“Hidden” Momentum of a Magnetized Sphere in an Electric Field

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1 Problem

What is the electromagnetic field momentum of a nonconducting sphere (at rest) with uniform magnetization density $\mathbf{M}$ when in a uniform electric field $\mathbf{E}$? The sphere has unit (relative) permittivity $\epsilon = 1$ such that the uniform electric field penetrates the sphere

2 Solution

We use Gaussian units in this note, which is closely related to Sec. 2.2 of [1].

The magnetic fields of the sphere of radius a, with total magnetic moment $\mathbf{m}$

$$\mathbf{m} = \frac{4\pi a^3}{3}\mathbf{M}, \tag{1}$$

are[1]

$$\mathbf{B}(r < a) = \frac{8\pi}{3}\mathbf{M} = \frac{2\mathbf{m}}{a^3}, \qquad \mathbf{H}(r < a) = \mathbf{B} - 4\pi\mathbf{M} = -\frac{4\pi}{3}\mathbf{M}, \tag{2}$$

$$\mathbf{B}(r > a) = \mathbf{H}(r > a) = \frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{r}}}{r^3}. \tag{3}$$

The density $\mathbf{p}$ of electromagnetic field momentum is

$$\mathbf{p} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}, \tag{4}$$

where c is the speed of light in vacuum (which is the medium outside the magnetized sphere). The total electromagnetic-field momentum $\mathbf{P}_{\rm EM}$ can be computed as,

$$\mathbf{P}_{\rm EM} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol}. \tag{5}$$

To obtain a well defined result, the electric field must drop to zero at large distances, so we suppose (following Romer [3]) the electric field is due to a spherical shell of radius $b > a$ with charge density proportional to $\cos\theta$ (with respect to the $\mathbf{d}$-axis), such that the charge distribution has electric dipole moment $\mathbf{d}$ and the electric field has the form,

$$\mathbf{E}(r < b) = -\frac{\mathbf{d}}{b^3}, \qquad \mathbf{E}(r > b) = \frac{3(\mathbf{d}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{d}}{r^3}. \tag{6}$$

[1] See, for example, Sec. 5.10 of [2],

Then,

$$\begin{aligned}
\mathbf{P}_{\text{EM}} &= \int_{r<a} \frac{-\mathbf{d}\times 2\mathbf{m}}{4\pi a^3 b^3 c}\, d\text{Vol} + \int_{a<r<b} \frac{-\mathbf{d}\times[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}]}{4\pi b^3 r^3 c}\, d\text{Vol} \\
&\quad + \int_{r>b} \frac{[3(\mathbf{d}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{d}]\times[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}]}{4\pi r^6 c}\, d\text{Vol} \\
&= -\frac{2\mathbf{d}\times\mathbf{m}}{3b^3 c} - \frac{\mathbf{d}\times\mathbf{m}}{b^3 c}\ln\frac{b}{a} + \frac{\mathbf{d}\times\mathbf{m}}{b^3 c}\ln\frac{b}{a} - \frac{2\mathbf{d}\times\mathbf{m}}{3b^3 c} + \frac{\mathbf{d}\times\mathbf{m}}{3b^3 c} \\
&= \frac{\mathbf{m}\times\mathbf{d}}{b^3 c} = \frac{\mathbf{E}_{\text{on }\mathbf{m}}\times\mathbf{m}}{c}\,. \qquad (7)
\end{aligned}$$

However, we expect that the total momentum of a system at rest is zero, so we infer that there exists a “hidden” mechanical momentum in the system equal and opposite to that of eq. (7),[2]

$$\mathbf{P}_{\text{mech,hidden}} = -\frac{\mathbf{E}_{\text{on }\mathbf{m}}\times\mathbf{m}}{c}\,. \qquad (8)$$

If the magnetization $\mathbf{M}$ is associated with electron spin, a quantum effect, a “classical” mechanical model of this “hidden” mechanical momentum cannot be given.[3]

2.1 The Abraham Momentum

In 1903 Max Abraham noted [6] that the Poynting vector [7], which describes the flow of energy in the electromagnetic field,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}, \qquad (9)$$

when divided by c^2 has the additional significance of being the density of momentum stored in the electromagnetic field,

$$\mathbf{p}_{\text{EM}}^{(\text{A})} = \frac{\mathbf{E}\times\mathbf{H}}{4\pi c} \qquad (\text{Abraham}). \qquad (10)$$

The corresponding total Abraham momentum is,

$$\mathbf{P}_{\text{EM}}^{(\text{A})} = \int \frac{\mathbf{E}\times\mathbf{H}}{4\pi c}\, d\text{Vol}. \qquad (\text{Abraham}). \qquad (11)$$

Recalling that $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$, we have that

$$\mathbf{P}_{\text{EM}}^{(\text{A})} = \int \frac{\mathbf{E}\times(\mathbf{B}-4\pi\mathbf{M})}{4\pi c}\, d\text{Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\text{Vol} - \int \frac{\mathbf{E}\times\mathbf{M}}{c}\, d\text{Vol}, \qquad (12)$$

[2] In the author’s view, the electromagnetic field momentum (7) in this examples should also be called a ”hidden” momentum, as discussed in [4],

[3] A case where a “classical” model for “hidden” mechanical momentum can be given is discussed in [5].

If the electric field is uniform over the magnetization density $\mathbf{M}$, as in the present example, we can write

$$\mathbf{P}_{\rm EM}^{\rm (A)} = \mathbf{P}_{\rm EM} - \frac{\mathbf{E}_{\rm on\ \mathbf{m}} \times \mathbf{m}}{c} = \mathbf{P}_{\rm EM} + \mathbf{P}_{\rm mech,hidden}, \tag{13}$$

where $\mathbf{m} = \int \mathbf{M}\, d{\rm Vol}$ is the total magnetic moment of the system, recalling eq. (8).

This illustrates that the Abraham momentum is not strictly a field momentum, but includes a "mechanical" component.

The insight of eq. (13) for examples like the present was noted by Mansuripur in [8], although he subsequently badly misconstrued an effect of "hidden" momentum [9]-[14].

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