Canonical Electromagnetic Momentum and the Electromagnetic Momentum of Abraham and Poynting

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1 Problem

This note was originally a letter to David Jackson. His response is in [25].

The usual lore of electrodynamics gives us two approaches to the calculation of the interaction energy/momentum of a charged particle in an external electromagnetic field.

We should expect these two approaches to be equivalent, and indeed they are in all the examples that I have tried. Yet, the formal equivalence of these two approaches is never discussed, to my knowledge, in the literature.

I will label these two approaches as "Maxwell" and "Poynting".

Maxwell: The canonical 4-momentum P_μ of a particle of electric charge e and mechanical 4-momentum p_μ in an external electromagnetic field with 4-potential $A_\mu^{\rm ext}$ is given by

$$P_\mu = p_\mu + \frac{e}{c} A_\mu^{\rm ext}. \tag{1}$$

It seems a reasonable interpretation that the term $eA_\mu^{\rm ext}/c$ be regarded as the interaction energy/momentum of the particle in the external field.

Poynting: The electromagnetic energy $U_{\rm EM}$ and momentum $\mathbf{P}_{\rm EM}$ of an electromagnetic field (ignoring effects of macroscopic media) are given by

$$U_{\rm EM} = \int \frac{E^2 + B^2}{8\pi}\, d{\rm Vol}, \qquad \mathbf{P}_{\rm EM} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d{\rm Vol}. \tag{2}$$

For a single charge e in an external field, we write $\mathbf{E} = \mathbf{E}_e + \mathbf{E}_{\rm ext}$, $\mathbf{B} = \mathbf{B}_e + \mathbf{B}_{\rm ext}$, and the interaction parts of the electromagnetic energy and momentum are

$$U_{\rm EM,int} = \int \frac{\mathbf{E}_e \cdot \mathbf{E}_{\rm ext} + \mathbf{B}_e \cdot \mathbf{B}_{\rm ext}}{4\pi}\, d{\rm Vol}, \qquad \mathbf{P}_{\rm EM,int} = \int \frac{\mathbf{E}_e \times \mathbf{B}_{\rm ext} + \mathbf{E}_{\rm ext} \times \mathbf{B}_e}{4\pi c}\, d{\rm Vol}. \tag{3}$$

Equivalence between eqs. (2) and (3) requires that

$$e\phi_{\rm ext}(\mathbf{r}_e) = \int \frac{\mathbf{E}_e \cdot \mathbf{E}_{\rm ext} + \mathbf{B}_e \cdot \mathbf{B}_{\rm ext}}{4\pi}\, d{\rm Vol}, \tag{4}$$

and

$$e\mathbf{A}_{\rm ext}(\mathbf{r}_e) = \int \frac{\mathbf{E}_e \times \mathbf{B}_{\rm ext} + \mathbf{E}_{\rm ext} \times \mathbf{B}_e}{4\pi}\, d{\rm Vol}. \tag{5}$$

This is to be true independent of the motion of the charge!

I check the equivalence for the case of a charge that moves slowly with constant velocity $\mathbf{v}$, where $v \ll c$. The charge is at the origin at the time of the calculation. Then,

$$\mathbf{E}_e = e\frac{\hat{\mathbf{r}}}{r^2} = -e\boldsymbol{\nabla}\frac{1}{r}\,, \qquad \mathbf{B}_e = \frac{e}{c}\mathbf{v} \times \frac{\hat{\mathbf{r}}}{r^2} = -\frac{e}{c}\mathbf{v} \times \boldsymbol{\nabla}\frac{1}{r} = \frac{e}{c}\boldsymbol{\nabla} \times \frac{\mathbf{v}}{r}\,. \tag{6}$$

Of course, the external fields can be related to the corresponding potentials by

$$\mathbf{E}_{\rm ext} = -\boldsymbol{\nabla}\phi_{\rm ext} - \frac{1}{c}\frac{\partial \mathbf{A}_{\rm ext}}{\partial t}\,, \qquad \mathbf{B}_{\rm ext} = \boldsymbol{\nabla} \times \mathbf{A}_{\rm ext}. \tag{7}$$

As the particle is moving, the derivative $\partial \mathbf{A}_{\rm ext}/\partial t$ should be replaced by the convective derivative:

$$\frac{\partial \mathbf{A}_{\rm ext}}{\partial t} \to \frac{\partial \mathbf{A}_{\rm ext}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A}_{\rm ext}. \tag{8}$$

Then, the interaction energy of eq. (3) is

$$\begin{aligned}
U_{\rm EM,int} &= \int \frac{\mathbf{E}_e \cdot \mathbf{E}_{\rm ext} + \mathbf{B}_e \cdot \mathbf{B}_{\rm ext}}{4\pi}\, d{\rm Vol} \\
&= \frac{e}{4\pi}\int \boldsymbol{\nabla}\frac{1}{r} \cdot \boldsymbol{\nabla}\phi_{\rm ext}\, d{\rm Vol} + \frac{e}{4\pi c}\int \boldsymbol{\nabla}\frac{1}{r} \cdot \frac{\partial \mathbf{A}_{\rm ext}}{\partial t}\, d{\rm Vol} \\
&\quad - \frac{e}{4\pi c}\int \frac{\hat{\mathbf{r}}}{r^2} \cdot (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A}_{\rm ext}\, d{\rm Vol} + \frac{e}{4\pi c}\int \mathbf{v} \times \frac{\hat{\mathbf{r}}}{r^2} \cdot \boldsymbol{\nabla} \times \mathbf{A}_{\rm ext}\, d{\rm Vol} \\
&= -\frac{e}{4\pi}\int \boldsymbol{\nabla} \cdot \frac{\phi_{\rm ext}\hat{\mathbf{r}}}{r^2}\, d{\rm Vol} - \frac{e}{4\pi}\int \phi_{\rm ext}\nabla^2\frac{1}{r}\, d{\rm Vol} \\
&\quad + \frac{e}{4\pi c}\frac{\partial}{\partial t}\int \boldsymbol{\nabla} \cdot \frac{\mathbf{A}_{\rm ext}}{r}\, d{\rm Vol} - \frac{e}{4\pi c}\frac{\partial}{\partial t}\int \frac{\boldsymbol{\nabla} \cdot \mathbf{A}_{\rm ext}}{r}\, d{\rm Vol} \\
&\quad - \frac{e}{4\pi c}\int \frac{\hat{\mathbf{r}}}{r^2} \cdot (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A}_{\rm ext}\, d{\rm Vol} \\
&\quad + \frac{e}{4\pi c}\int \frac{\hat{\mathbf{r}}}{r^2} \cdot (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A}_{\rm ext}\, d{\rm Vol} - \frac{e}{4\pi c}\int \frac{\hat{\mathbf{r}}}{r^2} \cdot \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}_{\rm ext})\, d{\rm Vol} \\
&= -\frac{e}{4\pi}\oint \frac{\phi_{\rm ext}\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{Area} + e\int \phi_{\rm ext}\delta^3(\mathbf{r})\, d{\rm Vol} \\
&\quad + \frac{e}{4\pi c}\frac{\partial}{\partial t}\oint \frac{\mathbf{A}_{\rm ext}}{r} \cdot d\mathbf{Area} - \frac{e}{4\pi c}\frac{\partial}{\partial t}\int \frac{\boldsymbol{\nabla} \cdot \mathbf{A}_{\rm ext}}{r}\, d{\rm Vol} \\
&\quad - \frac{e}{4\pi c}\int \boldsymbol{\nabla} \cdot (\mathbf{v} \cdot \mathbf{A}_{\rm ext})\frac{\hat{\mathbf{r}}}{r^2}\, d{\rm Vol} - \frac{e}{4\pi c}\int (\mathbf{v} \cdot \mathbf{A}_{\rm ext})\nabla^2\frac{1}{r}\, d{\rm Vol} \\
&= e\phi_{\rm ext}(0) - \frac{e}{4\pi c}\oint (\mathbf{v} \cdot \mathbf{A}_{\rm ext})\frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{Area} + \frac{e}{c}\int (\mathbf{v} \cdot \mathbf{A}_{\rm ext})\delta^3(\mathbf{r})\, d{\rm Vol} \\
&= e\phi_{\rm ext}(0) + \frac{e}{c}\mathbf{v} \cdot \mathbf{A}_{\rm ext}(0).
\end{aligned} \tag{9}$$

In the above, we have supposed that we work in the Coulomb gauge so that we may set $\boldsymbol{\nabla}\cdot\mathbf{A}_{\rm ext}=0$, and we note that for r much larger than the length scale of the sources of the external fields the longitudinal part of the vector potential falls off at least as fast as $1/r^2$.

It seems that we obtain an equivalence between the electromagnetic energies of eqs. (2) and (3) only if the charge is at rest.

We now turn to the interaction momentum. I rewrite eq. (3) as

$$\mathbf{P}_{\rm EM,int}=\mathbf{P}_1+\mathbf{P}_2, \tag{10}$$

where

$$\mathbf{P}_1=\int\frac{\mathbf{E}_e\times\mathbf{B}_{\rm ext}}{4\pi c}\,d{\rm Vol}=-\frac{e}{4\pi c}\int\boldsymbol{\nabla}\frac{1}{r}\times(\boldsymbol{\nabla}\times\mathbf{A}_{\rm ext})\,d{\rm Vol}, \tag{11}$$

and

$$\mathbf{P}_2=\int\frac{\mathbf{E}_{\rm ext}\times\mathbf{B}_e}{4\pi c}\,d{\rm Vol}=-\frac{e}{4\pi c}\int\mathbf{E}_{\rm ext}\times\left(\frac{\mathbf{v}}{c}\times\boldsymbol{\nabla}\frac{1}{r}\right)\,d{\rm Vol} \tag{12}$$

We would like to show that $\mathbf{P}_1=e\mathbf{A}_{\rm ext}/c$ and that $\mathbf{P}_2=0$.

Using various vector calculus identities, and working in the Coulomb gauge, the argument of integral $\mathbf{P}_1$ can be written as

$$\begin{aligned}
\boldsymbol{\nabla}\frac{1}{r}\times(\boldsymbol{\nabla}\times\mathbf{A}_{\rm ext}) &= \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\frac{1}{r}\cdot\mathbf{A}_{\rm ext}\right)-\mathbf{A}_{\rm ext}\times\left(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\frac{1}{r}\right)-\left(\boldsymbol{\nabla}\frac{1}{r}\cdot\boldsymbol{\nabla}\right)\mathbf{A}_{\rm ext}-(\mathbf{A}_{\rm ext}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\frac{1}{r}\\
&=\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\frac{1}{r}\cdot\mathbf{A}_{\rm ext}\right)+\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\frac{1}{r}\times\mathbf{A}_{\rm ext}\right)-(\boldsymbol{\nabla}\cdot\mathbf{A}_{\rm ext})\boldsymbol{\nabla}\frac{1}{r}+\mathbf{A}_{\rm ext}\nabla^2\frac{1}{r}-2(\mathbf{A}_{\rm ext}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\frac{1}{r}\\
&=-\boldsymbol{\nabla}\left(\frac{\mathbf{A}_{\rm ext}\cdot\hat{\mathbf{r}}}{r^2}\right)-\boldsymbol{\nabla}\times\left(\frac{\hat{\mathbf{r}}}{r^2}\times\mathbf{A}_{\rm ext}\right)-4\pi\mathbf{A}_{\rm ext}\delta^3(\mathbf{r})+2(\mathbf{A}_{\rm ext}\cdot\boldsymbol{\nabla})\frac{\hat{\mathbf{r}}}{r^2}\,.
\end{aligned} \tag{13}$$

Hence,

$$\begin{aligned}
\mathbf{P}_1 &= \frac{e}{4\pi c}\int\boldsymbol{\nabla}\left(\frac{\mathbf{A}_{\rm ext}\cdot\hat{\mathbf{r}}}{r^2}\right)\,d{\rm Vol}+\frac{e}{4\pi c}\int\boldsymbol{\nabla}\times\left(\frac{\hat{\mathbf{r}}}{r^2}\times\mathbf{A}_{\rm ext}\right)\,d{\rm Vol}\\
&\quad+\frac{e}{c}\int\mathbf{A}_{\rm ext}\delta^3(\mathbf{r})\,d{\rm Vol}-\frac{e}{2\pi c}\int(\mathbf{A}_{\rm ext}\cdot\boldsymbol{\nabla})\frac{\hat{\mathbf{r}}}{r^2}\,d{\rm Vol}\\
&= \frac{e}{4\pi c}\oint\left(\frac{\mathbf{A}_{\rm ext}\cdot\hat{\mathbf{r}}}{r^2}\right)\,d\mathbf{Area}+\frac{e}{4\pi c}\oint d\mathbf{Area}\times\left(\frac{\hat{\mathbf{r}}}{r^2}\times\mathbf{A}_{\rm ext}\right)\\
&\quad+\frac{e}{c}\mathbf{A}_{\rm ext}(0)+\frac{e}{2\pi c}\int\frac{(3\mathbf{A}_{\rm ext}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{A}_{\rm ext}}{r^3}\,d{\rm Vol}\\
&= \frac{e}{c}\mathbf{A}_{\rm ext}(0)+\frac{e}{2\pi c}\int\frac{(3\mathbf{A}_{\rm ext}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{A}_{\rm ext}}{r^3}\,d{\rm Vol}.
\end{aligned} \tag{14}$$

I hope I have made a mistake, because the 2nd term in the last line of eq. (14) appears to be nonzero.

For the integral $\mathbf{P}_2$ we have

$$\begin{aligned}
\mathbf{P}_2 &= = -\frac{e\mathbf{v}}{4\pi c^2}\int \mathbf{E}_{\rm ext}\cdot\boldsymbol{\nabla}\frac{1}{r}\, d{\rm Vol} - \frac{e}{4\pi c}\int\left(\mathbf{E}_{\rm ext}\cdot\frac{\mathbf{v}}{c}\right)\boldsymbol{\nabla}\frac{1}{r}\, d{\rm Vol} \\
&= -\frac{e\mathbf{v}}{4\pi c^2}\int \boldsymbol{\nabla}\cdot\frac{\mathbf{E}_{\rm ext}}{r}\, d{\rm Vol} + \frac{e\mathbf{v}}{4\pi c^2}\int\frac{\boldsymbol{\nabla}\cdot\mathbf{E}_{\rm ext}}{r}\, d{\rm Vol} + \frac{e}{4\pi c}\int\left(\mathbf{E}_{\rm ext}\cdot\frac{\mathbf{v}}{c}\right)\frac{\hat{\mathbf{r}}}{r^2}\, d{\rm Vol} \\
&= -\frac{e\mathbf{v}}{4\pi c^2}\oint\frac{\mathbf{E}_{\rm ext}}{r}\cdot d\mathbf{Area} + \frac{e\mathbf{v}}{c^2}\int\frac{\rho_{\rm ext}}{r}\, d{\rm Vol} + \frac{e}{4\pi c}\int\left(\mathbf{E}_{\rm ext}\cdot\frac{\mathbf{v}}{c}\right)\frac{\hat{\mathbf{r}}}{r^2}\, d{\rm Vol} \\
&= \frac{e\mathbf{v}\phi_{\rm ext}(0)}{c^2} + \frac{e}{4\pi c}\int\left(\mathbf{E}_{\rm ext}\cdot\frac{\mathbf{v}}{c}\right)\frac{\hat{\mathbf{r}}}{r^2}\, d{\rm Vol}, \qquad (15)
\end{aligned}$$

recalling that we are working in Coulomb gauge. Very odd!

A different approach to the calculation of the interaction momentum follows Page and Adams [11]. We first suppose that the external fields are due to a single charge e'. We work only to accuracy $1/c^2$, and convert the Lienard-Wiechert fields [12, 13] of the charges from retarded quantities into present quantities.

Then the interaction momentum of the fields is

$$\begin{aligned}
\mathbf{P}_{\rm EM,int} &= \frac{1}{4\pi c}\int(\mathbf{E}_e\times\mathbf{B}_{e'} + \mathbf{E}_{e'}\times\mathbf{B}_e)\, d{\rm Vol} = \frac{ee'}{4\pi c^2}\int\frac{\hat{\mathbf{r}}_{e'}\times(\mathbf{v}_e\times\hat{\mathbf{r}}_e) + \hat{\mathbf{r}}_e\times(\mathbf{v}_{e'}\times\hat{\mathbf{r}}_{e'})}{r_e^2 r_{e'}^2}\, d{\rm Vol} \\
&= \frac{ee'}{2c^2 r_{ee'}}[\mathbf{v}+\mathbf{v}' + ((\mathbf{v}+\mathbf{v}')\cdot\hat{\mathbf{r}}_{ee'})\hat{\mathbf{r}}_{ee'}]. \qquad (16)
\end{aligned}$$

In the same approximation (first employed by C.G. Darwin [20], the vector potential at charge e due to charge e' is

$$\mathbf{A}_{e'}(\text{at } e) = \frac{e'}{2cr_{ee'}}[\mathbf{v}' + (\mathbf{v}'\cdot\hat{\mathbf{r}}_{ee'})\hat{\mathbf{r}}_{ee'}]. \qquad (17)$$

Hence, the interaction momentum (16) can also be written as

$$\mathbf{P}_{\rm EM,int} = \frac{e}{c}\mathbf{A}_{e'}(\text{at } e) + \frac{e'}{c}\mathbf{A}_e(\text{at } e'). \qquad (18)$$

If the external fields are due to a collection of charges $\{e'\}$ then the interaction momentum, to accuracy $1/c^2$ is

$$\mathbf{P}_{\rm EM,int} = \frac{e}{c}\mathbf{A}_{\rm ext}(\text{at } e) + \sum_{e'}\frac{e'}{c}\mathbf{A}_e(\text{at } e'). \qquad (19)$$

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This meeting featured an inspirational address by W. Thomson as a memorial to Herschel; among many other topics Thomson speculates on the size of atoms, on the origin of life on Earth as due to primitive organisms arriving in meteorites, and on how the Sun's source of energy cannot be an influx of matter as might, however, explain the small advance of the perihelion of Mercury measured by LeVerrier.
Æthereal Friction, Brit. Assoc. Reports, 43rd Meeting, Notes and Abstracts, pp. 32-35 (1873), http://kirkmcd.princeton.edu/examples/EM/stewart_bar_43_32_73.pdf Stewart argued that the radiation resistance felt by a charge moving through blackbody radiation should vanish as the temperature of the bath went to zero, just as he expected the electrical resistance of a conductor to vanish at zero temperature.
The 43$^{\rm rd}$ *meeting was also the occasion of reports by Maxwell on the exponential atmosphere as an example of statistical mechanics (pp. 29-32), by Rayleigh on the diffraction*

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