Magnetars

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1 Problem

The x-ray pulsar SGR1806-20 has recently been observed to have a period $T = 2\pi/\Omega$ of 7.5 s and a relatively large "spindown" rate $\left|\dot{T}\right| = 8 \times 10^{-11}$ [1].

Calculate the maximum magnetic field at the surface of this pulsar, assuming it to be a standard neutron star of mass $1.4M_\odot = 2.8 \times 10^{30}$ kg and radius 10 km, that the mass density is uniform, that the spindown is due to electromagnetic radiation, and that the angular velocity vector is perpendicular to the magnetic dipole moment of the pulsar.

Compare the surface magnetic field strength to the so-called QED critical field strength $m_e^2c^3/e\hbar = 4.4\times10^{13}$ gauss, at which electron-positron-pair-creation processes become highly probable.[1]

2 Solution

In Gaussian units, the rate of magnetic dipole radiation is,

$$\frac{dU}{dt} = \frac{2}{3c^3}\ddot{\mathbf{m}}^2 = \frac{2}{3c^3}\left|\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{m})\right|^2 = \frac{2m^2\Omega^4 \sin^4 \alpha}{3c^3}\,, \tag{1}$$

where $\Omega = 2\pi/T$ is the angular velocity, and α is the angle between the magnetic dipole moment $\mathbf{m}$ and the axis of rotation $\hat{\mathbf{\Omega}}$, which will be taken at 90° in the rest of this note.

The radiated power (1) is derived from a decrease in the rotational kinetic energy, $U = I\Omega^2/2$, of the pulsar,

$$\frac{dU}{dt} = -I\Omega\dot{\Omega} = \frac{2}{5}MR^2\Omega\left|\dot{\Omega}\right|, \tag{2}$$

where the moment of inertia I is taken to be that of a sphere of uniform mass density. Combining eqs. (1) and (2) for $\alpha = 90°$, we have,

$$m^2 = \frac{3}{5}\frac{MR^2\left|\dot{\Omega}\right|c^3}{\Omega^3}. \tag{3}$$

[1]The critical field strength $E_{\rm crit} = m^2c^3/e\hbar = (mc^2)/e\lambda\!\!\!^{-}_C$ (= electric field in which the energy gain of an electron over its (reduced) Compton wavelength $\lambda\!\!\!^{-}_C = \hbar/mc$ equals its rest mass/energy mc^2) was introduced in eq. (43) of [2] (1931), where the symbol v stands for the force eE on an electric charge e in an electric field E. The context was a resolution of Klein's paradox [3] by noting that an electron in a sufficiently strong field, associated with an abrupt step in the electric potential, leads to the production of electron-positron pairs such that the electron reflection coefficient can exceed unity.

The QED critical field strength was considered to be well known in 1936, when it was mentioned in the abstract of [4].

Substituting $\Omega = 2\pi/T$, and $\left|\dot{\Omega}\right| = 2\pi \left|\dot{T}\right| / T^2$, we find,

$$m^2 = \frac{3}{20\pi^2} M R^2 T \left|\dot{T}\right| c^3. \tag{4}$$

The static magnetic field $\mathbf{B}$ due to dipole $\mathbf{m}$ is,

$$\mathbf{B} = \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}, \tag{5}$$

so the peak field at radius R is,

$$\mathbf{B} = \frac{2\mathbf{m}}{R^3}. \tag{6}$$

Inserting this in eq. (4), the peak surface magnetic field is related by,

$$B^2 = \frac{3}{5\pi^2} \frac{MT\left|\dot{T}\right| c^3}{R^4} = \frac{3}{5\pi^2} \frac{(2.8 \times 10^{33})(7.5)(8 \times 10^{-11})(3 \times 10^{10})^3}{(10^6)^4} = 2.8 \times 10^{30} \text{ gauss}^2. \tag{7}$$

Thus, $B_{\rm peak} = 1.7 \times 10^{15}$ G $= 38 B_{\rm crit}$, where $B_{\rm crit} = 4.4 \times 10^{13}$ G.

When electrons and photons of kinetic energies greater than 1 MeV exist in a magnetic field with $B > B_{\rm crit}$, they can rapidly lose this energy via electron-positron pair creation.[2]

Kouveliotou *et al.* [1] report that $B_{\rm peak} = 8 \times 10^{14}$ G without discussing details of their calculation.

2.1 Force-Free Electrodynamics of the Magnetosphere

The preceding discussion tacitly assumes there to be vacuum outside the surface of the neutron star. This is not a good approximation, and it is better to suppose the neutron star possesses a magnetosphere of relativistic plasma that supports so-called force-free electrodynamics,[3] meaning that the Lorentz force on the charged particles of the plasma is negligible. An interesting prediction of the force-free magnetohydrodynamics is that even if the neutron star were not rotating it would emit radiation of strength approximately that of eq. (2) [11].

The Lorentz 4-force for a particle of electric charge e and 4-momentum p^μ can be written as $eF_{\mu\nu}p^\nu$, where $F_{\mu\nu}$ is the electromagnetic field 4-tensor. In force-free electrodynamics where the Lorentz force is negligible, the dimensionless parameter,[4]

$$\Upsilon = \frac{e\hbar}{m^3 c^5} \sqrt{(F_{\mu\nu}p^\nu)^2} = \frac{\sqrt{(F_{\mu\nu}p^\nu)^2}}{mc^2 B_{\rm crit}}, \tag{8}$$

where m is the rest mass of, say an electron or positron and $B_{\rm crit} = m^2c^3/e\hbar = 4.4 \times 10^{13}$ G, is also negligible. In more ordinary electrodynamics, the parameter Υ would be roughly

[2] This energy loss may be suppressed in the force-free electrodynamics mentioned in sec. 2.1.

[3] The force-free electrodynamics of neutron-star magnetospheres was pioneered in [5], and in [6] for black holes. Among the many subsequent discussions, see, for example, [7, 8, 9, 10].

[4] The parameter Υ was introduced in noninvariant notation in eq. (3.2-1) of [12], and in 4-vector notation as χ in [13]. See [14] for an experiment (by the author) in which $\Upsilon \approx 1$.

$(U/mc^2)(B/B_{\rm crit})$ (where U is the energy of the electron/positron), which is much larger than unity near the surface of a neutron star.

When the parameter $\Upsilon \gtrsim 1$, nonlinear, strong-field QED effects such as "spontaneous" electron-positron-pair creation ("sparking the vacuum") become important. If the force-free condition $F_{\mu\nu} p^\nu = 0$ holds down to the scale of the electron Compton wavelength ($\approx 10^{-11}$ cm) in the magnetosphere of a neutron star, then electron-positron-pair creation will be very weak even though $B/B_{\rm crit} \gg 1$ there.

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