# An Alternative Magnetic Field Based on Ampère's Force Law

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# 1 Problem

## 1.1 Historical Background

In 1820-1822, Ampère examined the force between two circuits, carrying steady currents $I_1$ and $I_2$, and inferred that this could be written (here in vector notation, which Ampère did not use) as (pp. 21-24 of [4]),[1]

$$\mathbf{F}_{\text{on }1} = \oint_1 \oint_2 d^2\mathbf{F}_{\text{on }1}, \quad d^2\mathbf{F}_{\text{on }1} = \frac{\mu_0}{4\pi} I_1 I_2 [3(\hat{\mathbf{r}}\cdot d\mathbf{l}_1)(\hat{\mathbf{r}}\cdot d\mathbf{l}_2) - 2\, d\mathbf{l}_1\cdot d\mathbf{l}_2]\frac{\hat{\mathbf{r}}}{r^2} = -d^2\mathbf{F}_{\text{on }2}, \quad (1)$$

where $\mathbf{r} = \mathbf{l}_1 - \mathbf{l}_2$ is the distance from a current element $I_2\, d\mathbf{l}_2$ at $\mathbf{r}_2 = \mathbf{l}_2$ to element $I_1\, d\mathbf{l}_1$ at $\mathbf{r}_1 = \mathbf{l}_1$.[2,3] The integrand $d^2\mathbf{F}_{\text{on }1}$ of eq. (1) has the appeal that it changes sign if elements 1 and 2 are interchanged, and so suggests a force law for current elements that obeys Newton's third law.

[1] A historical survey of the development of electrodynamics in the 1800's by one of the authors is the Appendix to [35]. A thoughtful online site about Ampère is http://www.ampere.cnrs.fr/parcourspedagogique/index-en.php.

[2] Ampère sometimes used the notation that the angles between $d\mathbf{l}_i$ and $\mathbf{r}$ are $\theta_i$, and the angle between the plane of $d\mathbf{l}_1$ and $\mathbf{r}$ and that of $d\mathbf{l}_2$ and $\mathbf{r}$ is $\omega$. Then, $d\mathbf{l}_1\cdot d\mathbf{l}_2 = dl_1\, dl_2(\sin\theta_1\sin\theta_2\cos\omega + \cos\theta_1\cos\theta_2)$, and the force element of eq. (1) can be written as,

$$d^2\mathbf{F}_{\text{on }1} = \frac{\mu_0}{4\pi} I_1 I_2\, dl_1\, dl_2(\cos\theta_1\cos\theta_2 - 2\sin\theta_1\sin\theta_2\cos\omega)\frac{\hat{\mathbf{r}}}{r^2} = -d^2\mathbf{F}_{\text{on }2}. \quad (2)$$

[3] Ampère also noted the equivalents to,

$$d\mathbf{l}_1 = \frac{\partial\mathbf{r}}{\partial l_1} dl_1, \qquad \mathbf{r}\cdot d\mathbf{l}_1 = \mathbf{r}\cdot\frac{\partial\mathbf{r}}{\partial l_1} dl_1 = \frac{1}{2}\frac{\partial r^2}{\partial l_1} dl_1 = r\frac{\partial r}{\partial l_1} dl_1, \qquad d\mathbf{l}_2 = -\frac{\partial\mathbf{r}}{\partial l_2} dl_2, \qquad \mathbf{r}\cdot d\mathbf{l}_2 = -r\frac{\partial r}{\partial l_2} dl_2, \quad (3)$$

where $l_1$ and $l_2$ measure distance along the corresponding circuits in the directions of their currents. Then,

$$d\mathbf{l}_1\cdot d\mathbf{l}_2 = -d\mathbf{l}_1\cdot\frac{\partial\mathbf{r}}{\partial l_2} dl_2 = -\frac{\partial}{\partial l_2}(\mathbf{r}\cdot d\mathbf{l}_1)\, dl_2 = -\frac{\partial}{\partial l_2}\left(r\frac{\partial r}{\partial l_1}\right) dl_1\, dl_2 = -\left(\frac{\partial r}{\partial l_1}\frac{\partial r}{\partial l_2} + r\frac{\partial^2 r}{\partial l_1\partial l_2}\right) dl_1\, dl_2, \quad (4)$$

and eq. (1) can also be written in forms closer to those used by Ampère,

$$d^2\mathbf{F}_{\text{on }1} = \frac{\mu_0}{4\pi} I_1 I_2\, dl_1\, dl_2\left[2r\frac{\partial^2 r}{\partial l_1\partial l_2} - \frac{\partial r}{\partial l_1}\frac{\partial r}{\partial l_2}\right]\frac{\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} 2 I_1 I_2\, dl_1\, dl_2\frac{\partial^2\sqrt{r}}{\partial l_1\partial l_2}\frac{\hat{\mathbf{r}}}{\sqrt{r}} = -d^2\mathbf{F}_{\text{on }2}. \quad (5)$$

In 1825, Ampère noted, p. 214 of [7], p. 29 of [4], p. 366 of the English translation in [33], that for a closed circuit, eq. (1) can be rewritten as,[4]

$$\mathbf{F}_{\text{on }1} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})\, d\mathbf{l}_2 - (d\mathbf{l}_1 \cdot d\mathbf{l}_2)\, \hat{\mathbf{r}}}{r^2} = \oint_1 I_1 d\mathbf{l}_1 \times \frac{\mu_0}{4\pi} \oint_2 \frac{I_2\, d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}\,, \tag{7}$$

in vector notation (which, of course, he did not use).[5] Ampère made very little comment on this result.[6] However, in retrospect, we see that the form (7) lends itself to the interpretation that the force between closed circuits with steady currents can be written in terms of a magnetic field $\mathbf{B}$ as,

$$\mathbf{F} = \oint I\, d\mathbf{l} \times \mathbf{B}, \qquad \text{where} \qquad \mathbf{B} = \frac{\mu_0}{4\pi} \oint \frac{I\, d\mathbf{l} \times \hat{\mathbf{r}}}{r^2}\,, \tag{8}$$

both equations of which are often called the Biot-Savart law.[7]

Ampère had no concept of a magnetic field, which originated with Faraday, inspired in part by patterns of iron filings on a sheet near a magnet.[8] Of particular interest here is

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[4]Note that for a fixed point 2, $d\mathbf{l}_1 = d\mathbf{r}$, and $dr = d\mathbf{r} \cdot \hat{\mathbf{r}} = d\mathbf{l}_1 \cdot \hat{\mathbf{r}}$. Then, for any function $f(r)$, $df = (df/dr)\, dr = (df/dr)\, d\mathbf{l}_1 \cdot \hat{\mathbf{r}}$. In particular, for $f = -1/r$, $df = d\mathbf{l}_1 \cdot \hat{\mathbf{r}}/r^2$, so the first term of the first form of eq. (6) is a perfect differential with respect to $\mathbf{l}_1$. Hence, when integrating around a closed loop 1, the first term does not contribute, and it is sufficient to write (as first argued by Neumann, p. 67 of [12]),

$$\mathbf{F}_{\text{on }1} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r^2}\, \hat{\mathbf{r}} = -\mathbf{F}_{\text{on }2}. \tag{6}$$

[5]Ampère's force law for closed circuits with steady currents can be written in many other ways as well. Maxwell gave an early survey of this in Arts. 510-526 of [23]. A review by one of the authors is in [32].

[6]As a consequence, the form (7) is generally attributed to Grassmann [11], as in [31], for example.

[7]Biot and Savart [1, 2] actually studied on the force due to an electric current $I$ in a wire on one pole, $p$, of a long, thin magnet. Their initial interpretation of the results was somewhat incorrect, which was remedied by Biot in 1821 and 1824 [3, 6] with a form that can be written in vector notation (and in SI units) as,

$$\mathbf{F} = \frac{\mu_0\, p}{4\pi} \oint \frac{I\, d\mathbf{l} \times \hat{\mathbf{r}}}{r^2}\,, \tag{9}$$

where $\mathbf{r}$ is the distance from a current element $I\, d\mathbf{l}$ to the magnetic pole. There was no immediate interpretation of eq. (9) in terms of a magnetic field, $\mathbf{B} = \mathbf{F}/p$.

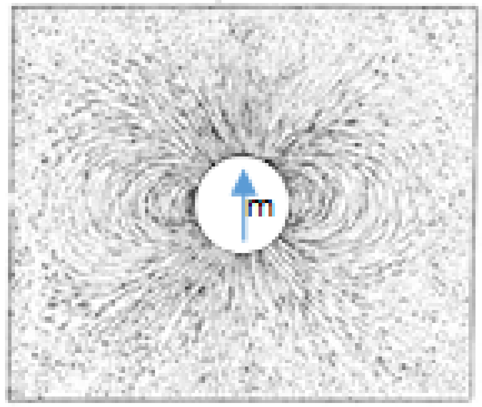
[8]Faraday first mentioned *magnetic lines of force* in Art. 114 of [8] (1831): *By magnetic curves, I mean the lines of magnetic forces, however modified by the juxtaposition of poles, which would be depicted by iron filings; or those to which a very small magnetic needle would form a tangent.*

In 1845, Art. 2247 of [13], the term *magnetic field* appears for the first time in print: *The ends of these bars form the opposite poles of contrary name; the magnetic field between them can be made of greater or smaller extent, and the intensity of the lines of magnetic force be proportionately varied.*

In 1852, Faraday published a set of speculative comments [18] in the Phil. Mag. (rather than Phil. Trans. Roy. Soc. London, the usual venue for his *Experimental Researches*), arguing more strongly for the physical reality of the lines of force.

In Art. 3258 he considered the effect of a magnet in vacuum, concluding (perhaps for the first time) that the lines of force have existence independent of a material medium:
*A magnet placed in the middle of the best vacuum we can produce...acts as well upon a needle as if it were surrounded by air, water or glass; and therefore these lines exist in such a vacuum as well as where there is matter.*

Fig. 3 from Art. 3295 of [18], in which Faraday showed the pattern of iron filings in a plane containing the axis of a small dipole magnet, as shown below.




This pattern corresponds to the lines of force of a magnetic dipole $\mathbf{m}$ on a hypothetical magnetic pole $p$ as deduced by Poisson, eq. (9), p. 507 of [5] (1824),

$$\mathbf{F} = -p\,\boldsymbol{\nabla}\frac{\mathbf{m}\cdot\hat{\mathbf{r}}}{r^2} = p\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\mathbf{m}}{r^3}\,. \tag{10}$$

where $\mathbf{r}$ is the vector from the center of the dipole $\mathbf{m}$ to the pole $p$. This was regarded by Poisson as an action-at-a-distance force, and he did not consider the possibility of a magnetic force field such as $\mathbf{B} = \mathbf{F}/p$ that existed in vacuum at points unoccupied by magnetic poles.

Our present view is that iron filings are not magnetic poles, but magnetic dipoles, which align themselves along lines of the magnetic field $\mathbf{B}$.

The first to adopt Faraday's concept of a magnetic field was Thomson (later Lord Kelvin), who discussed the magnetic field of a magnetic dipole $\mathbf{m}$ in sec. II of [15] (1846). However, he did not follow the path of Poisson (to write $\mathbf{B} = -\boldsymbol{\nabla}(\mathbf{m}\cdot\hat{\mathbf{r}}/r^2)$, but simply stated that,

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \qquad \text{where} \qquad \mathbf{A} = \frac{\mathbf{m}\times\mathbf{r}}{r^3}\,, \tag{11}$$

in his eq. (II) where $\mathbf{B} = (X, Y, Z)$ and his eq. (3) where $\mathbf{A} = (\alpha, \beta, \gamma)$. This is the first appearance of a vector potential in print.[9] Like Poisson, Thomson provided no figure, but

[9]In 1867 Gauss posthumously published an analysis that he dated to 1835 (p. 609 of [21]), in which he stated that a time-dependent electric current leads to an electric field which is the time derivative of what we now called the vector potential. English translation from [27]:

> The Law of Induction
> Found out Jan. 23, 1835, at 7 a.m. before getting up.
>
> 1. The electricity producing power, which is caused in a point $P$ by a current-element $\boldsymbol{\gamma}$, at a distance from $P$, $= r$, is during the time $dt$ the difference in the values of $\boldsymbol{\gamma}/r$ corresponding to the moments $t$ and d$t$, divided by d$t$. where $\boldsymbol{\gamma}$ is considered both with respect to size and direction. This can be expressed briefly and clearly by
>
> $$-\frac{\mathrm{d}(\boldsymbol{\gamma}/r)}{\mathrm{d}t}\,. \tag{12}$$

Gauss' unpublished insight that electromagnetic induction is related to the negative time derivative of a scalar quantity was probably communicated in the late 1830's to his German colleagues, of whom Weber was the closest.

gave a brief verbal description that suggests he was aware of the form (10) given by Poisson, which agrees with his eq. (11).

Meanwhile, in 1845, Neumann followed the examples of Lagrange, Laplace and Poisson in relating forces of gravity and electrostatics to (scalar) potentials, and sought a potential for Ampère's force law between two (closed) current loops. For this, he noted that this force law can be rewritten in the form (6), which permits us to write $\mathbf{F}_{\rm on1} = -\boldsymbol{\nabla} U$ where, $U$ is the scalar potential (energy) given on p. 67 of [12],[10]

$$U = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r} \tag{13}$$

in SI units. We now also write this as,

$$U = I_i \oint_i d\mathbf{l}_i \cdot \mathbf{A}_j = I_i \int d\mathbf{Area}_i \cdot \boldsymbol{\nabla} \times \mathbf{A}_j = I_i \int d\mathbf{Area}_i \cdot \mathbf{B}_j = I_i \Phi_{ij}, \tag{14}$$

where $\Phi_{ij}$ is the magnetic flux through circuit $i$ due to the current $I_j$ in circuit $j$, and,

$$\mathbf{A}_j = \frac{\mu_0}{4\pi} \oint_j \frac{I_j\, d\mathbf{l}_j}{r}\,, \tag{15}$$

such that Neumann is often credited for inventing the vector potential $\mathbf{A}$, although he appears not to have written his eq. (13) in any of the forms of eq. (14).

In 1870, Helmholtz made a review of electrodynamics, and in eq. (1), p. 76 of [22],[11] he deduced that a general form for the magnetic interaction energy (his $P$, but our $U$) of two current elements, which are part of closed circuits of steady currents, could be written as a combination of the forms he attributed to Neumann [12, 16] and to Weber [14, 17],[12]

$$d^2U = \frac{\mu_0}{4\pi} \left( \frac{1+k}{2} \frac{I_1\, d\mathbf{l}_1 \cdot I_2\, d\mathbf{l}_2)}{r} + \frac{1-k}{2} \frac{(I_1\, d\mathbf{l}_1 \cdot \hat{\mathbf{r}})(I_2\, d\mathbf{l}_2 \cdot \hat{\mathbf{r}})}{r} \right), \tag{16}$$

where $k = 1$ for Neumann's form and $k = -1$ for Weber's. Then, in eq. ($1^a$) he noted that the scalar $U$ is related to a vector potential (his $(U, V, W)$ but our $\mathbf{A}$) as $U = \int \mathbf{J} \cdot \mathbf{A}\, d\mathrm{Vol}/2$, noting that $I\, d\mathbf{l} \leftrightarrow \mathbf{J}\, d\mathrm{Vol}$ where $\mathbf{J}$ is the (steady) current density (which obeys $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$),[13]

$$\mathbf{A} = \frac{1+k}{2} \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{r}\, d\mathrm{Vol} + \frac{1-k}{2} \frac{\mu_0}{4\pi} \int \frac{(\mathbf{J} \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}}}{r}\, d\mathrm{Vol} = \frac{1+k}{2} \mathbf{A}_{\rm N} + \frac{1-k}{2} \mathbf{A}_{\rm W}, \tag{17}$$

$$\mathbf{A}_{\rm N} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{r}\, d\mathrm{Vol}, \qquad \mathbf{A}_{\rm W} = \frac{\mu_0}{4\pi} \int \frac{(\mathbf{J} \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}}}{r}\, d\mathrm{Vol}, \tag{18}$$

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On p. 612 (presumably also from 1835), Gauss noted a relation (here transcribed into vector notation) between the vector $\mathbf{A} = \oint d\mathbf{l}/r$ and the magnetic scalar potential $\Omega$ of a circuit with unit electrical current (which he related to the solid angle subtended by the circuit on p. 611), $-\boldsymbol{\nabla}\Omega = \boldsymbol{\nabla} \times \mathbf{A}$. While we would identify $-\boldsymbol{\nabla}\Omega$ with the magnetic field $\mathbf{H}$, Gauss called it the "electricity-generating force". In any case, this is the earliest (claimed) appearance of the curl operator (although published later than MacCullagh's use of it, p. 22 of [10] (1839)).

[10]If we write eq. (13) as $U = I_1 I_2 M_{12}$, then $M_{12}$ is the mutual inductance of circuits 1 and 2.

[11]For comments by one of the authors on this paper, see [34]. See also commentaries in [26, 28, 29, 30].

[12]See also sec. IIB of [31], and [32]. The energy that Helmholtz associated with Weber was never actually advocated by the latter, who had a different vision of magnetic energy, as discussed in sec. A.23 of [35].

[13]Thus, we cannot write for an isolated current element that $d\mathbf{A} = \mu_0 I[(1+k)d\mathbf{l} + (1-k)(d\mathbf{l} \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}}]/8\pi r$.

although neither Neumann nor Weber ever wrote the forms called $\mathbf{A}_{\rm N}$ and $\mathbf{A}_{\rm W}$ here.[14,15]

### 1.2 The Problem

Consider the magnetic field,

$$\mathbf{B}_{\rm A-W} = \frac{\mu_0}{4\pi}\oint I(d\mathbf{l}\cdot\hat{\mathbf{r}})\frac{\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\int(\mathbf{J}\cdot\hat{\mathbf{r}})\frac{\hat{\mathbf{r}}}{r^2}\,d{\rm Vol}\,, \tag{19}$$

which has the form of $\mathbf{A}_{\rm W}$ of eq. (18), but with the factor $r$ in the denominator replaced by $r^2$. Show that the Ampère force law (1) for $d^2\mathbf{F}_{\rm on\ 1}$ on current element $I_1\,d\mathbf{l}_1$ due to element $I_2\,d\mathbf{l}_2$ can be related to this magnetic field by,

$$d^2\mathbf{F}_{\rm on\ 1} = -I_1\,d\mathbf{l}_1\cdot\{d\mathbf{B}_{\rm A-W}(I_2\,d\mathbf{l}_2) + 2\boldsymbol{\nabla}[d\mathbf{B}_{\rm A-W}(I_2\,d\mathbf{l}_2)\cdot\mathbf{r}]\}\,\hat{\mathbf{r}}, \quad d\mathbf{B}_{\rm A-W} = \frac{\mu_0}{4\pi}I_2(d\mathbf{l}_2\cdot\mathbf{r})\frac{\mathbf{r}}{r^4}\,. \tag{20}$$

Show also that $\boldsymbol{\nabla}\cdot\mathbf{B}_{\rm A-W} = 0$, consistent with Ampère's view that magnetism is due to electric currents rather than magnetic charges/poles.

Compute the magnetic field (19) due to a magnetic dipole $\mathbf{m}$, a small current loop of radius $a$, current $I$, with $\mathbf{m}$ along the axis of the loop, and $m = \pi a^2 I$. Compare the field lines for this model of a magnetic dipole with the figure from Faraday on p. 3.

Would Faraday have accepted the field $\mathbf{B}_{\rm A-W}$ as a valid physical description of the magnetic field?

## 2 Solution

From eq. (20), we have,

$$\begin{aligned}\boldsymbol{\nabla}[d\mathbf{B}_{\rm A-W}(I_2\,d\mathbf{l}_2)\cdot\mathbf{r}] &= \frac{\mu_0}{4\pi}I_2\boldsymbol{\nabla}\left(\frac{d\mathbf{l}_2\cdot\mathbf{r}}{r^2}\right) = \frac{\mu_0}{4\pi}I_2\left(-\frac{2d\mathbf{l}_2\cdot\mathbf{r}}{r^3}\boldsymbol{\nabla}r + \frac{\boldsymbol{\nabla}(d\mathbf{l}_2\cdot\mathbf{r})}{r^2}\right)\\ &= \frac{\mu_0}{4\pi}I_2\left(-\frac{2d\mathbf{l}_2\cdot\mathbf{r}}{r^4}\mathbf{r} + \frac{d\mathbf{l}_2}{r^2}\right) = \frac{\mu_0}{4\pi}I_2\left(-\frac{2d\mathbf{l}_2\cdot\hat{\mathbf{r}}}{r^2}\hat{\mathbf{r}} + \frac{d\mathbf{l}_2}{r^2}\right).\end{aligned} \tag{21}$$

Then,

$$\begin{aligned}&-I_1\,d\mathbf{l}_1\cdot\{d\mathbf{B}_{\rm A-W}(I_2\,d\mathbf{l}_2) + 2\boldsymbol{\nabla}[d\mathbf{B}_{\rm A-W}(I_2\,d\mathbf{l}_2)\cdot\mathbf{r}]\}\,\hat{\mathbf{r}}\\ &= -\frac{\mu_0}{4\pi}I_1\,d\mathbf{l}_1\cdot\left[I_2(d\mathbf{l}_2\cdot\hat{\mathbf{r}})\frac{\hat{\mathbf{r}}}{r^2} + 2I_2\left(-\frac{2d\mathbf{l}_2\cdot\hat{\mathbf{r}}}{r^2}\hat{\mathbf{r}} + \frac{d\mathbf{l}_2}{r^2}\right)\right]\hat{\mathbf{r}}\\ &= \frac{\mu_0}{4\pi}I_1\,d\mathbf{l}_1\cdot\left(3I_2(d\mathbf{l}_2\cdot\hat{\mathbf{r}})\frac{\hat{\mathbf{r}}}{r^2} - \frac{2I_2\,d\mathbf{l}_2}{r^2}\right)\hat{\mathbf{r}} = \frac{\mu_0}{4\pi}I_1I_2\,[3(d\mathbf{l}_1\cdot\hat{\mathbf{r}})(d\mathbf{l}_2\cdot\hat{\mathbf{r}}) - 2\,d\mathbf{l}_1\cdot d\mathbf{l}_2]\,\frac{\hat{\mathbf{r}}}{r^2}\\ &= d^2\mathbf{F}_{\rm on\ 1},\end{aligned} \tag{22}$$

[14] Both $\mathbf{A}_{\rm N}$ and $\mathbf{A}_{\rm W}$ lead to the same magnetic field $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, namely the Biot-Savart form (8), which is an early example of gauge invariance.

[15] Helmholtz' discussion was tacitly restricted to electro- and magnetostatics, such that his eq. ($3^a$), p. 80, that $\boldsymbol{\nabla}\cdot\mathbf{A} = k\,dV/dt$, where $V$ is the instantaneous electric scalar potential, led him to identify $k = 0$ with Maxwell's theory [20] with its emphasis on $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$, although we would now consider $k = 1$ to be compatible with Maxwell's theory for static electromagnetism. Maxwell was more interested in electrodynamics than electro/magnetostatics, such that his only mention of the "Neumann" magnetostatic vector potential, our eq. (13), was in eq. (9), Art. 422 of [23].

according to Ampère's form (1).

Thus, Ampère's force law (1) can be related to a magnetic field $\mathbf{B}_{\mathrm{A-W}}$ if we allow the force law (20) to depend on the spatial derivatives of $\mathbf{B}_{\mathrm{A-W}}$ as well as $\mathbf{B}_{\mathrm{A-W}}$ itself. Such a derivative coupling is not favored in the simplest implementation of a field theory, but cannot be excluded altogether. However, the force (20) on current element $I_1\,d\mathbf{l}_1$ is not a function only of this element and the field $\mathbf{B}_{\mathrm{A-W}}$ at the element, so it not in the spirit of Faraday's vision of a field theory.

The divergence of $\mathbf{B}_{\mathrm{A-W}}$ is,

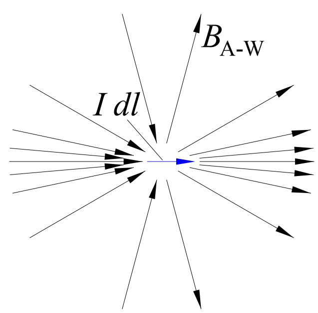
$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{B}_{\mathrm{A-W}}(\mathbf{x}) &= \frac{\mu_0}{4\pi}\int \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{J}(\mathbf{x}')\cdot\mathbf{r}}{r^4}\mathbf{r}\right)\,d\mathrm{Vol}' \\
&= \frac{\mu_0}{4\pi}\int\left[\frac{\mathbf{J}(\mathbf{x}')\cdot\mathbf{r}}{r^4}\boldsymbol{\nabla}\cdot\mathbf{r}+\mathbf{r}\cdot\boldsymbol{\nabla}\left(\frac{\mathbf{J}(\mathbf{x}')\cdot\mathbf{r}}{r^4}\right)\right]\,d\mathrm{Vol}' \\
&= \frac{\mu_0}{4\pi}\int\left[\frac{3\mathbf{J}(\mathbf{x}')\cdot\mathbf{r}}{r^4}+\mathbf{r}\cdot\left(\frac{\boldsymbol{\nabla}(\mathbf{J}(\mathbf{x}')\cdot\mathbf{r})}{r^4}-\frac{4(\mathbf{J}(\mathbf{x}')\cdot\mathbf{r})}{r^5}\boldsymbol{\nabla}r\right)\right]\,d\mathrm{Vol}' \\
&= \frac{\mu_0}{4\pi}\int\left[\frac{3\mathbf{J}(\mathbf{x}')\cdot\mathbf{r}}{r^4}+\mathbf{r}\cdot\left(\frac{\mathbf{J}(\mathbf{x}')}{r^4}-\frac{4(\mathbf{J}(\mathbf{x}')\cdot\mathbf{r})\mathbf{r}}{r^6}\right)\right]\,d\mathrm{Vol}' = 0, \qquad (23)
\end{aligned}$$

away from $r = 0$, *i.e.*, for source currents away from the observation point.

To ascertain the behavior of $\mathbf{B}_{\mathrm{A-W}}$ for small $r$, it is useful to consider its flux across the surface of a sphere of radius $r$, within which the current $\mathbf{J}$ is approximately constant.

$$\Phi = \int(\mathbf{B}_{\mathrm{A-W}}\cdot\hat{\mathbf{r}})\,d\mathrm{Area} = \frac{\mu_0}{4\pi}\int_{-1}^{1}\frac{\mathbf{J}\cdot\hat{\mathbf{r}}}{r^2}\,2\pi r^2\,d\cos\theta = \frac{\mu_0 J}{2}\int_{-1}^{1}\cos\theta\,d\cos\theta = 0, \qquad (24)$$

taking the $z$-axis to be along the direction of $\mathbf{J}$ at the center of the sphere. That is, the magnetic field (19) for a current element $\mathbf{J}\,d\mathrm{Vol} = I\,d\mathbf{l}$ has lines of $\mathbf{B}_{\mathrm{A-W}}$ diverging from the current element in one hemisphere, and converging on it in the other, such that the total flux into/out of the current element is zero. Then, together with eq. (23), we see that $\boldsymbol{\nabla}\cdot\mathbf{B}_{\mathrm{A-W}} = 0$ everywhere.




Hence, there exists a vector potential $\mathbf{A}_{\mathrm{A-W}}$ such that $\mathbf{B}_{\mathrm{A-W}} = \boldsymbol{\nabla}\times\mathbf{A}_{\mathrm{A-W}}$. A particular form of the vector potential is,[16]

$$\mathbf{A}_{\mathrm{A-W}}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')\times\mathbf{r}}{2r^2}\,d\mathrm{Vol}'. \qquad (26)$$

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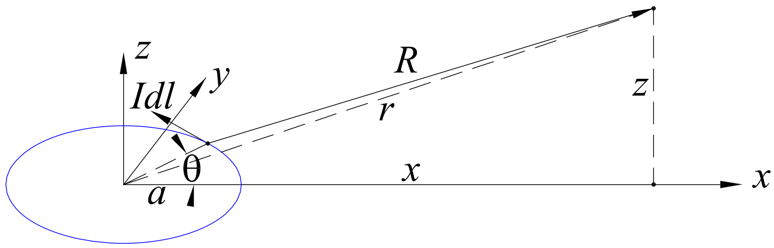
$$\begin{aligned}
\boldsymbol{\nabla}\times\frac{\mathbf{J}\times\mathbf{r}}{r^2} &= \frac{\mathbf{J}}{r^2}(\boldsymbol{\nabla}\cdot\mathbf{r}-\mathbf{r}\left(\boldsymbol{\nabla}\cdot\frac{\mathbf{J}}{r^2}\right)+(\mathbf{r}\cdot\boldsymbol{\nabla})\frac{\mathbf{J}}{r^2}-\left(\frac{\mathbf{J}}{r^2}\cdot\boldsymbol{\nabla}\right)\mathbf{r} \\
&= \frac{3\mathbf{J}}{r^2}-\mathbf{r}\left(\mathbf{J}\cdot\boldsymbol{\nabla}\frac{1}{r^2}\right)-\frac{2\mathbf{J}}{r^3}(\mathbf{r}\cdot\boldsymbol{\nabla})r-\frac{\mathbf{J}}{r^2} = \frac{2\mathbf{J}}{r^2}+2\mathbf{r}\frac{\mathbf{J}\cdot\mathbf{r}}{r^4}-\frac{2\mathbf{J}}{r^2} = \frac{2(\mathbf{J}\cdot\mathbf{r})\mathbf{r}}{r^4}\propto d\mathbf{B}_{\mathrm{A-W}}. \qquad (25)
\end{aligned}$$

## 2.1 Magnetic Dipole

We now consider a magnetic dipole $\mathbf{m} = \pi a^2 I\, \hat{\mathbf{z}}$, *i.e.*, a small, circular loop of radius $a$, centered on the origin, that carries steady current $I$.



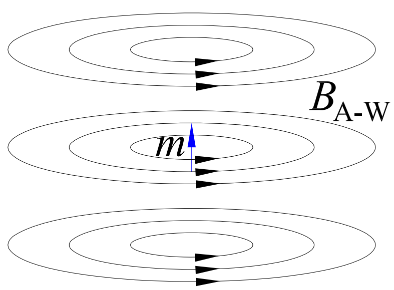

We calculate $\mathbf{B}_{\text{A−W}}$ at the point $\mathbf{r} = (x \gg a, 0, z)$, with $r = \sqrt{x^2 + z^2}$. For a current element $I\, dl = I a\, d\theta$ located at angle $\theta$ to the $x$-axis, *i.e.*, at $\mathbf{r}' = (a\cos\theta, a\sin\theta, 0)$, we have, with $\mathbf{R} = \mathbf{r} - \mathbf{r}'$,

$$d\mathbf{l} = a, d\theta(-\sin\theta, \cos\theta, 0), \qquad \mathbf{R} = (x - a\cos\theta, -a\sin\theta, z), \qquad d\mathbf{l}\cdot\mathbf{R} = -ax\, d\theta \sin\theta, \tag{27}$$

$$R = \sqrt{x^2 - 2ax\cos\theta + a^2 + z^2} \approx \sqrt{x^2 + z^2}\left(1 - \frac{ax\cos\theta}{x^2 + z^2}\right) = r\left(1 - \frac{ax\cos\theta}{r^2}\right), \tag{28}$$

$$\begin{aligned} \mathbf{B}_{\text{A−W}} &= \frac{\mu_0}{4\pi}\oint I\frac{(d\mathbf{l}\cdot\mathbf{R})\mathbf{R}}{R^4} \\ &\approx \frac{\mu_0}{4\pi}\frac{aI}{r^4}\int_0^{2\pi} d\theta\,(-x\sin\theta)\left(1 + \frac{4ax\cos\theta}{r^2}\right)(x - a\cos\theta, -a\sin\theta, z) \\ &= \frac{\mu_0}{4\pi}\frac{aI}{r^4}(0, \pi a x, 0) = \frac{\mu_0}{4\pi}\frac{mx}{r^4}\hat{\mathbf{y}} = \frac{\mu_0}{4\pi}\frac{mx}{r^4}\hat{\boldsymbol{\theta}} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^4} = \boldsymbol{\nabla}\times\frac{\mu_0}{4\pi}\frac{\mathbf{m}}{2r^2} = \boldsymbol{\nabla}\times\mathbf{A}_{\text{A−W}}. \end{aligned} \tag{29}$$

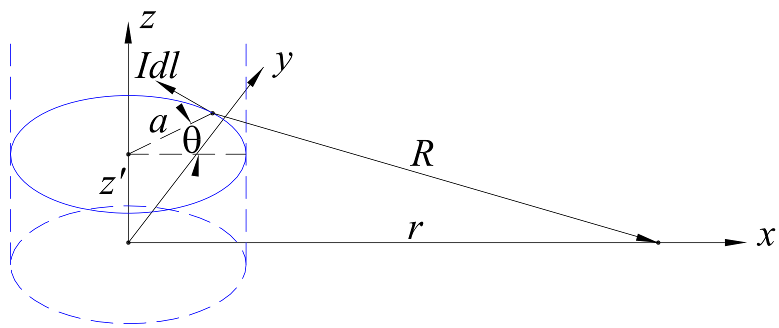
Lines of the $\mathbf{B}_{\text{A−W}} \propto \mathbf{m}\times\hat{\mathbf{r}}/r^3$ are circles centered on the axis $\mathbf{m}$, and do not at all resemble the pattern of iron filings found by Faraday for a small dipole magnet (p. 3 above).




Hence, while the field $\mathbf{B}_{\text{A−W}}$ is mathematically consistent with Ampère's force law (1) between two circuits with steady currents, it seems unappealing physically, and would not have been accepted by Faraday.

## 2.2 Infinite Solenoid (Aug. 20, 2021)

In this section, we consider an infinite solenoid of radius $a$ along the $z$-axis, with steady, azimuthal surface current $I$ per unit length in $z$.




We calculate $\mathbf{B}_{\text{A-W}}$ at the point $\mathbf{r} = (r, 0, 0)$. For a current element $I\,dl = I\,a\,d\theta$ located at angle $\theta$ to the $x$-axis at height $z'$, *i.e.*, at $\mathbf{r}' = (a\cos\theta, a\sin\theta, z')$, we have, with $\mathbf{R} = \mathbf{r} - \mathbf{r}'$,

$$d\mathbf{l} = a, d\theta(-\sin\theta, \cos\theta, 0), \quad \mathbf{R} = (r - a\cos\theta, -a\sin\theta, -z'), \quad d\mathbf{l}\cdot\mathbf{R} = -ar\,d\theta\sin\theta, \tag{30}$$

$$R = \sqrt{r^2 - 2ar\cos\theta + a^2 + z'^2}, \tag{31}$$

$$\begin{aligned}\mathbf{B}_{\text{A-W}} &= \frac{\mu_0}{4\pi}\oint I\frac{(d\mathbf{l}\cdot\mathbf{R})\mathbf{R}}{R^4} = -\frac{\mu_0}{4\pi}arI\int_{-\infty}^{\infty}dz'\int_0^{2\pi}d\theta\,\frac{\sin\theta rx - a\cos\theta, -a\sin\theta, -z')}{(r^2 - 2ar\cos\theta + a^2 + z'^2)^2} \\ &= \frac{\mu_0}{4\pi}\frac{\pi a^2 xI}{4}\int_0^{2\pi}d\theta\,\frac{(0, \sin^2\theta, 0)}{(r^2 - 2ar\cos\theta + a^2)^{3/2}}\,,\end{aligned} \tag{32}$$

using Dwight 120.2 [36]. This is a nonzero function for any value of the distance $x$ of the observer from the axis of the infinite solenoid. By symmetry, $\mathbf{B}_{\text{A-W}}$ is in the $\hat{\boldsymbol{\theta}}$-direction. The charactoer of $\mathbf{B}_{\text{A-W}}$ is in contrast to the Biot-Savart magnetic field which is zero outside the solenoid and constant inside (with value $\mu_0 I\,\hat{\mathbf{z}}$). For $r \gg a$ (outside the solenoid),

$$\mathbf{B}_{\text{A-W}}(x \gg a) \to \frac{\mu_0}{4\pi}\frac{\pi a^2 rI}{4r^3}\,\hat{\boldsymbol{\theta}} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{4r^3}\,, \tag{33}$$

where $\mathbf{r} = (r, 0, 0)$ and $\mathbf{m} = \pi a^2 I\,\hat{\mathbf{z}}$ is the magnetic dipole moment per unit length of the infinite solenoid. That is, the field lines of $\mathbf{B}_{\text{A-W}}$ for an infinite solenoid are of the same form as those for a magnetic dipole (sec. 2.1 above).

## 2.3 Long, Straight Wire (Aug. 21, 2021)

In this section, we consider a wire along the $z$-axis, carrying current $I$.

The Ampère-Weber magnetic field (19) at the point $\mathbf{r} = (r, 0, 0)$ is, integrating over current elements $I\,d\mathbf{l}$ at $\mathbf{r}' = (0, 0, z')$,

$$d\mathbf{l} = (0, 0, dz'), \qquad \mathbf{R} = \mathbf{r} - \mathbf{r}' = (r, 0, -z'), \qquad d\mathbf{l}\cdot\mathbf{R} = -z'\,dz', \qquad R = \sqrt{r^2 + z'^2}, \tag{34}$$

$$\mathbf{B}_{\text{A-W}} = \frac{\mu_0}{4\pi}\oint I\frac{(d\mathbf{l}\cdot\mathbf{R})\mathbf{R}}{R^4} = -\frac{\mu_0}{4\pi}I\int_{-\infty}^{\infty}dz'\,\frac{z'(r, 0, -z')}{(r^2 + z'^2)^2} = \frac{\mu_0 I}{16r}\,\hat{\mathbf{z}}, \tag{35}$$

using Dwight 122.2 [36]. That is, $\mathbf{B}_{\text{A-W}}$ is parallel to the wire and falls off inversely with the distance from it.

This result contrasts with Faraday's vision that the lines of magnetic field circle about long, straight, current-carrying wires (Art. 233 of [9]; see also sec. A.17.4 of [35]).

# A Appendix: The Biot-Savart-Lorentz Force Law

(Aug. 15, 2021)

The preceding discussion has largely emphasized magnetostatics, but in eq. (8) we mentioned the Biot-Savart force law, that was later generalized to describe the magnetic force on an electric charge $q$ with velocity $\mathbf{v}$, in the form now commonly known as the Lorentz force law,[17]

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \tag{36}$$

This force law would have appealed to Faraday as it expresses the force on an electric charge due to distant electric currents in terms of the magnetic field at the charge rather than the distant currents.[18]

We note that if this force law applied to the Ampère-Weber magnetic field discussed above, eq. (19), then the magnetic field due to a "dipole" magnet driven by a pair of coaxial (Helmholtz) coils would have field lines between the coils essentially as shown in the figure immediately above. As such, a charged particle moving between the coils would scarcely be deflected according the Lorentz force "law" of eq. (36), in contrast to the substantial deflection observed in practice.

This further reinforces that the Ampère-Weber magnetic field does not correspond to observed effects of magnetism.

## A.1 An Alternative to the Lorentz Force Law?

Can we generalize the alternative force law of eq. (20), which we now call the Ampère-Weber force law, to describe the magnetic force on an electric charge $q$ with velocity $\mathbf{v}$ in an alternative manner to the Lorentz force law (36)?

We could replace $I_1\, d\mathbf{l}_1$ by $q\mathbf{v}$ to write,

$$d\mathbf{F}_{\mathrm{A-W}} = -q\mathbf{v} \cdot \{d\mathbf{B}_{\mathrm{A-W}}(I_2\, d\mathbf{l}_2) + 2\boldsymbol{\nabla}[d\mathbf{B}_{\mathrm{A-W}}(I_2\, d\mathbf{l}_2) \cdot \mathbf{r}]\}\, \hat{\mathbf{r}}, \tag{37}$$

but to integrate this to obtain the total magnetic force on the charge $q$ we must know both the field $\mathbf{B}_{\mathrm{A-W}}$ at the charge as well as the details of the distant current $I_2$.[19] This prescription would not have appealed to Faraday, since despite the appearance of the field $\mathbf{B}_{\mathrm{A-W}}$ in eq. (37), we must also include "action-at-a-distance" contributions from $I_2\, d\mathbf{l}_2$.

# B Appendix: Forces on Magnetic Poles (Aug. 21, 2021)

An important insight of Ampère was that all magnetism is due to electric current, rather than to magnetic poles, as had been assumed by all previous workers. In particular (as mentioned

[17]This "law" was anticipated by Maxwell in Arts. 598-599 of [23], as reviewed in sec. A.28.4.7 of [35].

[18]Of course, the magnetic field at the charge is related to the distant currents by the Biot-Savart expression (8) for the magnetic field.

[19]The Ampère-Weber force law (20) is equivalent to Ampère's original force law (1), which also does not well generalize to an expression for the force on a moving electric charge, or on an isolated current element. Ampère understood this, and inferred that isolated current elements (isolated, moving electric charges) could not exist.

in footnote 7 above), Biot and Savart studied the interaction of a magnetic needle with an electric current, supposing that a magnetic pole $p$ resided on the tip of the needle, such that the force law they proposed can be written in vector form as,

$$\mathbf{F} = p\,\mathbf{B}, \qquad \text{where} \qquad \mathbf{B} = \frac{\mu_0}{4\pi}\oint \frac{I\,d\mathbf{l}\times\hat{\mathbf{r}}}{r^2}. \tag{38}$$

In particular, for the case of steady current $I\,\hat{\mathbf{z}}$ in a long straight wire, the Biot-Savart magnetic field is $\mathbf{B} = \mu_0 I\,\hat{\mathbf{z}}\times\hat{\mathbf{r}}/2\pi r$ at the magnetic pole $p$ at (transverse) distance $\mathbf{r}$ from the wire.

If we consider that an alternative magnetic field must also describe the force on a magnetic pole according to $\mathbf{F} = p\,\mathbf{B}_{\rm alt}$, then it is clear that the form $\mathbf{B}_{\rm A-W}$ of eq. (19) does not satisfy this. In particular, for the case of a magnetic field (35) due to the current in a long, straight wire (as in the experiment of Biot and Savart [1]), $\mathbf{B}_{\rm A-W} = \mu_0 I\,\hat{\mathbf{z}}/16r$ is parallel to the wire, which would imply a force on the pole parallel to the wire, rather than in the direction $\mathbf{I}\times\mathbf{r}$ as observed experimentally.

We could propose that the Ampère-Weber force law for a magnetic pole $p$ is,

$$\mathbf{F}_{\rm A-W} = \frac{8}{\pi}p\,\mathbf{B}_{\rm A-W}\times\hat{\mathbf{r}}, \tag{39}$$

which predicts the result of the original experiment of Biot and Savart [1]. However, this form is somewhat unsatisfactory as a field-theory result in that it requires knowledge of the vector $\mathbf{r}$ from the pole to the wire, in addition to that of the field $\mathbf{B}_{\rm A-W}$.[20]

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