the German literature.

[7]Maxwell wrote the potential energy of a magnetization density $\mathbf{M}$ in an external magnetic field as $U_{\rm potential} = -\int \mathbf{M}\cdot\mathbf{H}_{\rm ext}\, d{\rm Vol}$ in eq. (6), Art. 389 of [20], based on a model of Gilbertian magnetic moments.

external field, while the internal energy of the "batteries" does change. Hence, it is more pertinent to consider the case of two permanent magnetic moments.

The discussion above assumes that the "external" magnetic field $\mathbf{B}_{\rm ext}$ is static, so for a system of two permanent magnetic moments $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ we presume that moment $\boldsymbol{\mu}_2$ is held at rest by some force, which does no work. Then, eq. (9) becomes (with $\boldsymbol{\mu} \to \boldsymbol{\mu}_1$),

$$U_{\boldsymbol{\mu}_1,\rm internal,f} + U_{\boldsymbol{\mu}_2,\rm internal,f} = U_{\boldsymbol{\mu}_1,\rm internal,0} + U_{\boldsymbol{\mu}_2,\rm internal,0} - \boldsymbol{\mu}_1 \cdot \mathbf{B}_2 - U_{\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}, \qquad (10)$$

where the final magnetic-field interaction energy is,

$$U_{\boldsymbol{\mu}_1,\boldsymbol{\mu}_2} = \boldsymbol{\mu}_1 \cdot \mathbf{B}_2 = \boldsymbol{\mu}_2 \cdot \mathbf{B}_1 = \frac{3(\boldsymbol{\mu}_1 \cdot \hat{\mathbf{r}})(\boldsymbol{\mu}_2 \cdot \hat{\mathbf{r}}) - \boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2}{r^3}\,, \qquad (11)$$

with $\mathbf{r}$ as the position vector from moment 1 to moment 2. Then, eq. (10) takes the form,

$$m_1 c^2 + m_2 c^2 = m_{0,1} c^2 - \boldsymbol{\mu}_1 \cdot \mathbf{B}_2 + m_{0,2} c^2 - \boldsymbol{\mu}_2 \cdot \mathbf{B}_1, \qquad (12)$$

which suggests that the rest mass m of a permanent magnetic moment $\boldsymbol{\mu}$ in an external, static magnetic field $\mathbf{B}_{\rm ext}$ experiences the "mass shift" (1).

3 Comments

3.1 Can the Rest Mass Be Negative?

Since an electron has a permanent magnetic moment, eq. (1) should apply to it. If so, the rest mass of an electron would be negative if its magnetic moment, of magnitude $e\hbar/2m_0c$, is parallel to a magnetic field that exceeds (twice) the so-called QED critical field $B_{\rm critical} = m_0^2 c^3/e\hbar = 4.4 \times 10^{13}$ gauss. Such fields exist at the surface of magnetars [21].

On p. 15 of [7] it was argued that when the rest mass of an electron is negative, the direction of the cyclotron motion of the electron about the magnetic field vector is reversed in a very strong field, compared to that in lower magnetic fields.

The "mass shift" (1) of an electron in a laboratory magnetic field of 1 T is less than a part per billion, so would have little effect on "everyday" electrodynamics.

3.2 Does the "Mass Shift" Affect the Dirac Equation?

The view of Wald (private communication) is that the "mass shift", eq. (1), does not affect the Dirac equation, where the mass in that equation is the rest mass m_0 in zero magnetic field, even when the (spin-1/2) Dirac particle is in an external, static magnetic field. In this view, the permanent magnetic moment $e\hbar/2m_0c$ of an electron, is not affected by the "mass shift" (1).[8]

[8] An electron in an electromagnetic plane wave experiences a different kind of mass shift, first noted by Volkov [22, 23, 24], that $m = m_0\sqrt{1+\eta^2} \ge m_0$, where $\eta = eE/m_0\omega c$ for an electron in a circularly polarized plane wave with (rms) electric field strength E and angular frequency ω. The Dirac equation for an electron in a plane electromagnetic wave involves the shifted mass m, rather than the unshifted mass m_0.

In a classical view the shifted mass m is the "transverse mass" associated with the transverse oscillations (of amplitude less than the wavelength of the electromagnetic wave) of the electron in the wave, while in a quantum view the electron is a "quasiparticle", "dressed" by photons of the electromagnetic wave.

The "mass shift" (1), if it existed, would affect the kinetic energy, momentum, and orbital angular momentum of an electron, and hence the energy levels of an atom, and the Landau levels of an electron,[9] in an external, static magnetic field should be shifted by a part per billion in a static field of a few Tesla (which is not predicted in standard quantum theory).

3.3 Renormalization of the Mass Shift

Since the magnetic-field interaction energy $\boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}$ is "tied" to the electron, the gravitational interaction of an electron in a magnetic field includes this mass/energy, and it seems that the "gravitational mass" of an electron (for low velocities [25]) is $m+\boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}/c^2 = m_0$. That is, the gravitational interaction is not sensitive to the "mass shift" (1).

Then, if we "renormalize" the magnetic-field interaction energy $\boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}$ into the effective rest energy of an electron, such that the effective mass of the electron in an external magnetic field is just its rest mass m_0 in zero magnetic field, the "mass shift" (1) plays no dynamical role in electromagnetic interactions (or the gravitational interaction). It may remain comforting to imagine the "mass shift" (1) as an explanation for where the magnetic-field interaction energy (24) comes from, but it has no direct physical consequence.

However, this leaves the consideration of the energetics of a permanent magnetic moment in an external, static magnetic field somewhat ambiguous. As noted in sec. 2 above, if a permanent magnetic moment $\boldsymbol{\mu}$ is drawn into an external, static magnetic field $\mathbf{B}_{\rm ext}$ with $\boldsymbol{\mu}$ parallel to $\mathbf{B}_{\rm ext}$, then the moment takes on positive kinetic energy. To conserve energy, some other energy must have decreased. The magnetic-field interaction energy $\boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}$ has increased, while the magnetic potential energy $-\boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}$ has decreased. As noted in secs. 4.3 and 5.1 of [9], the rate of exchange of energy between electric current density $\mathbf{J}$ and the electromagnetic fields is $\mathbf{J}\cdot\mathbf{E}$, with the implication that the magnetic field $\mathbf{B}$ does not participate in the exchange of electromagnetic energy. Hence, the magnetic-field interaction energy $\boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}$ is of little relevance here, in contrast to the magnetic potential energy $-\boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}$. We can take the view that the effective magnetic-field interaction energy in work-energy considerations is $-\boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}$ rather than $\boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}$. This notion is presented in most textbooks, without discussion of the subtleties considered here.

Our attitude towards possible "renormalization" of the "mass shift" (1) into the rest mass of the magnetic moment depends on how independent we consider the magnetic-field interaction energy (8) to be from the "physical" magnetic moment.

[9]A semiclassical analysis of the lowest Landau level of an electron of rest mass m and velocity $\mathbf{v}$ in a circular orbit of radius r in magnetic field $\mathbf{B}$ is that its angular momentum is $\gamma m v r = \hbar$, where m is the rest mass of the electron in the $\mathbf{B}$ field, not necessarily the electron rest mass m_0 in zero magnetic field, and $\gamma = 1/\sqrt{1-v^2/c^2}$. Then, Newton's 2$^{\rm nd}$ law tells us that $\gamma m v^2/r = evB/c$ for an orbit in a plane perpendicular to $\mathbf{B}$. We now have that $\gamma m v = eBr/c$, and hence that $r^2 = \hbar c/eB$, independent of the value of m. For $B = B_{\rm critical} = m_0^2 c^3/e\hbar$, the radius of the orbit is $r = \hbar/m_0 c$, the Compton wavelength of an electron of rest mass m_0. For $B > B_{\rm critical}$, where the rest mass m becomes negative according to eq. (1), there is no classical description of the electron.

3.3.1 The Magnetic-Field Interaction Energy for Two Permanent Magnetic Moments

We consider further the case of two permanent magnetic moments, mentioned at the end of sec. 2 above, now assuming that they are electrically neutral (for example, neutrons). Again, moment 2 is somehow held at rest, while moment 1 moves towards it. Then, there is no electric field associated with moment 2, although the moving moment 1, with velocity $\mathbf{v}_1$, does have an electric field of order $\mu_1 v_1/c$ (as well as corrections of order $1/c$ to the static approximation to the magnetic field of moment 1).

The density of energy in the electromagnetic field is $u_{\rm EM} = (E^2 + B^2)/8\pi$. As usual in classical electromagnetism, we "renormalize" the self-field energies of the moments into their rest masses, leaving the interaction energy density as,

$$u_{\rm EM,int}(\mathbf{r}) = \frac{\mathbf{E}_1 \cdot \mathbf{E}_2 + \mathbf{B}_1 \cdot \mathbf{B}_2}{4\pi} \tag{13}$$

In the case of the two magnetic moments, with $\mathbf{E}_2 = 0$, the interaction energy density is just,

$$u_{\rm EM,int}(\mathbf{r}) = \frac{\mathbf{B}_1 \cdot \mathbf{B}_2}{4\pi}\,. \tag{14}$$

This energy density is large only extremely close to the two moments, so it is not very distinct from the rest energy of the moments.

In particular, we consider the region close to moment 1, taking it to be at the origin at the time of interest, with moment 2 at position $-R\,\hat{\mathbf{x}}$. We also suppose both magnetic moment vectors are parallel to $\hat{\mathbf{z}}$. Then, the magnetic fields close to moment 1 are,

$$\mathbf{B}_1(\mathbf{r}) \approx \frac{3(\boldsymbol{\mu}_1 \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} - \boldsymbol{\mu}_1}{r^3} + \frac{8\pi}{3}\boldsymbol{\mu}_1\,\delta^3(\mathbf{r}), \qquad \mathbf{B}_2(\mathbf{r}) \approx \frac{3(\boldsymbol{\mu}_2 \cdot \hat{\mathbf{z}})\,\hat{\mathbf{z}} - \boldsymbol{\mu}_2}{R^3} = \frac{2\mu_2\,\hat{\mathbf{z}}}{R^3}, \tag{15}$$

recalling that for an Ampèrian "point" magnetic moment there is a delta function in the magnetic field at the moment.[10] The interaction energy density close to moment 1 is, in spherical coordinates r, θ, ϕ with polar axis z,

$$u_{\rm EM,int}(\mathbf{r}) \approx \boldsymbol{\mu}_1 \cdot \mathbf{B}_2(\mathbf{r}=0)\left(\frac{3\cos^2\theta - 1}{4\pi r^3} + \frac{2}{3}\,\delta^3(\mathbf{r})\right) \tag{16}$$

The volume integral of the first term of eq. (16) is zero for any sphere of radius a about moment 1 such that eq. (16) is valid, and hence,

$$U_{\rm EM,int}(r < a) = \int_{r<a} u_{\rm EM,int}(\mathbf{r})\, d{\rm Vol} = \frac{2}{3}\boldsymbol{\mu}_1 \cdot \mathbf{B}_2(\text{at } \boldsymbol{\mu}_1). \tag{17}$$

That is, 2/3 of the interaction energy (11) is considered to be "inside" a permanent magnetic moment in classical electrodynamics.

The suggestion here is to consider that the other 1/3 of that energy, while nominally outside the moment, should be "renormalized" into the moment, and all of the interaction energy (11) be regarded as part of the rest energy of the moment.

[10] See, for example, eq. (5.64) of [26].

3.4 Magnetic Moment Maintained by a "Battery"

Is is less clear that the magnetic-field interaction energy should be "renormalized" into the rest energy of the system when the electric currents (and associated magnetic moments) are driven by a "battery", when in an external magnetic field, as we have a better classical model of such currents than of a permanent magnetic moment.

We note that Joule heating of the resistive medium depletes the stored energy of the "battery", which does experiences a tiny reduction of mass as a result, independent of the "mass shift (1). For example, a 1.5-V AA battery has a mass of about 20 grams, and stored electrical energy of about 2 Ah $\approx 10^4$ joule $= 10^{11}$ erg. The mass corresponding to this stored energy is $10^{11}/c^2 \approx 10^{-10}$ g, leading to a downward "mass shift" of about 5 parts per trillion of the mass of the battery as it is depleted.

A Appendix: The Magnetic-Field Interaction Energy

In general, the magnetic field energy can be computed as,

$$U_{\rm B} = \int \frac{\mathbf{B}^2}{8\pi}\, d{\rm Vol}. \tag{18}$$

In the present example, $\mathbf{B} = \mathbf{B}_{\boldsymbol{\mu}} + \mathbf{B}_{\rm ext}$, so the magnetic-field interaction energy is,

$$U_{\boldsymbol{\mu},\mathbf{B}_{\rm ext}} = \int \frac{\mathbf{B}_{\boldsymbol{\mu}} \cdot \mathbf{B}_{\rm ext}}{4\pi}\, d{\rm Vol}. \tag{19}$$

For static magnetic $\mathbf{B}_{\rm ext}$ this energy can be rewritten as,

$$\begin{aligned} U_{\boldsymbol{\mu},\mathbf{B}_{\rm ext}} = \int \frac{\mathbf{B}_{\rm ext} \cdot \boldsymbol{\nabla} \times \mathbf{A}_{\boldsymbol{\mu}}}{4\pi}\, d{\rm Vol} = \int \frac{\mathbf{A}_{\boldsymbol{\mu}} \cdot \boldsymbol{\nabla} \times \mathbf{B}_{\rm ext}}{4\pi}\, d{\rm Vol} + \int \frac{\boldsymbol{\nabla} \cdot (\mathbf{A}_{\boldsymbol{\mu}} \times \mathbf{B}_{\rm ext})}{4\pi}\, d{\rm Vol} \\ = \int \frac{\mathbf{J}_{\rm ext} \cdot \mathbf{A}_{\boldsymbol{\mu}}}{c}\, d{\rm Vol}, \end{aligned} \tag{20}$$

where $\mathbf{A}_{\boldsymbol{\mu}}$ is the vector potential for moment $\boldsymbol{\mu}$, $\mathbf{J}_{\rm ext}$ is the electric current density that is the source of $\mathbf{B}_{\rm ext}$, and we suppose that $\mathbf{A}$ and $\mathbf{J}$ fall off sufficiently quickly at large distances so that $\int \boldsymbol{\nabla} \cdot (\mathbf{A}_{\boldsymbol{\mu}} \times \mathbf{B}_{\rm ext})\, d{\rm Vol} = \oint_\infty \mathbf{A}_{\boldsymbol{\mu}} \times \mathbf{B}_{\rm ext} \cdot d\mathbf{Area}$ is negligible.

The vector potential of a magnetic moment $\boldsymbol{\mu}$ was deduced by Thomson (Lord Kelvin) in 1846 [27] as,[11]

$$\mathbf{A}_{\boldsymbol{\mu}} = \frac{\boldsymbol{\mu} \times \hat{\mathbf{r}}}{r^2}\,. \tag{21}$$

The external magnetic field $\mathbf{B}_{\rm ext}$ is nearly uniform in the region of the (small) magnetic moment $\boldsymbol{\mu}$ so for purposes of computation of the interaction field energy, it suffices to consider a magnetic field that is uniform with a sphere of radius a centered on $\boldsymbol{\mu}$. As in Prob. 12(a)

[11]This was the first published use of a vector potential. Equation (21) also follows from a multipole expansion of the vector potential, as on p. 84 of [28].

of [29], this uniform magnetic field could be due to a uniform magnetization density $\mathbf{M} = 3\mathbf{B}_{\rm ext}/8\pi$, which is equivalent to a surface current density on the sphere of radius a,

$$\mathbf{K_e} = cM \sin\theta\, \hat{\boldsymbol{\phi}} = \frac{3cB}{8\pi} \sin\theta\, \hat{\boldsymbol{\phi}}, \tag{22}$$

in a spherical coordinate system (r, ϕ, θ) with its z-axis along $\mathbf{B}_{\rm ext} = B_{\rm ext}\,\hat{\mathbf{z}}$. The interaction field energy is then,

$$\begin{aligned} U_{\boldsymbol{\mu},\mathbf{B}_{\rm ext}} &= \int \frac{\mathbf{J_e}\cdot\mathbf{A}_{\boldsymbol{\mu}}}{c}\, d\text{Vol} = \int \frac{\mathbf{K_e}\cdot\mathbf{A}_{\boldsymbol{\mu}}}{c}\, d\text{Area} = \frac{a^2}{c}\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta\, \frac{3cB_{\rm ext}}{8\pi}\sin\theta\,\hat{\boldsymbol{\phi}}\cdot\frac{\boldsymbol{\mu}\times\hat{\mathbf{r}}}{a^2} \\ &= \frac{3B_{\rm ext}}{8\pi}\int_0^{2\pi} d\phi \int_{-1}^{1} \sin\theta\, d\cos\theta\, \boldsymbol{\mu}\cdot\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = -\frac{3B_{\rm ext}}{8\pi}\int_0^{2\pi} d\phi \int_{-1}^{1} \sin\theta\, d\cos\theta\, \boldsymbol{\mu}\cdot\hat{\boldsymbol{\theta}}. \end{aligned} \tag{23}$$

We have that $-\hat{\boldsymbol{\theta}} = -\cos\theta\cos\phi\,\hat{\mathbf{x}} - \cos\theta\sin\phi\,\hat{\mathbf{y}} + \sin\theta\,\hat{\mathbf{z}}$, which leads to,[12]

$$U_{\boldsymbol{\mu},\mathbf{B}_{\rm ext}} = \frac{3\boldsymbol{\mu}\cdot B_{\rm ext}\,\hat{\mathbf{z}}}{8\pi}\int_0^{2\pi} d\phi \int_{-1}^{1} \sin^2\theta\, d\cos\theta = \boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}, \tag{24}$$

independent of the radius a of the sphere within which the external field $\mathbf{B}_{\rm ext}$ is uniform.[13]

Of course, a must be large enough to contain the magnetic moment $\boldsymbol{\mu}$, so that the external field is uniform over the entire moment.

B Appendix: $\mathbf{J}\cdot\mathbf{E} = 0$ for Two Parallel, Permanent Magnetic Moments

We consider further the case of two permanent magnetic moments, mentioned at the end of sec. 2 above, now assuming that they are electrically neutral (for example, neutrons). Again, moment 2 is somehow held at rest, while moment 1 moves towards it. Then, there is no electric field associated with moment 2, although the moving moment 1, with velocity $\mathbf{v}_1$, does have an electric field of order $\mu_1 v_1/c$ (as well as corrections of order $1/c$ to the static approximation to the magnetic field of moment 1).

The magnetic moments have no intrinsic electric-dipole moments when at rest, but in general a moving magnetic moment appears to have an electric-dipole moment $\mathbf{p} = \mathbf{v}/c\times\boldsymbol{\mu}$ for low velocity $\mathbf{v}$, while the magnetic moment remains $\boldsymbol{\mu}$ in the low-velocity limit.[14] For

[12] Swapping subscripts $\boldsymbol{\mu}$ and ext in eq. (20) leads to the alternative form, $U_{\boldsymbol{\mu},\mathbf{B}_{\rm ext}} = \int \mathbf{J}_{\boldsymbol{\mu}}\cdot\mathbf{A}_{\rm ext}\, d\text{Vol}/c$. A derivation of $U_{\boldsymbol{\mu},\mathbf{B}_{\rm ext}} = \boldsymbol{\mu}\cdot\mathbf{B}_{\rm ext}$ via this form is given in Sec. 4.3 of [9].

[13] A naïve inference is that the interaction field energy (24) is located within the magnetic moment $\boldsymbol{\mu}$ if the latter is spherical, and occupies the same volume as does the internal energy of the moment. However, we should recall that there is nonzero magnetic field outside the sphere, corresponding to a magnetic dipole of moment $4\pi a^3\mathbf{M}/3 = a^3\mathbf{B}_{\rm ext}/2$, and this field contributes to the interaction energy (19). The portions of the interaction energy, of density $\mathbf{B}_{\boldsymbol{\mu}}\cdot\mathbf{B}_{\rm ext}/4\pi$, inside and outside the sphere of radius a varies with the radius, but the total interaction energy is independent of a.

[14] The special relativistic transformation of electric and magnetic dipole moments was first given by Lorentz (1910) [30]. Some elaboration of this theme is given in [31].

the present example, where $\boldsymbol{\mu}_1$ and $\mathbf{v}$ are parallel, the apparent electric-dipole moment of the moving moment $\boldsymbol{\mu}_1$ is zero, and there is no electric field due to this effect.

We can also consider the electric field of the accelerating magnetic moment $\boldsymbol{\mu}_1$, which is given by eq. (24) of [32] as,[15]

$$\mathbf{E}_1(\mathbf{r},t) = \frac{1}{4\pi c^2}\frac{d^2}{dt^2}\left[\frac{\hat{\mathbf{R}}\times(\hat{\mathbf{R}}\times\boldsymbol{\mu}_1)}{R}\left(1-\frac{1}{c}\frac{dR}{dt}\right)\right] + \frac{1}{4\pi c}\frac{d}{dt}\left[\frac{\boldsymbol{\mu}_1\times\hat{\mathbf{R}}}{R^2}\left(1-\frac{1}{c}\frac{dR}{dt}\right)\right], \quad (25)$$

where the magnetic moment $\boldsymbol{\mu}_1$ is at $\mathbf{r}'$, with $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. This field vanishes at the moment $\boldsymbol{\mu}_2$ as $\boldsymbol{\mu}_1$, $\dot{\boldsymbol{\mu}}_1$ and $\ddot{\boldsymbol{\mu}}_1$ are parallel to $\mathbf{R}$ in the present example.

Hence,

$$\mathbf{J}\cdot\mathbf{E} = \mathbf{J}_2\cdot\mathbf{E}_1 + \mathbf{J}_1\cdot\mathbf{E}_2 = 0, \quad (26)$$

noting that $\mathbf{E}_2 = 0$ everywhere.

A general argument of Poynting [33] is that,

$$\frac{\partial u_{\rm EM}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} = -\mathbf{J}\cdot\mathbf{E}, \quad (27)$$

where $u_{\rm EM} = (E^2 + B^2)/8\pi$ is the density of energy in the electromagnetic field.

From eqs. (26) and (27) we have that,

$$\frac{\partial u_{\rm EM,int}}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{S}_{\rm int}, \quad (28)$$

where the interaction Poynting vector, $\mathbf{S}_{\rm int} = (c/4\pi)\,\mathbf{E}_1\times\mathbf{B}_2$ since $\mathbf{E}_2 = 0$, has a first-order term, as well ones of order $1/c$ and higher.[16]

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[15] The second term of eq. (25) corresponds to the field of the apparent electric-dipole moment of the moving magnetic moment. The magnetic field of the accelerated magnetic moment includes terms of order $1/c$ and $1/c^2$ that were neglected in eq. (11) above.

[16] If we consider that the magnetic moment of a neutron is of order $e\hbar/m_0 c$, then $\boldsymbol{\mu}\propto 1/c$ and $\mathbf{B}_{\boldsymbol{\mu}}\propto 1/c$, such that both $u_{\rm EM,int}$ and $\mathbf{S}_{\rm int}$ are of order $1/c^2$.

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