Mansuripur's Paradox

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1 Problem

An electrically neutral current-loop,[1] with magnetic dipole $\mathbf{m}_0$, that is at rest in a static, uniform electric field $\mathbf{E}_0$ experiences no force or torque. However, if that system is observed in the lab frame where the loop has velocity $\mathbf{v}$ parallel to $\mathbf{E}_0$, and $v \ll c$, where c is the speed of light, then there appears to be an electric dipole moment,[2] $\mathbf{p} = \mathbf{v}/c \times \mathbf{m}_0$ associated with the loop (in Gaussian units). Naïvely, the torque on this moment due to the electric field (which has strength $\mathbf{E} = \mathbf{E}_0 + \mathcal{O}(v^2/c^2)$ in the lab frame) is $\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} \approx E_0 v \mathbf{m}_0/c$.

Can/should the torque be different in different frames of reference?

This paradox was first posed by Spavieri [3][3] *and more recently by Mansuripur [7].*[4,5] *It is a conceptual variant of a famous problem by Shockley [16] that introduced the concept of "hidden mechanical momentum".*[6]

The paradox is compounded by supposing the static, "uniform" electric field is due to a single electric charge q at large, fixed distance from the magnetic moment in the rest frame of the latter, and the lab-frame velocity $\mathbf{v}$ is along the line of centers of the charge and moment. Discuss the force on charge q in the lab frame.[7]

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[1] This problem avoids the delicate issue of whether a current loop actually is electrically neutral, naïvely assuming this to be so. For discussion of the small departures from this assumption required so that the transverse force on the conduction charges equal the centripetal force, see [1].

[2] See, for example, eq. (2) of [2].

[3] Spavieri built on earlier examples of Bedford and Krumm [4] that were also discussed by Namias [5] and by Vaidman [6].

[4] Mansuripur's main concern seems to be with an old debate about the electrodynamics of moving media, and the validity of the Lorentz force law in such media, being unaware that there is no generally valid form of the Lorentz force law that uses the total $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{H}$ fields of macroscopic electrodynamics. Among the extensive literature on this subject, see, for example [8]. See also [9] by the author. Mansuripur claims to resolve the paradox by use of the so-called Einstein-Laub force density [10, 11], which was derived from an invalid form of the Lorentz force law and leads to nonzero self forces, as discussed in [12] and in sec. 2.3.1 of [13]. Mansuripur also becomes enmeshed in the so-called Abraham-Minkowski debate concerning the nature of "electromagnetic" momentum in media at rest (see, for example, [14], where Table 1 summarizes five variants of the electromagnetic force law). Mansuripur considers that magnetic dipoles based on conduction-current loops are equivalent to (quantum) magnetization, and notes that if one uses the Abraham field momentum rather than the Minkowski momentum, then there is no field momentum for magnetization in an electric field, and no "hidden" mechanical momentum. However, this argument does not apply to a magnetic dipole based on a loop of conduction current, which appears to be the subject of [7].

[5] A related issue is that the torque on a pair of charged particles with relative motion is nonzero in general, but is zero if one particle is fixed in a particular inertial frame [15].

[6] See [17] for commentary on the relation between "hidden" mechanical momentum and the Abraham-Minkowski debate.

[7] This configuration is that of the Aharonov-Bohm effect [18], which has the classical paradox that if the charge is in motion in the rest frame of the magnetic moment, then the magnetic moment, but not the charge, is subject to a force [19].

2 Solution

The magnetic moment of a loop of current I of radius a has the magnitude $m_0 = \pi a^2 I/c$, so the supposed lab-frame torque, of magnitude $\tau = E_0 v m_0/c = \pi a^2 I E_0 v/c^2$, is an effect of order $1/c^2$. Hence, discussion of the problem should include all relevant effects at order $1/c^2$, even though the restriction that $v \ll c$ suggests that terms of order $1/c^2$ could be ignored.[8]

The presence of a torque $\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}$ would suggest that the lab-frame electric dipole moment $\mathbf{p}$ will "precess" about $\mathbf{m}_0$ so as to bring it into alignment with the electric field $\mathbf{E}$.

However, the apparent electric dipole moment $\mathbf{p} = \mathbf{v}/c \times \mathbf{m}_0$ is independent of $\mathbf{E}$, and so cannot move into alignment with that field.[9] It must be that the torque, if it exists, has no effect on the mechanical configuration of the system.

A key to the resolution of the paradox is that moving dipoles do not have the properties that one might naïvely associate with them, based on understanding in their rest frame (sec. 2.1). Analysis of a moving dipole in the lab frame is best obtained via transformation from its rest frame, which implies that the torque is nonzero in the lab frame in the present example.

Before detailed analysis of the lab-frame torque in sec. 2.3, we first discuss a force paradox sec. 2.2) which illustrates that the lab-frame fields of a moving dipole are not those naïvely associated with the lab-frame moments $\mathbf{p}$ and $\mathbf{m}$. The need for a nonzero torque in the lab frame is related to the existence of time-dependent "hidden" angular momentum in the lab frame, are discussed in secs. 2.4-5. The complementary problem of an electric dipole in an external magnetic field is considered in sec. 2.6. Finally, in sec. 2.7 we note that the present example cannot be realized in systems with "ordinary" (or superconducting) currents, but could only arise in rather "academic" thought experiments if conduction-current loops are involved.

2.1 Ambiguity as to the Meaning of Moving Dipole Moments

The concepts of dipole moments arose in examples of systems with charges at rest, and/or where all current densities $\mathbf{J}$ are divergence free (*i.e.*, $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$). In particular, the definition of the magnetic moment $\mathbf{m}$ as,

$$\mathbf{m} = \int \frac{\mathbf{r} \times \mathbf{J}}{2c}\, d\text{Vol} \tag{1}$$

assumes that $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$, which does not hold for the present example in the lab frame where the current density has bulk motion.

One can define electric and magnetic dipole moments $\mathbf{p}$ and $\mathbf{m}$ in any frame in terms of volume integrals of the densities $\mathbf{P}$ and $\mathbf{M}$ of electric and magnetic polarization (in that frame), respectively,

$$\mathbf{p} = \int \mathbf{P}\, d\text{Vol}, \qquad \mathbf{m} = \int \mathbf{M}\, d\text{Vol}. \tag{2}$$

[8] This point was emphasized by Coleman and Van Vleck [20] in their commentary on [16].

[9] When $\mathbf{v}$ is parallel to $\mathbf{E}$ (and $\mathbf{B} = 0$), there is no precession of the magnetic moment $\mathbf{m}_0$. See, for example, [21, 22]. This result applies also to spin-1/2 elementary particles, whose magnetic moment is the quantum equivalent of current loops, rather than pairs of equal and opposite magnetic charges, as first noted by Fermi [23] based on considerations of the hyperfine interaction.

The relativistic transformations of densities $\mathbf{P}$ and $\mathbf{M}$ were first discussed by Lorentz [24], who noted that they follow the same transformations as do the magnetic and electric fields $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ and $\mathbf{E} = \mathbf{D} - 4\pi\mathbf{P}$, respectively,

$$\mathbf{P} = \gamma\left(\mathbf{P}_0 + \frac{\mathbf{v}}{c} \times \mathbf{M}_0\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{P}_0)\hat{\mathbf{v}}, \quad \mathbf{M} = \gamma\left(\mathbf{M}_0 - \frac{\mathbf{v}}{c} \times \mathbf{P}_0\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{M}_0)\hat{\mathbf{v}}, \tag{3}$$

where the inertial lab frame moves with velocity $\mathbf{v}$ with respect to the (inertial) rest frame of the polarization densities, and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Considerations of the fields moving electric dipoles perhaps first arose in the context of Čerenkov radiation when the dipole velocity v exceeds the speed of light c/n in a medium of index of refraction n [25]. Later discussions by Frank of his pioneering work are given in [26, 27].[10]

As noted by Frank [27],[11] specialization of eq. (3) to point electric and magnetic dipole moments $\mathbf{p}$ and $\mathbf{m}$, and to low velocities, leads to the forms,

$$\mathbf{p} \approx \mathbf{p}_0 + \frac{\mathbf{v}}{c} \times \mathbf{m}_0, \qquad \mathbf{m} \approx \mathbf{m}_0 - \frac{\mathbf{v}}{c} \times \mathbf{p}_0, \qquad \mathbf{p}_0 \approx \mathbf{p} - \frac{\mathbf{v}}{c} \times \mathbf{m}, \qquad \mathbf{m}_0 \approx \mathbf{m} + \frac{\mathbf{v}}{c} \times \mathbf{p}, \tag{4}$$

where $\mathbf{p}_0$ and $\mathbf{m}_0$ and the moments in the rest frame of the point particle, while $\mathbf{p}$ and $\mathbf{m}$ are the moments when the particle has velocity $\mathbf{v}$ in the lab frame. However, the fields associated with an electric and/or magnetic dipole moving at low velocity are not simply the instantaneous fields of the moments $\mathbf{p}$ and $\mathbf{m}$, which leads to ambiguities in interpretations of the physical significance of the dipole moments $\mathbf{p}$ and $\mathbf{m}$ in the lab frame.

In sec. III of [32] it is claimed that the magnetic moment of a point electric dipole $\mathbf{p}_0$ which moves with velocity $\mathbf{v}$ is,

$$\mathbf{m} = -\frac{\mathbf{v}}{2c} \times \mathbf{p}_0, \tag{5}$$

which differs from eq. (4) by a factor of 2. Additional support for this form appears to be given in probs. 6.21, 6.22 and 11.27 of [33]. The relation (5) (and an infinite set of other definitions) can be used (with care) in the lab frame, but we use the convention of eq. (4) in the following.

As discussed in sec. 2.3, the forces and torques experienced by moving moments in the lab frame are not equal to those that might be expected on the lab-frame values of moments $\mathbf{p}$ and $\mathbf{m}$. This situation is somewhat anomalous in the theory of relativity, and it is therefore not surprising that it has led to various confusions over the years, apparently beginning with the Trouton-Noble experiment [35].

The approach taken in the rest of this note is that consistent results in the lab frame are best obtained by transformation of the corresponding results in the rest frame of the system.

2.2 The External Field is Due to a Single Distant Charge

Suppose that the external field $\mathbf{E}_0 = E_0\,\hat{\mathbf{x}}$ near the dipole is due to a single charge q at $x = -d_0$ for large d_0 (in the rest frame).

[10] The relation that $\mathbf{p} \approx \mathbf{v}/c \times \mathbf{m}_0$ when $\mathbf{p}_0 = 0$ appears in eq. (13) of [28], which was inspired by sec. 600 of [29]. See also [30]. That a moving current leads to an apparent charge separation was noted in [31].

[11] Ref. [27] recounts a past controversy that these transformations might involve the index n if the magnetic dipole were not equivalent to an Ampèrian current loop.

In the lab frame we might argue that the force on q is due to both the electric field from the apparent electric dipole $\mathbf{p} \approx \mathbf{v}/c \times \mathbf{m}_0$ and the magnetic field of the magnetic moment $\mathbf{m} \approx \mathbf{m}_0$,

$$\mathbf{F}_q \stackrel{?}{=} q\left(\mathbf{E}_p + \frac{\mathbf{v}}{c} \times \mathbf{B}_m\right) \approx q\left(-\frac{\mathbf{p}}{d_0^3} + \frac{\mathbf{v}}{c} \times \frac{-\mathbf{m_0}}{d_0^3}\right) = -2q\frac{\mathbf{v}}{c} \times \frac{\mathbf{m_0}}{d_0^3} = -\frac{2qvm_0}{d_0^3}\,\hat{\mathbf{y}}. \tag{6}$$

How can this be, as the force on the charge is zero in the rest frame, and 3-force is invariant under low-velocity transformations? Is the Lorentz force law incompatible with relativity?

The issue is that the fields of a dipole moving at low velocity are not the same as the (instantaneous) fields of the moments obtained by transformation of the moments from their rest frame. That is, the meaning of a moving dipole must be considered with care.

The proper calculation is that,

$$\mathbf{F}_q = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \tag{7}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the Lorentz transformations of the fields of the magnetic moment $\mathbf{m}_0$ in its rest frame, where,

$$\mathbf{E}_m = 0, \qquad \mathbf{B}_m = -\frac{\mathbf{m_0}}{d_0^3}, \tag{8}$$

at charge q. The (low-velocity) transforms of these to the lab frame are,

$$\mathbf{E} \approx \mathbf{E}_m - \frac{\mathbf{v}}{c} \times \mathbf{B}_m = -\frac{\mathbf{v}}{c} \times \mathbf{B}_m, \qquad \mathbf{B} \approx \mathbf{B}_m + \frac{\mathbf{v}}{c} \times \mathbf{E}_m = \mathbf{B}_m. \tag{9}$$

Using these in eq. (7) we find $\mathbf{F}_q = 0$ as expected (and that the Lorentz force law works fine).

It remains disconcerting that the electric field in the lab frame at charge q is the negative of that inferred from the relation $\mathbf{p} \approx \mathbf{v}/c \times \mathbf{m}_0$. For additional discussion of these matters, see [2], particularly sec. 3.[12]

2.3 Torque in the Lab Frame

2.3.1 A Naïve Analysis

To calculate the torque on the dipole in the lab frame, we might presume that the apparent electric dipole moment $\mathbf{p} \approx \mathbf{v}/c \times \mathbf{m}_0$ can be associated with charges $\pm Q$ with small separation, such that $\mathbf{p} = Q(\mathbf{r}_+ - \mathbf{r}_-)$. In this case, we could write (following [5]),

$$\begin{aligned}\boldsymbol{\tau}_p &\stackrel{?}{=} \mathbf{r}_+ \times Q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) + \mathbf{r}_- \times -Q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) = Q(\mathbf{r}_+ - \mathbf{r}_-) \times \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \\ &= \mathbf{p} \times \mathbf{E} + \mathbf{p} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right),\end{aligned} \tag{10}$$

in the limit of a point dipole. Similarly, if the moving particle has magnetic dipole $\mathbf{m}$ in the lab frame, it seems reasonable (again following [5]) that the partial torque $\boldsymbol{\tau}_p$ due to external $\mathbf{E}$ and $\mathbf{B}$ fields can be deduced by supposing that the dipole consists of a pair of magnetic

[12] Related discussion is given in [34].

charges $\pm Q_m$ subject to the Lorentz force $Q_m(\mathbf{B} - \mathbf{v}/c \times \mathbf{E})$, which leads (in the limit of a point dipole) to,

$$\begin{aligned}
\boldsymbol{\tau}_m &\overset{?}{=} \mathbf{r}_+ \times Q_m \left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right) + \mathbf{r}_- \times -Q_m \left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right) = Q_m(\mathbf{r}_+ - \mathbf{r}_-) \times \left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right) \\
&= \mathbf{m} \times \mathbf{B} - \mathbf{m} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{E}\right). \qquad (11)
\end{aligned}$$

The total torque (in case of a Gilbert magnetic dipole) is then,

$$\boldsymbol{\tau} = \boldsymbol{\tau}_p + \boldsymbol{\tau}_m \overset{?}{\approx} \mathbf{p} \times \mathbf{E} + \mathbf{m} \times \mathbf{B} + \mathbf{p} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right) - \mathbf{m} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{E}\right). \qquad (12)$$

In the present example, the external electric field in the lab frame is just $\mathbf{E} \approx \mathbf{E}_0$ to order v/c, and the external magnetic field is $\mathbf{B} \approx -\mathbf{v}/c \times \mathbf{E}_0 \approx -\mathbf{v}/c \times \mathbf{E}$. Also, the magnetic moment is $\mathbf{m}_0$ and the electric dipole moment is $\mathbf{p} \approx \mathbf{v}/c \times \mathbf{m}_0$ in this frame, to order v/c. Then, to this order, the total torque on the moving dipole is,

$$\boldsymbol{\tau} \overset{?}{\approx} \mathbf{p} \times \mathbf{E} - \mathbf{m} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{E}\right). \qquad (13)$$

While this computation of the torque is not equal to $\mathbf{p} \times \mathbf{E}$ in general, it does equal this if $\mathbf{v}$ is parallel to $\mathbf{E}$ (or if $\mathbf{m}_0$ is parallel to $\mathbf{v} \times \mathbf{E}$).

2.3.2 Analysis via Transformation of Torque from the Rest Frame

However, a lesson of secs. 2.1-2 is that computations in the lab frame involving the apparent dipole moments $\mathbf{p}$ and $\mathbf{m}$ (in that frame) may give invalid results. Furthermore, we desire an analysis for an Ampèrian magnetic dipole. A better procedure is to transform computations in the rest frame to the lab frame.

For this, it is insightful to consider the (antisymmetric) torque tensor,

$$\tau_{\mu\nu} = \int (r_\mu f_\nu - r_\nu f_\mu)\, d\mathrm{Vol} = \sum r_\mu F_\nu - r_\nu F_\mu, \qquad (14)$$

where $r_\mu = (ct, \mathbf{r})$, the Lorentz 4-force density is $f_\mu = (\mathbf{J} \cdot \mathbf{E}/c, \rho\mathbf{E} + \mathbf{u}/c \times \mathbf{B})$ for a system with charge-current-density 4-vector $(c\rho, \mathbf{J})$, and the Lorentz 4-force is $F_\mu = \gamma_u(\mathbf{F} \cdot \mathbf{u}/c, \mathbf{F})$ for a system of particles with velocities $\mathbf{u}$ and $\gamma_u = 1/\sqrt{1 - u^2/c^2}$

The torque 3-vector $\boldsymbol{\tau}_0 = (\tau_{23}, -\tau_{13}, \tau_{12})$ on a general dipole with electric-charge dipole moment $\mathbf{p}_0$ and Ampèrian magnetic moment (due to electrical currents) $\mathbf{m}_0$ in its rest frame is,

$$\begin{aligned}
\boldsymbol{\tau}_0 &= \int \mathbf{r}_0 \times \left(\rho_0 \mathbf{E}_0 + \frac{\mathbf{J}_0 \times \mathbf{B}_0}{c}\right) d\mathrm{Vol}_0 \\
&= \int \rho_0 \mathbf{r}_0 \, d\mathrm{Vol}_0 \times \mathbf{E}_0 + \int \frac{(\mathbf{r}_0 \cdot \mathbf{B}_0)\mathbf{J}_0 - (\mathbf{r}_0 \cdot \mathbf{J}_0)\mathbf{B}_0}{c}\, d\mathrm{Vol}_0 \\
&= \mathbf{p}_0 \times \mathbf{E}_0 + \mathbf{m}_0 \times \mathbf{B}_{\mathbf{0}}, \qquad (15)
\end{aligned}$$

noting that for steady current $\mathbf{J}_0$,

$$\int \mathbf{r}_0 \cdot \mathbf{J}_0 \, d\mathrm{Vol}_0 = \sum_n q_n \mathbf{r}_{0n} \cdot \mathbf{u}_{0n} = \frac{1}{2}\frac{d}{dt}\sum_n q_n r_{0n}^2 = 0, \tag{16}$$

and,

$$\begin{aligned}
\int \frac{(\mathbf{r}_0 \cdot \mathbf{B}_0)\mathbf{J}_0}{c} \, d\mathrm{Vol}_0 &= \sum_n \frac{q_n(\mathbf{r}_{0n} \cdot \mathbf{B}_0)\mathbf{u}_{0n}}{c} \\
&= \sum_n \frac{q_n(\mathbf{r}_{0n} \cdot \mathbf{B}_0)\mathbf{u}_{0n}}{2c} - \sum_n \frac{q_n(\mathbf{u}_{0n} \cdot \mathbf{B}_0)\mathbf{r}_{0n}}{2c} + \frac{1}{4c}\frac{d}{dt}\sum_n q_n(\mathbf{r}_{0n} \cdot \mathbf{B}_0)\mathbf{r}_{0n} \\
&= \sum_n \frac{q_n(\mathbf{r}_{0n} \times \mathbf{u}_{0i})}{2c} \times \mathbf{B}_0 = \mathbf{m}_0 \times \mathbf{B_0}, \qquad (17)
\end{aligned}$$

where the magnetic moment in the rest frame is,

$$\mathbf{m}_0 = \sum_n \frac{q_n(\mathbf{r}_{0n} \times \mathbf{u}_{0n})}{2c} = \int \frac{\mathbf{r}_0 \times \mathbf{J}_0}{2c} \, d\mathrm{Vol}_0. \tag{18}$$

The rest-frame torque tensor also has components,

$$\tau_{0,0i} = -\tau_{0,i0} = ct_0 \int f_{0,i} \, d\mathrm{Vol} - \int \frac{r_{0,i}(\mathbf{J}_0 \cdot \mathbf{E}_0)}{c} \, d\mathrm{Vol} = [\mathbf{m}_0 \times \mathbf{E_0}]_i, \tag{19}$$

for a system subject to zero total Lorentz force, $\int \mathbf{f}_0 \, d\mathrm{Vol} = 0$, noting that,

$$\begin{aligned}
\int \frac{\mathbf{r}_0(\mathbf{J}_0 \cdot \mathbf{E}_0)}{c} \, d\mathrm{Vol}_0 &= \sum_n \frac{q_n(\mathbf{u}_{0n} \cdot \mathbf{E}_0)\mathbf{r}_{0n}}{c} \\
&= \sum_n \frac{q_n(\mathbf{u}_{0n} \cdot \mathbf{E}_0)\mathbf{r}_{0n}}{2c} - \sum_n \frac{q_n(\mathbf{r}_{0n} \cdot \mathbf{E}_0)\mathbf{u}_{0n}}{2c} + \frac{1}{4}\frac{d}{dt}\sum_n q_n(\mathbf{r}_{0n} \cdot \mathbf{E}_0)\mathbf{r}_{0n} \\
&= -\sum_n \frac{q_n(\mathbf{r}_{0n} \times \mathbf{u}_{0n})}{2c} \times \mathbf{E}_0 = -\mathbf{m}_0 \times \mathbf{E_0}. \qquad (20)
\end{aligned}$$

The result (19) may be surprising, in that for a case like the present example, in which the rest-frame 3-torque (15) is zero, the 4-torque is non-zero, and consequently the 3-torque is nonzero in other frames.[13] This peculiar result holds only for a particle with an (Ampèrian) magnetic dipole moment due to electric currents. If the magnetic dipole consisted of a pair of opposite magnetic charges (Gilbert dipole), $\tau_{0,0i}$ would be zero (assuming that the system does not have a magnetic current loop and associated Gilbertian electric dipole moment). Hence, the torque on a moving magnetic dipole in an external electric fields provides a classical distinction between Ampèrian and Gilbertian moments (which distinction is often considered to be relevant only in the quantum domain).

[13]This effect invalidates a claim in [36] that if the 3-torque $\boldsymbol{\tau}$ is zero in one (inertial) frame it is zero in all (inertial) frames.

For low velocities of the lab frame we have that,

$$\mathbf{r} \approx \mathbf{r}_0 + \mathbf{v}t, \qquad d\mathrm{Vol} \approx d\mathrm{Vol}_0, \tag{21}$$

$$\rho \approx \rho_0 + \frac{\mathbf{J}_0 \cdot \mathbf{v}}{c^2}, \qquad \mathbf{J} \approx \mathbf{J}_0 + \rho_0 \mathbf{v}, \tag{22}$$

$$\mathbf{E} \approx \mathbf{E}_0 - \frac{\mathbf{v}}{c} \times \mathbf{B}_0, \qquad \mathbf{B} \approx \mathbf{B}_0 + \frac{\mathbf{v}}{c} \times \mathbf{E}_0. \tag{23}$$

Hence, the lab-frame Lorentz force density is related to that in the rest frame by,

$$\rho\mathbf{E} + \frac{\mathbf{J}}{c} \times \mathbf{B} \approx \rho_0 \mathbf{E}_0 + \frac{\mathbf{J}_0}{c} \times \mathbf{B}_0 + \frac{(\mathbf{J}_0 \cdot \mathbf{E}_0)\mathbf{v}}{c^2}, \tag{24}$$

which can be confirmed explicitly using eqs. (22)-(23).[14]

The lab-frame 3-torque at time $t = 0$ is thus,

$$\begin{aligned}
\boldsymbol{\tau} &= \int \mathbf{r} \times \left(\rho\mathbf{E} + \frac{\mathbf{J} \times \mathbf{B}}{c}\right) d\mathrm{Vol} \\
&\approx \boldsymbol{\tau}_0 + \int \mathbf{r} \times \frac{(\mathbf{J}_0 \cdot \mathbf{E}_0)\mathbf{v}}{c^2}\, d\mathrm{Vol} \approx \boldsymbol{\tau}_0 + \int \frac{\mathbf{r}_0(\mathbf{J}_0 \cdot \mathbf{E}_0)}{c}\, d\mathrm{Vol}_0 \times \frac{\mathbf{v}}{c} \\
&= \mathbf{p}_0 \times \mathbf{E}_0 + \mathbf{m}_0 \times \mathbf{B}_0 + \frac{\mathbf{v}}{c} \times (\mathbf{m}_0 \times \mathbf{E}_0)
\end{aligned} \tag{26}$$

recalling eqs. (15) and (20).

We can express the lab-frame torque in terms of lab-frame quantities using eq. (4),

$$\begin{aligned}
\boldsymbol{\tau} &\approx \mathbf{p}_0 \times \mathbf{E}_0 + \mathbf{m}_0 \times \mathbf{B}_0 + \frac{\mathbf{v}}{c} \times (\mathbf{m}_0 \times \mathbf{E}_0) \\
&\approx \left(\mathbf{p} - \frac{\mathbf{v}}{c} \times \mathbf{m}\right) \times \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) + \left(\mathbf{m} + \frac{\mathbf{v}}{c} \times \mathbf{p}\right) \times \left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right) \\
&\quad + \frac{\mathbf{v}}{c} \times \left[\left(\mathbf{m} + \frac{\mathbf{v}}{c} \times \mathbf{p}\right) \times \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)\right] \\
&\approx \mathbf{p} \times \mathbf{E} + \mathbf{m} \times \mathbf{B} + \mathbf{p} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right) - \mathbf{B} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{p}\right) + \mathbf{E} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{m}\right) - \mathbf{m} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{E}\right) \\
&\quad + \frac{\mathbf{v}}{c} \times (\mathbf{m} \times \mathbf{E}) \\
&= \mathbf{p} \times \mathbf{E} + \mathbf{m} \times \mathbf{B} + \frac{\mathbf{v}}{c} \times (\mathbf{p} \times \mathbf{B}) \qquad \text{(Ampèrian magnetic dipole)},
\end{aligned} \tag{27}$$

noting the identity that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \times (\mathbf{a} \times \mathbf{c}) - \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$. This result differs somewhat from that of eq. (12).[15]

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[14]We could also represent the Lorentz force density in terms of polarization densities $\mathbf{P}$ and $\mathbf{M}$, replacing ρ by $-\boldsymbol{\nabla} \cdot \mathbf{P}$ and $\mathbf{J}$ by $\partial \mathbf{P}/\partial t + c\boldsymbol{\nabla} \times \mathbf{M}$, where we can associate the electric and magnetic dipole moments $\mathbf{p}$ and $\mathbf{m}$ of a "point" particle at the origin with "free" polarization densities $\mathbf{P}$ and $\mathbf{M}$ according to, $\mathbf{P} = \mathbf{p}\,\delta^3(\mathbf{r})$ and $\mathbf{M} = \mathbf{m}\,\delta^3(\mathbf{r})$. These relations are best established first in the rest frame, where, $\mathbf{P}_0 = \mathbf{p}_0\,\delta^3(\mathbf{r}_0)$ and $\mathbf{M}_0 = \mathbf{m}_0\,\delta^3(\mathbf{r}_0)$. and then transforming to the lab frame via eq. (3), with the result for low velocities,

$$\mathbf{P} \approx \mathbf{P}_0 + \frac{\mathbf{v}}{c} \times \mathbf{M}_0 = \left(\mathbf{p}_0 + \frac{\mathbf{v}}{c} \times \mathbf{m}_0\right) \delta^3(\mathbf{r}), \quad \mathbf{M} = \mathbf{M}_0 - \frac{\mathbf{v}}{c} \times \mathbf{P}_0 = \left(\mathbf{m}_0 - \frac{\mathbf{v}}{c} \times \mathbf{p}_0\right) \delta^3(\mathbf{r}). \tag{25}$$

Then, the inverse relations $\mathbf{p} = \int \mathbf{P}\, d\mathrm{Vol}$ and $\mathbf{m} = \int \mathbf{M}\, d\mathrm{Vol}$ are consistent with eq. (4).

See [37, 38] for use of this approach for the present example.

[15]Mansuripur states without apparent derivation in either [7] or [39] that $\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} + \mathbf{m} \times \mathbf{B}$ for a system on which the total force $\mathbf{F}$ is zero.

In the rest frame of the present example the 3-vector $\boldsymbol{\tau}_0$, eq. (15), is zero, while to order $1/c^2$ the lab-frame 3-torque is,

$$\boldsymbol{\tau} \approx \mathbf{p} \times \mathbf{E} \qquad \text{(present example, Ampèrian magnetic dipole)}, \tag{28}$$

since $\mathbf{B} = 0$ here. This confirms the existence of a nonzero lab-frame torque in the present example, so paradox remains as its physical significance in case of an Ampèrian magnetic dipole.

If instead the magnetic dipole $\mathbf{m}_0$ were made from opposite magnetic charges (and the electric dipole $\mathbf{p}_0$ is made from opposite electric charges), then $\tau_{0,0i} = 0$, and the lab-frame 3-torque is given by,

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{\tau}_0 = \mathbf{p}_0 \times \mathbf{E}_0 + \mathbf{m}_0 \times \mathbf{B}_0 \\ &\approx \mathbf{p} \times \mathbf{E} + \mathbf{m} \times \mathbf{B} + \frac{\mathbf{v}}{c} \times (\mathbf{p} \times \mathbf{B} - \mathbf{m} \times \mathbf{E}) \qquad \text{(Gilbert magnetic dipole)}. \end{aligned} \tag{29}$$

Thus, the naïve computation (12) is slightly wrong in general, which further illustrates the difficulty in interpreting lab-frame dipole moments.

In the rest frame of present example the torque 4-tensor is zero (for a Gilbert magnetic dipole), so this tensor is also zero in the lab frame, as is the lab-frame 3-torque,[16]

$$\boldsymbol{\tau} = 0 \qquad \text{(present example, Gilbert magnetic dipole)}. \tag{30}$$

Mansuripur's paradox vanishes for the case of a Gilbert magnetic dipole, but remains for an Ampèrian one.

2.3.3 Transformation in Two Steps

An analysis that uses rest-frame and lab-frame quantities in the same steps can be given by supposing that the moments in eqs. (10)-(11) are those in the rest frame (and that the magnetic moment is Gilbertian), while the positions, velocities and fields are those in the lab frame [5]. This analysis can be regarded as a partial transformation of the analysis in the rest frame (ignoring possible nonzero components τ_{0i} in that frame), in which the positions and fields are transformed to the lab frame, but the moments are not transformed. In this hybrid view, the lab-frame torques are,

$$\boldsymbol{\tau}_p = \mathbf{p}_0 \times \mathbf{E} + \mathbf{p}_0 \times \left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right), \qquad \boldsymbol{\tau}_m = \mathbf{m}_0 \times \mathbf{B} - \mathbf{m}_0 \times \left(\frac{\mathbf{v}}{c} \times \mathbf{E}\right). \tag{31}$$

In the present example, $\mathbf{p}_0$ and $\mathbf{B}$ are zero, while $\mathbf{v} \parallel \mathbf{E}$, so the total torque $\boldsymbol{\tau} = \boldsymbol{\tau}_p + \boldsymbol{\tau}_m$ in the lab frame is zero, according to eq. (31).

We can complete the partial transformations of eq. (31), recalling eq. (4),

$$\boldsymbol{\tau}_p \approx \left(\mathbf{p} - \frac{\mathbf{v}}{c} \times \mathbf{m}\right) \times \mathbf{E} + \mathbf{p} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right), \tag{32}$$

[16]In Mansuripur's example, $\mathbf{B} = 0$ in the lab frame where $\mathbf{p} = \mathbf{v}/c \times \mathbf{m}$, so eq. (29) becomes $\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} - \mathbf{v}/c \times (\mathbf{m} \times \mathbf{E}) = (\mathbf{v} \cdot \mathbf{m})\mathbf{E}/c - (\mathbf{E} \cdot \mathbf{m})\mathbf{v}/c = 0$, since $\mathbf{E}$ and $\mathbf{v}$ are parallel to one another and perpendicular to $\mathbf{m}$.

$$\boldsymbol{\tau}_m = \left(\mathbf{m} + \frac{\mathbf{v}}{c} \times \mathbf{p}\right) \times \mathbf{B} - \mathbf{m} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{E}\right). \tag{33}$$

Using the identity that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \times (\mathbf{a} \times \mathbf{c}) - \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ the total torque in the lab frame can be written as,

$$\boldsymbol{\tau} = \boldsymbol{\tau}_p + \boldsymbol{\tau}_m \approx \mathbf{p} \times \mathbf{E} + \mathbf{m} \times \mathbf{B} + \frac{\mathbf{v}}{c} \times (\mathbf{p} \times \mathbf{B} - \mathbf{m} \times \mathbf{E}) \qquad \text{(Gilbert magnetic dipole)}. \tag{34}$$

as previously found in eq. (29), where all quantities in this expression are in the lab frame.[17]

The torque (34) vanishes in the present example where $\mathbf{B} = 0$, $\mathbf{m} \approx \mathbf{m}_0$, $\mathbf{p} \approx \mathbf{v}/c \times \mathbf{m}_0$ and $\mathbf{v} \parallel \mathbf{E}$.

2.4 Field Momentum and "Hidden" Mechanical Momentum in the Rest Frame

The example of Mansuripur contains an additional subtlety that deserves comment. Namely, even in its rest frame the system possesses nonzero electromagnetic field momentum $\mathbf{P}_{\rm EM}$.[18]

For systems in which effects of radiation and of retardation can be ignored, the electromagnetic momentum can be calculated in various equivalent ways [41],

$$\mathbf{P}_{\rm EM} = \int \frac{\varrho \mathbf{A}}{c}\, d\mathrm{Vol} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d\mathrm{Vol} = \int \frac{V \mathbf{J}}{c^2}\, d\mathrm{Vol}, \tag{35}$$

where ϱ is the electric charge density, $\mathbf{A}$ is the magnetic vector potential (in the Coulomb gauge where $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$), $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, V is the electric (scalar) potential, and $\mathbf{J}$ is the electric current density. The first form is due to Faraday [42] and Maxwell [43], the second form is due to Poynting [44], Poincaré [45] and Abraham [46], and the third form was introduced by Furry [47].

Since a system at rest must have zero total momentum [20], it must also possess a "hidden" mechanical momentum $\mathbf{P}_{\rm hidden}$ equal and opposite to the field momentum.[19] This "hidden" momentum is a "relativistic" effect of order $1/c^2$.[20,21,22]

[17] Thus, the argument in [5] applies to a Gilbert dipole but not to an Ampèrian dipole, as noted in [40]. This argument is not self-evidently correct to this author, so this section provided the validation needed (in my view).

[18] This is also true for the configuration of the Aharonov-Bohm effect [18].

[19] For commentary on "hidden" momentum by the author, see [48].

[20] The electromagnetic field momentum $\mathbf{P}_{\rm EM}$ is also an effect of order $1/c^2$ in that the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B}$ are of order $1/c$, so all three forms of eq. (35) are of order $1/c^2$.

[21] As pointed out by Griffiths [40], and earlier by Haus [49], classical systems with nonzero "hidden" mechanical momentum must have moving parts. If magnetic charges existed, a system of static electric and magnetic charges would have no "hidden" mechanical momentum, and its total field momentum must also be zero, as explicitly verified in [40] for simple systems including dipoles. For example, a Gilbertian magnetic dipole involves no electric current, so eq. (36) would give zero field momentum, and hence zero "hidden" mechanical momentum.

[22] Mansuripur [39] denies the existence of "hidden" mechanical momentum, while apparently accepting the usual relations (35) for electromagnetic field momentum. This view seems to be based on the use of the Abraham rather than Minkowski field momentum, and the mistaken belief that all magnetic dipoles can be represented via a (quantum) magnetization density. Mansuripur may have been led to this view by his emphasis of the so-called Einstein-Laub force density [10] (which is more consistent with the Gilbertian result (34), as shown by Cross [38]).

We evaluate $\mathbf{P}_{\rm EM}$ for a magnetic moment $\mathbf{m}_0 = \pi a^2 I\,\hat{\mathbf{z}}/c$ due to current I which flows in a circular loop of radius a subject to external electric field $\mathbf{E}_0$ that makes angle α to $\mathbf{m}$, *i.e.*, $\mathbf{E}_0 = E_0(\sin\alpha\,\hat{\mathbf{x}} + \cos\alpha\,\hat{\mathbf{z}})$. The largest magnetic field is inside the loop, in the $\hat{\mathbf{z}}$ direction, so the second form of eq. (35) indicates that $\mathbf{P}_{\rm EM}$ will be in the $-\hat{\mathbf{y}}$ direction. This result is counterintuitive in that the direction of the momentum is not related to the direction of the velocity (if any). In the present problem $\mathbf{P}_{\rm EM}$ is perpendicular to $\mathbf{v}$, so the field angular momentum (39) is nonzero and position/time dependent for motion along the x-axis.

We use the third form of eq. (35) to compute the field momentum. The external electric field can be derived from the scalar potential $V = -E_0(x\sin\alpha + z\cos\alpha)$,and the y-component of $\mathbf{J}\,d$Vol is $Ia\cos\phi\,d\phi$ in cylindrical coordinates (ρ, ϕ, z) centered on the moment. Then, noting that $x = a\cos\phi$ and $z = 0$ on the loop, we find,

$$P_{{\rm EM},y} = \int \frac{VJ_y}{c^2}\,d{\rm Vol} = \int_0^{2\pi} \frac{(-E_0 a\cos\phi\sin\alpha)(Ia\cos\phi)}{c^2}\,d\phi = -\frac{\pi a^2 I E_0\sin\alpha}{c^2} = -\frac{m_0 E_0\sin\alpha}{c}\,. \tag{36}$$

That is,[23]

$$\mathbf{P}_{\rm EM} = \frac{\mathbf{E}_0\times\mathbf{m}_0}{c} \qquad \text{(Ampèrian magnetic dipole).} \tag{37}$$

As the total momentum of the system at rest must be zero, we infer that there exists "hidden" mechanical momentum given by,

$$\mathbf{P}_{\rm hidden} = -\mathbf{P}_{\rm EM} = -\frac{\mathbf{E}_0\times\mathbf{m}_0}{c}\,. \tag{38}$$

The momenta (37)-(38) are effects of order $1/c^2$. See, for example, [50] for a classical model of the "hidden" momentum (38).

2.5 Torque and Changing "Hidden" Angular Momentum

A classical (Ampèrian) magnetic moment $\mathbf{m}_0$ has intrinsic mechanical angular momentum $\mathbf{L}_0 = 2Mc\,\mathbf{m}_0/Q$ where M and Q are the mass and charge of the particles whose motion generates the moment. In addition, the moment is associated with "hidden" mechanical angular momentum given by,

$$\mathbf{L}_{\rm hidden} = \mathbf{r}\times\mathbf{P}_{\rm hidden}, \tag{39}$$

where $\mathbf{r}$ is the position of the center of the moment.

In the inertial frame where the magnetic moment has position $\mathbf{r} = \mathbf{v}t = vt\,\hat{\mathbf{x}}$, with $v \ll c$ (such that the electric field and the moment have the same values as in the moment's rest frame to order v/c, and the field momentum and the "hidden" mechanical momentum have their rest-frame values to order $1/c^2$), the mechanical angular momentum of the system is[24]

$$\mathbf{L}_{\rm mech} = \mathbf{L}_0 + \mathbf{L}_{\rm hidden} \approx \mathbf{L}_0 - \mathbf{v}t\times\frac{\mathbf{E}\times\mathbf{m}_0}{c}\,. \tag{40}$$

[23]The result (37) first appeared in eq. (37) of [47].

[24]The intrinsic mechanical angular momentum $\mathbf{L}_0$ has corrections at order v^2/c^2, but these are time-independent in the lab frame.

To support this time-varying mechanical angular momentum, we expect from classical mechanics that the system must be subject to a torque,

$$\boldsymbol{\tau} = \frac{d\mathbf{L}_{\rm mech}}{dt} \approx -\mathbf{v} \times \frac{\mathbf{E} \times \mathbf{m}_0}{c} \,. \tag{41}$$

When $\mathbf{E}$ and $\mathbf{v}$ are parallel, we can rewrite eq. (41) as,

$$\boldsymbol{\tau} = -\mathbf{E} \times \frac{\mathbf{v} \times \mathbf{m}_0}{c} = \mathbf{p} \times \mathbf{E} \qquad \text{(present example, Ampèrian magnetic dipole).} \tag{42}$$

This equals the lab-frame torque (28) (for $\mathbf{v} \parallel \mathbf{E}$) computed in sec. 2.3.2 via transformation from the rest frame, and the "paradoxical" torque is needed to "cause" the changes in the lab-frame "hidden" mechanical angular momentum.[25]

There also exists the electromagnetic field angular momentum,

$$\mathbf{L}_{\rm EM} = \mathbf{r} \times \mathbf{P}_{\rm EM} = -\mathbf{L}_{\rm hidden}, \tag{43}$$

whose time rate of change is equal and opposite to that of the "hidden" mechanical angular momentum,

$$\frac{d\mathbf{L}_{\rm EM}}{dt} = -\frac{d\mathbf{L}_{\rm hidden}}{dt} \,. \tag{44}$$

We can say that the "torque" $-d\mathbf{L}_{\rm hidden}/dt$ "causes" the change in $\mathbf{L}_{\rm EM}$.

2.5.1 Flow of Field Momentum and Angular Momentum

As first noted by Poincaré [45], an electromechanical system contains densities $\mathbf{p}_{\rm mech}$ of mechanical momentum as well as,

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} \tag{45}$$

of electromagnetic field momentum (in media where $\mathbf{E} = \mathbf{D}$ and $\mathbf{B} = \mathbf{H}$ as in the present example). Changes in these momentum densities can be related to a force density $\mathbf{f}$ which is the divergence of a stress tensor T,

$$\mathbf{f} = \boldsymbol{\nabla} \cdot \mathsf{T} = \frac{\partial \mathbf{p}_{\rm EM}}{\partial t} + \frac{\partial \mathbf{p}_{\rm mech}}{\partial t} \,, \tag{46}$$

where the stress tensor can be written as the sum of mechanical and electromagnetic stress tensors,

$$\mathsf{T} = \mathsf{T}^{\rm mech} + \mathsf{T}^{\rm EM}, \tag{47}$$

with,

$$\mathsf{T}^{\rm EM}_{ij} = \frac{E_i E_j + B_i B_j}{4\pi} - \delta_{ij} \frac{E^2 + B^2}{8\pi} \,, \tag{48}$$

[25]Similar arguments have been given by Griffiths and Hnizdo [37], Cross [38], Vanzella [51], Barnett [52], Saldanha [53], and Boyer [54]. Since eq. (28) was deduced from the Lorentz force law, this law is not brought into doubt by the present example, contrary to the claim in [7].

which is often called the Maxwell stress tensor. Outside of matter, where the density $\mathbf{p}_{\rm mech}$ and the tensor $\mathsf{T}^{\rm mech}$ are zero, eq. (46) can be written as,

$$\frac{\partial \mathbf{p}_{\rm EM}}{\partial t} - \boldsymbol{\nabla} \cdot \mathsf{T}^{\rm EM} = 0 \qquad \text{(outside matter)}, \tag{49}$$

which indicates that the (tensor) flux of electromagnetic momentum $\mathbf{p}_{\rm EM}$ is given by $-\mathsf{T}^{\rm EM}$ in "empty" space.[26]

We define the densities of electromagnetic and mechanical angular momenta as,

$$\mathbf{l}_{\rm EM} = \mathbf{r} \times \mathbf{p}_{\rm EM}, \qquad \mathbf{l}_{\rm mech} = \mathbf{r} \times \mathbf{p}_{\rm mech}. \tag{50}$$

Then, the vector moment of eq. (46) leads to the relation,

$$\mathbf{r} \times \mathbf{f} = \mathbf{r} \times \boldsymbol{\nabla} \cdot \mathsf{T} = \boldsymbol{\nabla} \cdot \mathbf{r} \times \mathsf{T} = \frac{\partial \mathbf{l}_{\rm EM}}{\partial t} + \frac{\partial \mathbf{l}_{\rm mech}}{\partial t}, \tag{51}$$

where the cross product $\mathbf{r} \times \mathsf{T}$ involves only the second index of T, and the i component of $\mathbf{r} \times \boldsymbol{\nabla} \cdot \mathsf{T}$ is, noting that the tensor T is symmetric,

$$\mathbf{r} \times \boldsymbol{\nabla} \cdot \mathsf{T}\big|_i = \epsilon_{ijk} x_j \frac{\partial \mathsf{T}_{lk}}{\partial x_l} = \frac{\partial}{\partial x_l} \epsilon_{ijk} x_j \mathsf{T}_{lk} = \boldsymbol{\nabla} \cdot \mathbf{r} \times \mathsf{T}\big|_i \,. \tag{52}$$

See, for example, [56].

Outside of matter the density $\mathbf{l}_{\rm mech}$ is zero, and eq. (51) can be written as,

$$\frac{\partial \mathbf{l}_{\rm EM}}{\partial t} - \boldsymbol{\nabla} \cdot \mathbf{r} \times \mathsf{T}^{\rm EM} = 0 \qquad \text{(outside matter)}, \tag{53}$$

which indicates that the tensor $-\mathbf{r} \times \mathsf{T}$ (whose components are $-\epsilon_{ikl} x_k T_{jl}$) equals the (tensor) flux of the density $\mathbf{l}_{\rm EM}$ of electromagnetic angular momentum in "empty" space.

We can regard the changes in the lab-frame electromagnetic-angular-momentum density as due to the flux of angular momentum, described by the tensor $\mathbf{r} \times \mathsf{T}^{\rm EM}$, out from the time-dependent "hidden" mechanical angular momentum in the currents of the moving magnetic moment.[27]

2.6 Electric Dipole in an External Magnetic Field

In the complementary example of a "point particle" with electric dipole moment $\mathbf{p}_0$ at rest in a constant, uniform magnetic field $\mathbf{B}_0$ there is no torque on the particle in either the rest frame or in a lab frame that has low velocity $\mathbf{v}$ parallel to $\mathbf{B}_0$, recalling eq. (26).

If the magnetic field were due to a distant magnetic monopole, there would be no "hidden" mechanical momentum in the system, as noted in [40]. Hence there would be no changing

[26] Another interpretation of eq. (49) is that the force density $\boldsymbol{\nabla} \cdot \mathsf{T}^{\rm EM}$ "causes" the time rate of change of the electromagnetic-momentum density $\mathbf{p}_{\rm EM}$. Such interpretation is delicate in that $\boldsymbol{\nabla} \cdot \mathsf{T}^{\rm EM} = \partial(\mathbf{E} \times \mathbf{B})/4\pi c\, \partial t$, so eq. (49) is more of a tautology than a cause/effect relation.

[27] The torque density $\mathbf{r} \times \mathbf{f}$ can be said to "cause" the changes in the densities $\mathbf{l}_{\rm EM}$ and $\mathbf{l}_{\rm mech}$ of electromagnetic and mechanical angular momenta. However, this interpretation has the same delicacy discussed in the previous footnote.

"hidden" mechanical angular momentum in the lab frame, and no torque would be needed there.

In a more realistic example the magnetic field is due to electrical currents in, say, an Ampèrian magnetic dipole. In this case the system at rest possesses nonzero electromagnetic field momentum $\mathbf{P}_{\rm EM} = \mathbf{B}_0 \times \mathbf{p}_0/2c$ [55]. The "hidden" mechanical momentum of the system is the negative of this, and in the lab frame the time rate of change of the corresponding "hidden" mechanical angular momentum is, $d\mathbf{L}_{\rm hidden}/dt = -\mathbf{v} \times (\mathbf{B}_0 \times \mathbf{p}_0/2c) = -vp_0 B_0\, \hat{\mathbf{z}}/2c$, for moment $\mathbf{p}_0$ in the z-direction and $\mathbf{B}_0$ and $\mathbf{v}$ in the x-direction.

It may appear paradoxical that the "hidden" mechancial angular momentum is time dependent in the lab frame, but there is no torque there on the dipole.

However, we should also consider the effect of the fields of the electric dipole on the source of the external magnetic field. For example, suppose the magnetic field $\mathbf{B}_0 = B_0\, \hat{\mathbf{x}}$ at the origin is due to a current loop at $x = -d_0$ on the x-axis with (Ampèrian) magnetic moment $\mathbf{m}_0 = B_0 d_0^3\, \hat{\mathbf{x}}/2$. If the electric dipole at the origin has moment $\mathbf{p}_0 = p_0\, \hat{\mathbf{z}}$, then the electric field on the magnetic moment is $\mathbf{E} = -\mathbf{p}_0/d_0^3$. The torque on the magnetic moment in the lab frame follows from eq. (26) as $\boldsymbol{\tau} \approx \mathbf{v}/c \times (\mathbf{m}_0 \times \mathbf{E}) = -vm_0 E\, \hat{\mathbf{z}}/c = -vp_0 B_0\, \hat{\mathbf{z}}/2c$. This torque equals the time rate of change of the "hidden" mechanical angular momentum found above, so the absence of a lab-frame torque on the electric dipole at the origin (at time $t = 0$) in this example is not paradoxical.[28]

2.7 Physical Realizations of Magnetic Moments

The behavior of a moving current loop in an external electric field depends on the physical nature of the current.

If the current flows in a resistive conductor, that conductor would "shield" the current from a constant, uniform external electric field $\mathbf{E}$ if the conductor is at rest or in uniform motion with respect to the field. In this case there would be no Lorentz force on the current due to the external field, and no torque in the frame where the current loop has velocity $\mathbf{v}$.

Similarly, if the current loop is a superconductor, the supercurrent is "shielded" from the external field, and there is no torque.

A model of a neutral current loop that could realize Mansuripur's paradox is a pair of nonconducting, coaxial disks with positive charge fixed to the rim of one and negative charge on the other, with the disks rotating in opposite senses with the same magnitude of angular velocity. The paradox applies also to models in which the current is a charged, compressible gas or liquid that flow inside a nonconducting tube (models i and iii of [6]).[29]

In sum, the present example can be realized only in rather "academic" thought experiments if the magnetic momenent is due to conduction current loops.

The most practical realization of the present example would involve magnetic fields due

[28] A delicate point is that the electromagnetic field momentum of the magnetic dipole in the electric field of the electric dipole is $\mathbf{P}_{\rm EM} = \mathbf{E} \times \mathbf{m}_0/c$, which equals field momentum $\mathbf{B}_0 \times \mathbf{p}_0/2c$ of the electric dipole in the field of the magnetic dipole. These are not different momenta, but two computations of the field momentum of the system. See [55] for a computation in which the magnetic field is due to a long solenoid magnet.

[29] To have an electrically neutral current loop, one must postulate a pair of such tubes that containing opposite charged gas/liquid flowing in opposite directions.

to intrinsic (Ampèrian) magnetic momentums, such as associated with a nonconducting permanent magnet, or a neutron.

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References

[1] K.T. McDonald, *Charge Density in a Current-Carrying Wire* (Dec. 23, 2010), http://kirkmcd.princeton.edu/examples/wire.pdf

[2] V. Hnizdo and K.T. McDonald, *Fields and Moments of a Moving Electric Dipole* (Nov. 29, 2011), http://kirkmcd.princeton.edu/examples/movingdipole.pdf

[3] G. Spavieri, *Proposal for Experiments to Detect the Missing Torque in Special Relativity*, Found. Phys. Lett. **3**, 291 (1990), http://kirkmcd.princeton.edu/examples/EM/spavieri_fpl_3_291_90.pdf

[4] D. Bedford and P. Krumm, *On the origin of magnetic dynamics*, Am. J. Phys. **54**, 1036 (1986), http://kirkmcd.princeton.edu/examples/EM/bedford_ajp_54_1036_86.pdf

[5] V. Namias, *Electrodynamics of moving dipoles: The case of the missing torque*, Am. J. Phys. **57**, 171 (1989), http://kirkmcd.princeton.edu/examples/EM/namias_ajp_57_171_89.pdf

[6] L. Vaidman, *Torque and force on a magnetic dipole*, Am. J. Phys. **58**, 978 (1990), http://kirkmcd.princeton.edu/examples/EM/vaidman_ajp_58_978_90.pdf

[7] M. Mansuripur, *Trouble with the Lorentz Law of Force: Incompatibility with Special Relativity and Momentum Conservation*, Phys. Rev. Lett. **108**, 193901 (2012), http://kirkmcd.princeton.edu/examples/EM/mansuripur_prl_108_193901_12.pdf

[8] D. Schieber, *Some Remarks on the Classical Electrodynamics of Moving Media*, Appl. Phys. **14**, 327 (1977), http://kirkmcd.princeton.edu/examples/EM/schieber_ap_a14_327_77.pdf

[9] K.T. McDonald, *Methods of Calculating Forces on Rigid Magnetic Media* (March 18, 2002), http://kirkmcd.princeton.edu/examples/magnetic_force.pdf

[10] A. Einstein and J. Laub, *Über die elektromagnetishen Grundgleichungen für bewegte Köper*, Ann. Phys. **26**, 532 (1908), http://kirkmcd.princeton.edu/examples/EM/einstein_ap_26_532_08.pdf
Über die im elektromagnetishen Felde auf ruhende Köper ausgeübten ponderomotorishen Kräfte, Ann. Phys. **26**, 541 (1908), http://kirkmcd.princeton.edu/examples/EM/einstein_ap_26_541_08.pdf

http://kirkmcd.princeton.edu/examples/EM/einstein_ap_26_541_08_english.pdf
Einstein and Laub on the Electrodynamics of Moving Media, in The Collected Papers of Albert Einstein, vol. 2, ed. J. Stachel (Princeton U. Press, 1989),
http://kirkmcd.princeton.edu/examples/EM/einstein_laub_08_commentary.pdf

[11] A. Einstein, *Sur les force pondéromotrices qui agissent sur des conducteurs ferromagnétique disposés un champ magnétique et parcourus par un courant*, Arch. Sci. Phys. Nat. **30**, 323 (1910), http://kirkmcd.princeton.edu/examples/EM/einstein_aspn_30_323_10.pdf

[12] K.T. McDonald, *Biot-Savart vs. Einstein-Laub Force Law* (May 21, 2013),
http://kirkmcd.princeton.edu/examples/laub.pdf

[13] K.T. McDonald, *Forces on Magnetic Dipoles* (Oct. 26, 2014),
http://kirkmcd.princeton.edu/examples/neutron.pdf

[14] B.A. Kemp, *Resolution of the Abraham-Minkowski debate: Implications for the electromagnetic wave theory of light in matter*, J. Appl. Phys. **109**, 111101 (2011),
http://kirkmcd.princeton.edu/examples/EM/kemp_jap_109_111101_11.pdf

[15] J.D. Jackson, *Torque or no torque? Simple charged particle motion observed in different inertial frames*, Am. J. Phys. **72**, 1484 (2004),
http://kirkmcd.princeton.edu/examples/EM/jackson_ajp_72_1484_04.pdf

[16] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on "Magnetic Currents"*, Phys. Rev. Lett. **18**, 876 (1967),
http://kirkmcd.princeton.edu/examples/EM/shockley_prl_18_876_67.pdf

[17] K.T. McDonald, *Abraham, Minkowski and "Hidden" Mechanical Momentum* (June 6, 2012), http://kirkmcd.princeton.edu/examples/abraham.pdf

[18] Y. Aharonov and D. Bohm, *Significance of Electromagnetic Potentials in Quantum Theory*, Phys. Rev. **115**, 485 (1959),
http://kirkmcd.princeton.edu/examples/QM/aharonov_pr_115_485_59.pdf

[19] K.T. McDonald, *Classical Aspects of the Aharonov-Bohm Effect* (Nov. 29, 2013),
http://kirkmcd.princeton.edu/examples/aharonov.pdf

[20] S. Coleman and J.H. Van Vleck, *Origin of "Hidden Momentum" Forces on Magnets*, Phys. Rev. **171**, 1370 (1968), http://kirkmcd.princeton.edu/examples/EM/coleman_pr_171_1370_68.pdf

[21] V. Bargmann, L. Michel and V.L. Telegdi, *Precession of the Polarization of Particles Moving in a Homogeneous Electromagnetic Field*, Phys. Rev. Lett. **2**, 435 (1959),
http://kirkmcd.princeton.edu/examples/QED/bargmann_prl_2_435_59.pdf

[22] K.T. McDonald, *Polarization Precession* (Jan. 14, 1970),
http://kirkmcd.princeton.edu/examples/polprecess.pdf

[23] E. Fermi, *Über die magnetischen Momente der Atomkerne*, Z. Phys. **60**, 320 (1930),
http://kirkmcd.princeton.edu/examples/QED/fermi_zp_60_320_30.pdf

[24] H.A. Lorentz, *Alte und Neue Fragen der Physik*, Phys. Z. **11**, 1234 (1910), http://kirkmcd.princeton.edu/examples/EM/lorentz_pz_11_1234_10.pdf

[25] I.M. Frank, Izv. Akad. Nauk USSR, Ser. Fiz. **6**, 3 (1942), http://kirkmcd.princeton.edu/examples/EM/frank_ian_6_3_42.pdf

[26] I.M. Frank, *Vavilov-Cherenkov radiation for electric and magnetic multipoles*, Sov. Phys. Usph. **27**, 772 (1984), http://kirkmcd.princeton.edu/examples/EM/frank_spu_27_772_84.pdf

[27] I.M. Frank, *On the moments of a magnetic dipole moving in a medium*, Sov. Phys. Usph. **32**, 456 (1989), http://kirkmcd.princeton.edu/examples/EM/frank_spu_32_456_89.pdf

[28] S.J. Barnett, *Electric Fields due to the Motion of Constant Electromagnetic Systems*, Phil. Mag. **44**, 1112 (1922), http://kirkmcd.princeton.edu/examples/EM/barnett_pm_44_1112_22.pdf

[29] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, 3$^{\rm rd}$ ed./ (Clarendon Press, 1892), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_92.pdf

[30] W.G.F. Swann, *Unipolar Induction*, Phys. Rev. **15**, 365 (1920), http://kirkmcd.princeton.edu/examples/EM/swann_pr_15_365_20.pdf

[31] E. Budde, *Das Clausius'sche Gesetz und die Bewegung der Erde in Raume*, Ann. Phys. **10**, 553 (1880), http://kirkmcd.princeton.edu/examples/EM/budde_ap_10_553_1880.pdf

[32] G.P. Fisher, *The Electric Dipole Moment of a Moving Magnetic Dipole*, Am. J. Phys. **39**, 1528 (1971), http://kirkmcd.princeton.edu/examples/EM/fisher_ajp_39_1528_71.pdf

[33] J.D. Jackson, *Classical Electrodynamics*, 3$^{\rm rd}$ ed. (Wiley, 1999), http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

[34] W.G.V. Rosser, *Classical electromagnetism and relativity: a moving magnetic dipole*, Am. J. Phys **61**, 371 (1993), http://kirkmcd.princeton.edu/examples/EM/rosser_ajp_61_371_93.pdf

[35] F.T. Trouton and H.R. Noble, *The Forces Acting on a Charged Condensor moving through Space*, Proc. Roy. Soc. London **72**, 132 (1903), http://kirkmcd.princeton.edu/examples/EM/trouton_prs_72_132_03.pdf
The Mechanical Forces Acting on a Charged Condensor moving through Space, Phil. Trans. Roy. Soc. London A **202**, 165 (1904), http://kirkmcd.princeton.edu/examples/EM/trouton_ptrsla_202_165_04.pdf

[36] S. Aranoff, *Torque and Angular Momentum on a System at Rest in Special Relativity*, Am. J. Phys. **37**, 453 (1969), http://kirkmcd.princeton.edu/examples/GR/aranoff_ajp_37_453_69.pdf

[37] D.J. Griffiths and V. Hnizdo, *Comment on "Trouble with the Lorentz Law of Force"* (May 21, 2012), http://arxiv.org/abs/1205.4646

[38] D.J. Cross, *Resolution of the Mansuripur Paradox*, http://arxiv.org/abs/1205.5451

[39] M. Mansuripur, *Maxwell's macroscopic equations, the energy-momentum postulates, and the Lorentz law of force*, Phys. Rev. E **79**, 026608 (2009),
http://kirkmcd.princeton.edu/examples/EM/mansuripur_pre_79_026608_09.pdf
Nature of electric and magnetic dipoles gleaned from the Poynting theorem and the Lorentz force law of classical electrodynamics, Opt. Comm. **283**, 4594 (2010),
http://kirkmcd.princeton.edu/examples/EM/mansuripur_oc_283_4594_10.pdf

[40] D.J. Griffiths, *Dipoles at rest*, Am. J. Phys. **60**, 979 (1992),
http://kirkmcd.princeton.edu/examples/EM/griffiths_ajp_60_979_92.pdf

[41] K.T. McDonald, *Three Expressions for Electromagnetic Field Momentum* (Apr. 10, 2006), http://kirkmcd.princeton.edu/examples/pem_forms.pdf

[42] Part I, sec. IX of M. Faraday, *Experimental Researches in Electricity* (Dover Publications, New York, 2004; reprint of the 1839 edition).

[43] Secs. 22-24 and 57 of J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865),
http://kirkmcd.princeton.edu/examples/EM/maxwell_ptrsl_155_459_65.pdf

[44] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884),
http://kirkmcd.princeton.edu/examples/EM/poynting_ptrsl_175_343_84.pdf

[45] H. Poincaré, *La Théorie de Lorentz et la Principe de Réaction*, Arch. Neer. **5**, 252 (1900), http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00.pdf
Translation: *The Theory of Lorentz and the Principle of Reaction*,
http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00_english.pdf

[46] M. Abraham, *Prinzipien der Dynamik des Elektrons*, Ann. Phys. **10**, 105 (1903),
http://physics.princeton.edu/~mcdonald/examples/EM/abraham_ap_10_105_03.pdf

[47] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621 (1969),
http://kirkmcd.princeton.edu/examples/EM/furry_ajp_37_621_69.pdf

[48] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012),
http://kirkmcd.princeton.edu/examples/hidden.pdf

[49] H.A. Haus, *Electrodynamics of Moving Media and the Force on a Current Loop*, Appl. Phys. A **27**, 99 (1982), http://kirkmcd.princeton.edu/examples/EM/haus_ap_a27_99_82.pdf

[50] K.T. McDonald, *"Hidden" Momentum in a Current Loop* (June 30, 2012),
http://kirkmcd.princeton.edu/examples/penfield.pdf

[51] D.A.T. Vanzella, *Comment on "Trouble with the Lorentz Law of Force: Incompatibility with Special Relativity and Momentum Conservation"*, Phys. Rev. Lett. **110**, 089401 (2013), http://kirkmcd.princeton.edu/examples/EM/vanzella_prl_110_089401_13.pdf
http://arxiv.org/abs/1205.1502

[52] S.M. Barnett, *Comment on "Trouble with the Lorentz Law of Force: Incompatibility with Special Relativity and Momentum Conservation"*, Phys. Rev. Lett. **110**, 089402 (2013), http://kirkmcd.princeton.edu/examples/EM/barnett_prl_110_089402_13.pdf

[53] P. Saldanha, *Comment on "Trouble with the Lorentz Law of Force: Incompatibility with Special Relativity and Momentum Conservation"*, Phys. Rev. Lett. **110**, 089403 (2013), http://kirkmcd.princeton.edu/examples/EM/saldanha_prl_110_089403_13.pdf
http://arxiv.org/abs/1205.6858

[54] T.H. Boyer, *Examples and comments related to relativity controversies*, Am. J. Phys. **80**, 962 (2012), http://kirkmcd.princeton.edu/examples/GR/boyer_ajp_80_962_12.pdf

[55] K.T. McDonald, *Electromagnetic Momentum of a Capacitor in a Uniform Magnetic Field* (June 18, 2006), http://kirkmcd.princeton.edu/examples/cap_momentum.pdf

[56] L. Levitov, *MIT 8.311 Problem Set #2* (Feb. 19, 2004),
http://ocw.mit.edu/courses/physics/8-311-electromagnetic-theory-spring-2004/assignments/ps2.pdf
Levitov's prob. 1 is taken from prob. 5, chap. 3 of J. Schwinger, L.L. DeRaad, Jr, K.A. Milton and W.-Y. Tsai, *Classical Electrodynamics* (Perseus Books, 1998),
http://kirkmcd.princeton.edu/examples/EM/schwinger_em_98.pdf