# Wave Amplification in a Magnetic Medium

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# 1 Problem

One way to prepare an optically active medium is to turn on a strong DC magnetic field at right angles to a static magnetic field that has initially aligned the dipoles of a magnetic medium. Then, the dipoles will precess about the direction of the strong magnetic field, before eventually relaxing into alignment with that field. During those intervals while the dipoles $\mathbf{m}$ are antialigned with the initial static field, they are in a state of high energy $U = -\mathbf{m} \cdot \mathbf{B}$. When in this state, the medium can give up energy to a probe electromagnetic wave (with magnetic field along the direction of the strong DC field), thereby amplifying it.

Deduce the equations of motion for the magnetization $\mathbf{M} = N\mathbf{m}$ of a medium that consists of $N$ permanent dipoles $\mathbf{m}$ (with angular momentum $\mathbf{L} = \Gamma\mathbf{m}$) per unit volume when the medium is immersed in a magnetic field $\mathbf{B}$. Consider the specific example of a static magnetic field $B_{0x}\hat{\mathbf{x}} + B_{0y}\hat{\mathbf{y}}$ where $B_{0x} \ll B_{0y}$, and an oscillatory field $B_y e^{-i\omega t}\hat{\mathbf{y}}$. You may suppose that $M \ll B_x$ and $M \ll B_y$.

A measure of the ability of the medium to amplify a probe wave is the frequency-dependent index of refraction $n(\omega) = \sqrt{\mu}$, where $\mu$ is the magnetic susceptibility related by $\mathbf{B} = \mu\mathbf{H} = \mathbf{H} + 4\pi\mathbf{M}$ (in Gaussian units, and in a medium of dielectric constant $\epsilon = 1$). In the present example, the wave field has magnetic field along the $y$ axis, so that you can write,

$$B_y(\omega) = \mu H_y(\omega) = H_y \left(1 + 4\pi \frac{M_y}{H_y}\right) , \tag{1}$$

Since we assume that $M \ll B_y$, we also have $M_y \ll H_y$, and the index of refraction is given by.

$$n(\omega) = \sqrt{\mu} \approx 1 + 2\pi \frac{M_y}{H_y} . \tag{2}$$

If the medium is to exchange energy with a wave, there must be additional processes occuring. For index of refraction to include absorption (or amplification), if suffices to suppose that there is a kind of damping mechanism that aligns the magnetic dipoles with the static magnetic field. A phenomenological form for this is,

$$\frac{d\mathbf{m}}{dt} = \gamma(\hat{\mathbf{m}} \times \hat{\mathbf{B}}) \times \mathbf{m} \approx -\gamma m(\hat{\mathbf{m}} - \hat{\mathbf{y}}), \tag{3}$$

where $\gamma$ is the damping factor, and the approximation notes that the static field is largely along the $y$ axis. Include this damping in the equations of motion, solve for the oscillatory behavior of $M_y \propto e^{-i\omega t}$ assuming the damping is slow so that $\gamma \ll \Gamma B_x$, and then calculate the index $n(\omega)$. Show that when $M_x$ has precessed to be opposite to $B_{0x}$, the index of refraction implies amplification of a traveling wave of $H_y$ (and $M_y$).

## 2 Solution

The merits of an oscillatory magnetic field transverse to a static magnet field in the study of individual magnetic moments were emphasized by Rabi [1]. Bloch [2] extended this approach to magnetic media, but it was perhaps Dicke [3] who realized that the optically active medium thereby created could lead to "super-radiance", *i.e.*, to laser beams.

When a magnetic dipole $\mathbf{m}$ is subject to a magnetic field $\mathbf{B}$ it experiences a torque $\mathbf{m}\times\mathbf{B}$ that precesses the angular momentum $\mathbf{L} = \mathbf{m}/\Gamma$, where $\Gamma = m/L$ is the gryomagnetic ratio of the dipole. If the magnetic dipoles are electrons, then $\Gamma = e/2m_e c \approx 10^7$ Hz/gauss, where $e$ and $m_e$ are the charge and mass of the electron, and $c$ is the speed of light. Thus,

$$\mathbf{m}\times\mathbf{B} = \frac{d\mathbf{L}}{dt} = \frac{1}{\Gamma}\frac{d\mathbf{m}}{dt}\,. \tag{4}$$

The precession frequency is $\Gamma B \approx 10^7 B$ for $B$ in gauss. We will consider magnetic fields $B_y(t)$ of optical frequencies, $\approx 10^{15}$ Hz, so the precession will be very slow compared to the wave frequency.

The equation of motion of a single moment, including the damping (3) of the moment to alignment with the static magnetic field that is predominantly along the $y$ axis, is,

$$\frac{d\mathbf{m}}{dt} = \Gamma\mathbf{m}\times\mathbf{B} - \gamma m(\hat{\mathbf{m}} - \hat{\mathbf{y}}). \tag{5}$$

The equation of motion for the magnetization $\mathbf{M} = N\mathbf{m}$ is therefore,

$$\frac{d\mathbf{M}}{dt} + \gamma\mathbf{M} = \Gamma\mathbf{M}\times\mathbf{B} + \gamma M\hat{\mathbf{y}}. \tag{6}$$

For a magnetic field $B_{0x}\hat{\mathbf{x}} + (B_{0y} + B_y(t))\hat{\mathbf{y}} = (H_x + 4\pi M_x)\hat{\mathbf{x}} + (H_y + 4\pi M_y)\hat{\mathbf{y}}$, the components of eq. (6) are,

$$\frac{dM_x}{dt} + \gamma M_x = -\Gamma M_z B_y, \tag{7}$$

$$\frac{dM_y}{dt} + \gamma M_y = \Gamma M_z B_x + \gamma M, \tag{8}$$

$$\frac{dM_z}{dt} + \gamma M_z = \Gamma(M_x B_y - M_y B_x). \tag{9}$$

The desired physical picture is that the magnetization $\mathbf{M}$ precesses around the $y$ axis (subject to the "slow" damping $\gamma$), with the oscillatory magnetization $M_y$ being only a small perturbation about this dominant motion. From eqs. (7) and (9) we see that this is a good approximation so long as $M_y B_x \ll M_x B_y$. We choose $B_{0x}$ to be small compared to $B_{0y}$, and prepare the medium in an initial state with $M_y \ll M_x$. The latter might be accomplished, for example, by starting with $B_{0y} = 0$ so the dipoles line up with $B_{0x}$, and then turning on the field $B_y$ quickly; if the damping time is long compared to the precession period, then there is a useful interval during which the desired behavior obtains.

We are principally interested in the behavior of $M_y$ for use in calculating the index of refraction, so we take the derivative of eq. (8), noting that $M$ is constant since the medium is comprised of permanent dipoles, and insert eq. (9) to find,

$$\begin{aligned} \frac{d^2M_y}{dt^2} + \gamma\frac{dM_y}{dt} &= \Gamma\frac{dM_z}{dt}B_x = \Gamma B_x[\Gamma(M_xB_y - M_yB_x) - \gamma M_z] \\ &= \Gamma^2 B_x(M_xH_y - M_yH_x) - \gamma\left(\frac{dM_y}{dt} + \gamma M_y - \gamma M\right). \end{aligned} \tag{10}$$

Assuming that $M \ll B_x$, then $H_x \approx B_x$ and we may approximate $\Gamma^2 B_xH_x \equiv \omega_0^2$ as being constant ($\omega_0 \approx \Gamma B_x$). Then,

$$\frac{d^2M_y}{dt^2} + 2\gamma\frac{dM_y}{dt} + (\gamma^2 + \omega_0^2)M_y = \Gamma^2 B_xH_x\frac{M_x}{H_x}H_y + \gamma^2 M = \omega_0^2\frac{M_x}{H_x}H_y + \gamma^2 M. \tag{11}$$

The term $\gamma^2 M$ leads to a constant component $M_y = \gamma^2 M/(\gamma^2 + \omega_0^2)$, which we can ignore since we assume that the damping constant $\gamma$ is small compared to the frequency $\omega_0 \approx \Gamma B_x$. Our main interest is the behavior of the system when a wave is present, $H_y = H_{0y}e^{-i\omega t}$ and $M_y = M_{0y}e^{-i\omega t}$, at frequency $\omega \gg \omega_0$, in which case we can regard $M_x$ as effectively constant over a few cycles of the high frequency wave. Inserting this hypothesis in eq. (11), we find that the high-frequency part of $M_y$ obeys,

$$M_y = \frac{M_x}{H_x}\frac{\omega_0^2 H_y}{\omega_0^2 - \omega^2 + \gamma^2 - 2i\gamma\omega}\,. \tag{12}$$

Recall that we need $M_yH_x \ll M_xH_y$ for the dominant behavior of the magnetization to be precession about the $y$ axis. From eq. (12) we see that this would not hold for frequency $\omega$ close to $\omega_0$ (since we assume that $\gamma \ll \omega_0$). But we consider $\omega$ of optical frequencies, so $\omega \gg \omega_0$ for any reasonable value of $B_x$, as noted previously.

The index of refraction for a wave propagating in the $z$ direction with magnetic field along the $y$ axis is therefore,

$$n(\omega) = \sqrt{\mu} \approx 1 + 2\pi\frac{M_y}{H_y} = 1 + 2\pi\frac{M_x}{H_x}\frac{\omega_0^2(\omega_0^2 - \omega^2 + \gamma^2 + 2i\gamma\omega)}{(\omega_0^2 - \omega^2 + \gamma^2)^2 + 4\gamma^2\omega^2}\,. \tag{13}$$

In particular, during the part of the precession cycle when the magnetization $M_x$ is antialigned with $B_x \approx H_x$, $Im(n) < 0$, and a propagating wave $H_{0y}e^{i\omega(nz/c-t)}$ is amplified during its passage through the medium.

It appears difficult to realize the desired precession of $\mathbf{M}$ about the $y$ axis as suggested, since $B_y$ would have to reach full strength in less that the damping time $1/\gamma$, and no actual laser has (I believe) been built utilizing a magnetic medium. The interest of this problem is in providing a classical viewpoint of how wave amplification is possible in principle by preparing a medium in an optically active state.

# References


[1] I.I. Rabi, *Space Quantization in a Gyrating Magnetic Field*, Phys. Rev. **51**, 4652 (1937), http://kirkmcd.princeton.edu/examples/QM/rabi_pr_51_652_37.pdf

[2] F. Bloch, *Nuclear Induction*, Phys. Rev. **70**, 460 (1946),
http://kirkmcd.princeton.edu/examples/QM/bloch_pr_70_460_46.pdf

[3] R.H. Dicke, *Coherence in Spontaneous Radiation Processes*, Phys. Rev. **93**, 99 (1954),
http://kirkmcd.princeton.edu/examples/QM/dicke_pr_93_99_54.pdf