Field and Kinetic Energies of a Pair of Permanent Magnetic Dipoles

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

Show that the gain in kinetic energy is equal and opposite to the change in the magnetic (interaction) field energy for a pair of magnetic dipoles with moment $\mathbf{m} = m\,\hat{\mathbf{z}}$ that lie along the z-axis, and hence the magnetic force does work in this example.

Note that the story is the same if "magnetic" is changed to "electric" everywhere in this example.

2 Solution

Since the magnetic field of a (point) magnetic dipole $\mathbf{m}$ has the form (in Gaussian units),

$$\mathbf{B} = \frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m}}{r^3}\,, \tag{1}$$

at distance $\mathbf{r}$ from the dipole, the torque $\boldsymbol{\tau} = \mathbf{m}' \times \mathbf{B}$ on a second magnetic dipole $\mathbf{m}'$ is zero when the two dipoles are (anti)parallel. The (interaction) magnetic field energy of the two permanent magnetic dipoles is,

$$U_B = -\mathbf{m}'\cdot\mathbf{B} = -\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})(\mathbf{m}'\cdot\hat{\mathbf{r}}) - \mathbf{m}\cdot\mathbf{m}'}{r^3}\,, \tag{2}$$

and the force on the second dipole is,

$$\mathbf{F}' = -\boldsymbol{\nabla} U_B = \boldsymbol{\nabla}(\mathbf{m}'\cdot\mathbf{B}). \tag{3}$$

In the present example, where the first dipole is at the origin, the torque on the dipoles is zero, and the (attractive) force on the second dipole (at $(0, 0, z)$) is along the z-axis,

$$F = -\frac{dU_B}{dz}\,. \tag{4}$$

As the second dipole moves towards the first (which latter is taken to be at rest), its kinetic energy increases according to,

$$\frac{d\mathrm{KE}}{dt} = \mathbf{F}\cdot\mathbf{v} = -\frac{dU_B}{dz}\frac{dz}{dt} = -\frac{dU_B}{dt}\,. \tag{5}$$

That is, the increase in kinetic energy of the second magnetic dipole is equal and opposite to the decrease in the magnetic (interaction) field energy. And, the work $\int_i^f \mathbf{F}\cdot d\mathbf{r} = U_{B_i} - U_{B_f}$ done on that dipole by the magnetic force is nonzero.

This note does not consider magnetic dipoles that are resistive current loops maintained by batteries. While the forces between such dipoles are the same as for permanent magnetic dipoles (of the same strengths), energy consideration must take into account the (chemical) energy stored in the batteries, as well as the Joule heating of the resistive conductors. The effect is to change the sign of the interaction energy (2), such that eq. (3) becomes $\mathbf{F}' = \boldsymbol{\nabla} U_B = \boldsymbol{\nabla}(\mathbf{m}' \cdot \mathbf{B})$. Hence, the gain in kinetic energy of the second, moving magnetic dipole is equal to the increase in the interaction field energy, and these two energies are compensated by a reduction in the energies of the batteries that maintain the constant currents (which also must supply the energy of the Joule heating of the resistive conductors).

A general discussion of force and energy for systems with constant currents is given, for example, in sec. 10-2 of [1].

3 Comments

The result (4) contradicts a claim on p. 21 of [2]:
Note that the failure of the rest mass m to be constant resolves a paradox concerning what one is taught in elementary physics courses: On one hand, one is (correctly) taught that an external magnetic field can do no work on a body, so a body moving in an external magnetic field cannot gain energy. On the other hand, one is (also correctly) taught that a magnetic dipole released in a nonuniform external magnetic field will gain kinetic energy. Where does this kinetic energy come from? Equation (B6) shows that it comes from the rest mass of the body.

It appears that something was not correctly taught in the elementary physics courses taken by the authors of [2]. The culprit appears to be the mantra "magnetic forces do no work", that appears on p. 215 of [3]. This claim applies to a single electric charge that has no intrinsic magnetic moment, but it does not apply to permanent magnetic moments,[1] and it also does not apply to collections of moving electric charges.[2]

A Appendix: Interaction Field Energy

Typically, the field energy (2) of a pair of magnetic dipoles is deduced from the force (3), so the argument of sec. 2 could be regarded as circular.

Here, we compute the interaction field energy of two sources,

$$U_B = \int \frac{\mathbf{B} \cdot \mathbf{B}'}{4\pi}\, d\mathrm{Vol}, \tag{6}$$

first for a pair of (hypothetical) magnetic charge q and q', then for a magnetic charge q and a magnetic dipole $\mathbf{m}'$, and finally for two magnetic dipoles $\mathbf{m}$ and $\mathbf{m}'$.

The goal is to deduce the interaction energy for a pair of permanent magnetic dipoles, which in Nature are not composed of (Gilbertian) magnetic charges (magnetic monopoles), but are Ampèrian, meaning that the magnetic field at the center of the dipole is in the same direction as its moment $\mathbf{m}$, as is the case for a current loop, rather than opposite to $\mathbf{m}$ as for

[1]For another illustration of this issue, see [4].

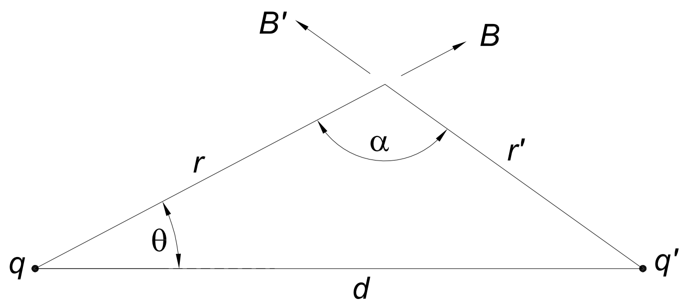
[2]The total magnetic force on a charge-current distribution can do work, although the magnetic force on each of the charges does no work. For an illustration of this conundrum, see [5].

a pair of equal and opposite magnetic charges. Of course, permanent magnetic dipoles are quantum entities, which are not well described as a classical current loop. However, when considering the interaction energy associated with magnetic dipoles, only the fields "outside" the dipole are relevant, and these are the same for Ampérian and Gilbertian dipoles. Hence, we compute using the latter for which the calculations are simpler.

A.1 Two Magnetic Charges

The magnetic field of a single (hypothetical) magnetic charge q is $\mathbf{B} = q\mathbf{r}/r^2$, where $\mathbf{r}$ points from the charge to the observer.

We calculate in a spherical coordinate system (r, θ, φ) whose origin is at charge q and whose z-axis points towards charge q', at distance d from q, as shown in the figure below.



By the law of cosines we have,

$$r' = \left(r^2 + d^2 - 2rd\cos\theta\right)^{1/2}, \tag{7}$$

and also,

$$\cos\alpha = \frac{r^2 + r'^2 - d^2}{2rr'} = \frac{r - d\cos\theta}{r'}. \tag{8}$$

The interaction energy of the two magnetic charges is,

$$\begin{aligned}
U_B &= \int \frac{\mathbf{B}\cdot\mathbf{B}'}{4\pi} d\mathrm{Vol} = \int \frac{qq'\cos\alpha}{4\pi r^2 r'^2} d\mathrm{Vol} = \frac{qq'}{2}\int_0^\infty dr \int_{-1}^1 d\cos\theta \frac{r - d\cos\theta}{(r^2 + d^2 - 2rd\cos\theta)^{3/2}} \\
&= \frac{qq'}{2}\int_0^\infty dr \left[\left(\frac{1}{|r-d|} - \frac{1}{r+d}\right)\left(\frac{1}{2d} - \frac{d}{2r^2}\right) - \frac{1}{2r^2 d}\left(|r-d| - r - d\right)\right] \\
&= \frac{qq'}{2}\int_0^d dr \left[\left(\frac{1}{d-r} - \frac{1}{r+d}\right)\frac{r^2 - d^2}{2r^2 d} + \frac{1}{rd}\right] + \frac{qq'}{2}\int_d^\infty dr \left[\left(\frac{1}{r-d} - \frac{1}{r+d}\right)\frac{r^2 - d^2}{2r^2 d} + \frac{1}{r^2}\right] \\
&= qq'\int_d^\infty \frac{dr}{r^2} = \frac{qq'}{d}.
\end{aligned} \tag{9}$$

Note how the contribution to the energy U_B at distances $r < d$ vanishes, and the energy is accounted for in Maxwell's view entirely by the contribution for $r > d$, *i.e.*, at relatively large distances.

A.2 Magnetic Dipole plus Magnetic Charge

We now consider magnetic charge q at the origin, plus magnetic charge q' at $\mathbf{r}$ and magnetic charge $-q'$ at $\mathbf{r}' = \mathbf{r} - \mathbf{m}'/q'$, where $\mathbf{m}' = q'(\mathbf{r} - \mathbf{r}')$ is the magnetic dipole moment of magnetic charges $\pm q'$. We ignore the self energy of the dipole $\mathbf{m}'$ and compute the interaction field energy of charge q and the magnetic dipole, which follows from eq. (9) as,

$$U_B = \frac{qq'}{r} - \frac{qq'}{r'} \approx \frac{qq'}{r}\left(1 - \frac{1}{1 - \mathbf{m}' \cdot \hat{\mathbf{r}}/q'r}\right) = -\frac{q\,\mathbf{m}' \cdot \hat{\mathbf{r}}}{r^2} \qquad (\mathbf{r} \text{ from } q \text{ to } \mathbf{m}'). \tag{10}$$

In this, $\mathbf{r}$ points from q to $\mathbf{m}'$. The usual convention for the interaction potential energy U_B/q of a dipole is that the vector $\mathbf{r}$ points from the dipole to the test charge. Therefore, we rewrite (10) as,

$$U_B = \frac{q\,\mathbf{m}' \cdot \hat{\mathbf{r}}}{r^2} = \frac{q\,\mathbf{m}' \cdot \mathbf{r}}{r^3} \qquad \text{from to} q \text{ to } . \tag{11}$$

A.3 Two Magnetic Dipoles

Finally, we consider a magnetic dipole $\mathbf{m}$ at the origin, plus magnetic charge q' at $\mathbf{r}$ and magnetic charge $-q'$ at $\mathbf{r}' = \mathbf{r} - \mathbf{m}'/q'$, where $\mathbf{m}'$ is the magnetic dipole moment of magnetic charges $\pm q'$. We ignore the self energy of the dipoles and compute the interaction field energy of magnetic dipole $\mathbf{m}$ plus the two charges $\pm q'$ that form magnetic dipole $\mathbf{m}'$. From eq. (10) we have,

$$\begin{aligned} U_B &= \frac{q'\,\mathbf{m} \cdot \mathbf{r}}{r^3} - \frac{q'\,\mathbf{m} \cdot \mathbf{r}'}{r'^3} \approx \frac{q'\,\mathbf{m} \cdot \mathbf{r}}{r^3} - \frac{q'\,\mathbf{m} \cdot \mathbf{r}}{r^3(1 - \mathbf{m}' \cdot \hat{\mathbf{r}}/q'r)^3} + \frac{\mathbf{m} \cdot \mathbf{m}'}{r'^3} \\ &\approx -\frac{3(\mathbf{m} \cdot \mathbf{r})(\mathbf{m} \cdot \hat{\mathbf{r}}')}{r^4} + \frac{\mathbf{m} \cdot \mathbf{m}'}{r^3} = -\frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})(\mathbf{m} \cdot \hat{\mathbf{r}}') - \mathbf{m} \cdot \mathbf{m}'}{r^3}. \end{aligned} \tag{12}$$

Hence, the form (2) can be deduced from the interaction field-energy (6), as well as from the force (3).

References

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