The Electric Self Force on a Current Loop

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1 Problem

Discuss the electric (and magnetic) self force on a loop of resistive wire that carries a steady electric current, taking into account the retarded electric field of the moving charges.[1,2]

2 Solution

If the total self force on a current loop were not zero, Newton's first law would be violated, and a current loop would be a "bootstrap spaceship".[3] However, the self forces includes the internal "mechanical" forces as well as the (macroscopic) electromagnetic forces, so the former should be considered as well as the latter.[4]

In a model of the steady current loop in which the (rigid) conductor contains conduction electrons, (positive) lattice ions, and surface charges of either sign, the lattice ions and surface charges are at rest with respect to the conductor, so the electromagnetic force on each of these is balanced by an equal and opposite "mechanical" force, which is ultimately a quantum electrodynamic effect. The conduction electrons are taken to have constant speed v, so they have no acceleration along the axial direction of the conductor/wire, but each has a transverse acceleration v^2/ρ, where ρ is the local radius of curvature of the wire. The required centripetal force for this acceleration is the sum of the electromagnetic and "mechanical" forces on a conduction electron.

[1]This problem was suggested by Vladimir Onoochin [1].

[2]In Maxwell's electrodynamics [2], this problem is not trivial. In contrast, the force between two moving charges e_1 and e_2 at positions $\mathbf{r}_1$ and $\mathbf{r}_2$ in Weber's electrodynamics (1846 [3], p. 144 of [4]) obeys Newton's third law,

$$\mathbf{F}_1 = \frac{e_1 e_2}{r^2}\,\hat{\mathbf{r}}\left(1 - A^2\dot{r}^2 + 2A^2 r\,\ddot{r}\right) = -\mathbf{F}_2, \qquad (1)$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, so the total self force on a current loop is zero. The constant A has dimensions of velocity^{-1}, and was later (1856) written by Weber and Kohlsrausch [5] as $1/C$, who noted that their C is the ratio of the magnetic units to electrical units in the description of static phenomenon, which they determined experimentally to have a value close to 4.4×10^8 m/s. Apparently, they regarded it as a coincidence that their C was roughly $\sqrt{2}$ times the speed c of light.

Weber was perhaps the last major physicist who did not use electric and magnetic fields to describe electromagnetism, preferring instead an (instantaneous) action-at-a-distance formulation for the forces between charges, eq. (1). This was the first published force law for moving charges (which topic Ampère refused to speculate upon).

For an extensive discussion of Weber's electrodynamics, see [6]. Maxwell gave a review of the German school of electrodynamics of the mid 19th century in the final chapter 23 of his *Treatise* [7].

[3]For additional comments by the author on "bootstrap spaceships", see Appendix A, and [8, 9].

[4]See, for example, [10, 11].

The total force on the system is equal to the sum of the mass times acceleration of the conduction electrons. The total force on the lattice ions and surface charges is zero, so these do not accelerate with respect to the rest frame of the current loop.

The steady current loop is not a "bootstrap spaceship".

A different accounting emphasizes the total electric force, and the total magnetic force, on the system, as discussed below.

2.1 Magnetic Self Force

Contemporary discussions of magnetic forces tend to assume as valid the Lorentz force law for an electric charge e with velocity $\mathbf{v}$ in external electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$,

$$\mathbf{F}_e = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \qquad (2)$$

in Gaussian units, with c being the speed of light in vacuum.[5] Applying this law to electrical circuits 1 and 2, the force on circuit 1 due to circuit 2 is the so-called Biot-Savart law,[6]

$$\begin{aligned} \mathbf{F}^{(\mathrm{B-S})}_{\mathrm{on}\ 1} &= \oint_1 d\mathbf{F}^{(\mathrm{B-S})}_{\mathrm{on}\ 1} = \oint_1 \frac{I_1\, d\mathbf{l}_1 \times \mathbf{B}_{2\ \mathrm{at}\ 1}}{c} = \oint_1 \oint_2 d^2\mathbf{F}^{(\mathrm{B-S})}_{\mathrm{on}\ 1}, \qquad \mathbf{B}_{2\ \mathrm{at}\ 1} = \oint_2 \frac{I_2\, d\mathbf{l}_2 \times \hat{\mathbf{r}}}{cr^2}, \\ d^2\mathbf{F}^{(\mathrm{B-S})}_{\mathrm{on}\ 1} &= I_1 d\mathbf{l}_1 \times \frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{c^2 r^2} = I_1 I_2 \frac{(\hat{\mathbf{r}} \cdot d\mathbf{l}_1)\, d\mathbf{l}_2 - (d\mathbf{l}_1 \cdot d\mathbf{l}_2)\, \hat{\mathbf{r}}}{c^2 r^2} \neq -d^2\mathbf{F}^{(\mathrm{B-S})}_{\mathrm{on}\ 2} \end{aligned} \qquad (3)$$

[5]Maxwell, Arts. 598-599 of [7], considered the "electromotive intensity" to be eq. (2) divided by e. However, he seems not to have made the inference that eq. (2) represents the force on a moving charge, as pointed out by FitzGerald [12].

The force law eq. (2) was more explicitly stated by Heaviside (1889), eq. (10) of [13], although Maxwell wrote it in a somewhat disguised form on p. 342 of [14] (1861), and in Art. 599 of [7] (for additional discussion, see secs. 1-2 of [17]). Like Heaviside, Lorentz (1892) gave the force law in the form $e(\mathbf{D} + \mathbf{v}/c \times \mathbf{H})$, eq. (113) of [15]. The debate as to whether the force depends on $\mathbf{B}$ or $\mathbf{H}$ was settled experimentally in favor of $\mathbf{B}$ only in 1944 [16].

In contemporary usage, as for Maxwell, the velocity $\mathbf{v}$ in the Lorentz force law is that of the charge in the (inertial) lab frame where $\mathbf{F}$, $\mathbf{E}$ and $\mathbf{B}$ are measured. However, in Lorentz' original view the velocity was to be measured with respect to the supposed rest frame of the ether. See, for example, [18].

[6]Biot and Savart [19, 20, 21] had no concept of the magnetic field $\mathbf{B}$ of an electric current I, and discussed only the force on a magnetic pole p, as $p \oint I\, d\mathbf{l} \times \hat{\mathbf{r}}/cr^2$, although not, of course, in vector form. The form (3) can be traced to Grassmann (1845) [23], still not in vector form. The vector relation $\mathbf{F}_{\mathrm{on}\ 1} = \oint_1 I_1\, d\mathbf{l}_1 \times \mathbf{B}_{2\ \mathrm{at}\ 1}/c$ appears without attribution as eq. (11) of Art. 603 of Maxwell's *Treatise* [7], while Einstein may have been the first to call this the Biot-Savart law, in sec. 2 of [24].

Heaviside (1886) [25, 26], discussed the form $d\mathbf{F} = \rho\mathbf{E} + \mathbf{\Gamma} \times \mathbf{H}$, where ρ is the electric charge density and $\mathbf{\Gamma} = \boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J} + \partial\mathbf{D}/\partial t$, where $\mathbf{J}$ is the conduction current density $\mathbf{J}$ and $\partial\mathbf{D}/\partial t$ is the "displacement current" density. However, the present view is that the "displacement current" does not experience a magnetic force.

The earliest description in English of eq. (3) as the Biot-Savart law may be in sec. 7-6 of [28].

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is the distance from a current element $I_2\, d\mathbf{l}_2$ at $\mathbf{r}_2$ to element $I_1\, d\mathbf{l}_1$ at $\mathbf{r}_1$.[7] Ampère held a rather different view [31], that,

$$\mathbf{F}^{(\mathrm{A})}_{\mathrm{on}\ 1} = \oint_1 \oint_2 d^2\mathbf{F}^{(\mathrm{A})}_{\mathrm{on}\ 1}, \qquad d^2\mathbf{F}^{(\mathrm{A})}_{\mathrm{on}\ 1} = I_1 I_2 [3(\hat{\mathbf{r}}\cdot d\mathbf{l}_1)(\hat{\mathbf{r}}\cdot d\mathbf{l}_2) - 2\, d\mathbf{l}_1\cdot d\mathbf{l}_2]\frac{\hat{\mathbf{r}}}{c^2 r^2} = -d^2\mathbf{F}^{(\mathrm{A})}_{\mathrm{on}\ 2}. \quad (4)$$

Ampère considered that the laws of electrodynamics should respect Newton's third law, of action and reaction, whereas the Biot-Savart/Lorentz law (3) does not. This "minor" detail is seldom discussed in textbooks,[8] and it held up acceptance of eq. (3) in preference to eq. (4) for about 70 years, 1820-1890.[9]

2.1.1 Equivalence of the Ampère and Biot-Savart Force Laws for Closed Circuits

According to the Biot-Savart form (3), the force on circuit element $I_1\, d\mathbf{l}_1$ due to current I_2 in circuit 2 is,

$$d\mathbf{F}^{(\mathrm{B-S})}_{\mathrm{on}\ d\mathbf{l}_1} = I_1\, d\mathbf{l}_1 \times \oint_2 \frac{I_2\, d\mathbf{l}_2 \times \mathbf{r}}{cr^3} = \frac{I_1 I_2}{c^2}\oint_2 \frac{(\mathbf{r}\cdot d\mathbf{l}_1)\, d\mathbf{l}_2 - (d\mathbf{l}_1\cdot d\mathbf{l}_2)\,\mathbf{r}}{r^3}. \quad (5)$$

To compare this with Ampère's form (4), it is useful to note the relations (given by Ampère),

$$d\mathbf{l}_2 = -\frac{\partial \mathbf{r}}{\partial l_2}\, dl_2, \qquad \mathbf{r}\cdot d\mathbf{l}_2 = \mathbf{r}\cdot\frac{\partial \mathbf{r}}{\partial l_2}\, dl_2 = -r\frac{\partial r}{\partial l_2}\, dl_2. \quad (6)$$

Then,

$$\begin{aligned} d\mathbf{F}^{(\mathrm{A})}_{\mathrm{on}\ d\mathbf{l}_1} &= \frac{I_1 I_2}{c^2}\oint_2 \frac{3(\mathbf{r}\cdot d\mathbf{l}_1)(\mathbf{r}\cdot d\mathbf{l}_2)\mathbf{r}}{r^5} - \frac{2(d\mathbf{l}_1\cdot d\mathbf{l}_2)\mathbf{r}}{r^3} \\ &= \frac{I_1 I_2}{c^2}\oint_2 \left(-\frac{3d\mathbf{l}_1}{r^4}\cdot\mathbf{r}\frac{\partial r}{\partial l_2}\, dl_2\right)\mathbf{r} - \frac{2 I_1 I_2}{c^2}\oint_2 \frac{(d\mathbf{l}_1\cdot d\mathbf{l}_2)\,\mathbf{r}}{r^3}. \end{aligned} \quad (7)$$

[7] If we follow Ampère in defining a "current element" as being electrically neutral, which is a good (but not exact [29]) approximation for currents in electrical circuits, then an isolated, moving charge is not a "current element" (contrary to remarks such as in [30]). A wire that is used to discharge a capacitor could be considered as an example of an Ampèrian current element when it carries the transient current. The magnetic forces on a pair of such current elements would not obey Newton's third law, but overall momentum conservation is observed when one takes into account the momentum stored in the electromagnetic fields of the system. This is easier to analyze for a pair of moving charges [33] than for a pair of discharging capacitors.

[8] One exception is sec. 7-5 of [28].

[9] Maxwell gave an intricate discussion in Arts. 502-526 of his *Treatise* [7], in which he pointed out that experiments on the forces between closed circuits cannot fully determine an expression for the magnetostatic forces, and that one arbitrary assumption is required to arrive at a "law". He considered (Art. 526) four such assumptions, including Ampère's that the force law obey Newton's third law, and Grassmann's that the force is zero between collinear current elements; Maxwell then expressed his preference of Ampère's form, although in Art. 599 he displayed the Lorentz force law without comment as to its relation to the forms of Ampère and Grassmann.

The ith component of the first integral on the second line of eq. (7) can be written as,

$$
\begin{aligned}
-\oint_2 \frac{3d\mathbf{l}_{1,j} r_i r_j}{r^4}\frac{\partial r}{\partial l_2}\, dl_2 &= \oint_2 d\mathbf{l}_{1,j}\left[\frac{\partial}{\partial l_2}\left(\frac{r_i r_j}{r^3}\right) - \frac{r_i}{r^3}\frac{\partial r_j}{\partial l_2} - \frac{r_j}{r^3}\frac{\partial r_i}{\partial l_2}\right] dl_2 \\
&= \oint_2 \frac{(d\mathbf{l}_1 \cdot d\mathbf{l}_2) r_i}{r^3} + \oint_2 \frac{(d\mathbf{l}_1 \cdot \mathbf{r})\, d\mathbf{l}_{2,i}}{r^3}\,. \qquad (8)
\end{aligned}
$$

Using this in eq. (7), we have that,

$$
d\mathbf{F}^{(\mathrm{A})}_{\mathrm{on}\ \mathrm{dl}_1} = \frac{I_1 I_2}{c^2}\oint_2 \frac{(\mathbf{r}\cdot d\mathbf{l}_1)\, d\mathbf{l}_2 - (d\mathbf{l}_1 \cdot d\mathbf{l}_2)\,\mathbf{r}}{r^3} = d\mathbf{F}^{(\mathrm{B-S})}_{\mathrm{on}\ \mathrm{dl}_1} \qquad (9)
$$

Considering circuit element $I_1\, d\mathbf{l}_1$ to be part of circuit 1, distinct from circuit 2, we find,

$$
\mathbf{F}^{(\mathrm{A})}_{\mathrm{on}\ 1} = \oint_1 d\mathbf{F}^{(\mathrm{A})}_{\mathrm{on}\ \mathrm{dl}_1} = \oint_1 d\mathbf{F}^{(\mathrm{B-S})}_{\mathrm{on}\ \mathrm{dl}_1} = \mathbf{F}^{(\mathrm{B-S})}_{\mathrm{on}\ 1} \qquad (10)
$$

Finally, since Ampère's force between a pair of circuit elements is along their line of centers,

$$
\mathbf{F}^{(\mathrm{A})}_{\mathrm{on}\ 1} = -\mathbf{F}^{(\mathrm{A})}_{\mathrm{on}\ 2} = \mathbf{F}^{(\mathrm{B-S})}_{\mathrm{on}\ 1} = -\mathbf{F}^{(\mathrm{B-S})}_{\mathrm{on}\ 2}\,. \qquad (11)
$$

When either the Biot-Savart form (3) or the Ampère form (4) is applied to a pair of circuits, the total forces on the circuits are the same, and Newton's third law is satisfied.[10,11]

When considering the force of a single circuit on itself, one can worry that the integrals in eqs. (5) and (7) might diverge, such that the magnetic self force might not be zero. Since Ampère's force law between pairs of current elements obeys Newton's third law, one has confidence that this leads to $\mathbf{F}^{(\mathrm{A})}_{\mathrm{self}} = -\mathbf{F}^{(\mathrm{A})}_{\mathrm{self}}$ such that the magnetic self force is zero, but the case for the Biot-Savart form cannot be argued so quickly. Stefan [37] considered that physical circuits have wires of finite diameter, for which it is convincing that the equivalence of the Ampère and the Biot-Savart force laws for closed, filamentary circuits implies that the magnetic self force is also zero for the latter form.[12]

A corollary to the above argument is that the total magnetic force on a current element due to currents in closed circuits is perpendicular to the current element, according to the (static) force laws of both Ampère and Biot-Savart.

An extrapolation of the static force laws of both Ampère and of Biot-Savart-Grassmann is that if a circuit moves in response to the (initially static) magnetic force on it, then that magnetic force does work on the moving circuit.[13] This extrapolation presumes that the static force laws are still approximately correct for examples where the motions have low velocity.

The bottom line of this section is that the magnetic self force is zero for a steady current loop.

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[10]A derivation something like the above was first given by Neumann in 1845 [34, 35, 36], and in more detail by Stefan in 1869 [37, 38, 39].

[11]A tacit assumption here is that effects of wave propagation can be ignored. For an example in which a pair of circuits emit radiation, with a resulting propulsive force on the circuits, see [40].

[12]This was argued (rather briefly) by Maxwell in Art. 687 of [7], and variants have been given in [36]-[63].

[13]As mentioned in footnote 2 above, Weber was the first to consider a force law for moving charges, but this involved instantaneous action at a distance. Effects of retardation, due to the finite speed of propagation of electromagnetic waves, on the electric and magnetic fields of a point electric charge were first considered by Liénard [64] and by Wiechert [65]. The computation of the retarded fields for electric charge and current distributions is reviewed in [66].

2.2 Electric Self Force

A naïve view is that a current loop is electrically neutral, such that there are no (macroscopic) electric forces, and the electric self force is trivially zero. However, if (conduction) current density $\mathbf{J}$ exists inside a medium with nonzero, but finite, electric conductivity σ, then there must be an electric field,

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma}\,, \tag{12}$$

inside the medium. In case of a steady current (and static magnetic field), this electric field is generated by electric charges on the surface of the medium.[14]

As remarked earlier, there is also a tiny difference, of order v^2/c^2, between bulk density of positive and negative charges inside a current-carrying medium [29], which we neglect here. In this approximation, the interior of the conductor (wire) of the current loop is electrically neutral, and the electric field (12) exerts no net force on interior of the loop (*i.e.*, on the conduction electrons and the lattice ions). However, the surface charges also experience the axial field (12) as well as an electric field component that is perpendicular to the surface of the conductor. While we can assume that the total surface charge is zero, its distribution around the loop is nonuniform, and it would seem that, in general, the total electric force on the surface charge is nonzero.

These surface charges are kept from leaving the surface (and from moving along the surface once the steady-state surface charge distribution has been established) by (quantum) "mechanical" effects, often summarized by the term "work function." That is, the total force on each surface charge is zero, and the total (self) force on the current loop is zero (as noted at the beginning of sec. 2 above)

We continue with discussion of a subtle effect.

2.2.1 The Retarded Electric Field of the Electric Current

The electric field of the (slowly) moving conduction electrons includes small terms, of order v^2/c^2 where v is their drift velocity, that are not radial, and leads to nonzero net force between pairs of conduction electrons, and between conduction electrons and static charges.

We recall that the Liénard-Wiechert fields [64, 65] observed at position $\mathbf{r}_1$ at time t due to a charge e_2 at position $\mathbf{r}_2$ with velocity $\mathbf{v}_2$ and acceleration $\mathbf{a}_2$ are,

$$\begin{aligned}
\mathbf{E}_1(\mathbf{r}_1, t) &= e_2 \left[\frac{1}{\{(1 - (\hat{\mathbf{n}} \cdot \mathbf{v}_2)/c)\}^3 r^2} \{(1 - v_2^2/c^2)(\hat{\mathbf{n}} - \mathbf{v}_2/c) + \mathbf{r}/c \times (\hat{\mathbf{n}} - \mathbf{v}_2/c) \times \mathbf{a}_2/c)\} \right] \\
&\approx e_2 \left[\frac{\hat{\mathbf{n}}}{r^2} - \frac{v_2^2}{c^2 r^2}\,\hat{\mathbf{n}} - \frac{1}{r^2}\{(1 - 3(\hat{\mathbf{n}} \cdot \mathbf{v}_2)/c)\}\frac{\mathbf{v}_2}{c} + \frac{1}{r^2}\left(\mathbf{r} \cdot \frac{\mathbf{a}_2}{c}\hat{\mathbf{n}} - \mathbf{r} \cdot \hat{\mathbf{n}}\frac{\mathbf{a}_2}{c} \right) \right] \\
&= e_2 \left[\frac{\hat{\mathbf{n}} - \mathbf{v}_2/c}{r^2} + \frac{3(\hat{\mathbf{n}} \cdot \mathbf{v}_2)\mathbf{v}_2 - v_2^2\,\hat{\mathbf{n}}}{c^2 r^2} - \frac{\mathbf{a}_2 - (\mathbf{a}_2 \cdot \hat{\mathbf{n}})\,\hat{\mathbf{n}}}{c^2 r} \right], \qquad (13) \\
\mathbf{B}_1(\mathbf{r}_1, t) &= [\hat{\mathbf{n}}] \times \mathbf{E}_1, \qquad (14)
\end{aligned}$$

[14]See, for example, [67]-[92]. A superconducting current loop will be considered in sec. 4 below.

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $\hat{\mathbf{n}} = \mathbf{r}/r$, quantities inside brackets, [...], are evaluated at the retarded time $t' = t - r/c$, and the approximations are to order $1/c^2$.

In static examples like the present, where there is no radiation [94], it is preferable to approximate the fields to order $1/c^2$ in terms of quantities at the present time, rather than at the retarded time. This was first done by Darwin (1920) [95], as reviewed in sec. 65 of [96] and sec. 12.6 of [97]. He worked in the Coulomb gauge, and kept terms only to order v^2/c^2. Then, the scalar and vector potentials at $\mathbf{r}_1$ due to a charge e_2 at $\mathbf{r}_2$ that has velocity $\mathbf{v}_2$ are,

$$\phi_1 = \frac{e_2}{r}\,, \qquad \mathbf{A}_1 = e_2 \frac{\mathbf{v}_2 + (\mathbf{v}_2 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}}{2cr}\,, \tag{15}$$

where $\hat{\mathbf{n}}$ is directed from the charge to the observer, whose (present) distance is r from the observer. The electric and magnetic fields in the Darwin approximation follow from the potentials (15),

$$\mathbf{E}_1 = -\boldsymbol{\nabla}\phi_1 - \frac{\partial \mathbf{A}_1}{\partial ct} = e_2 \frac{\hat{\mathbf{n}}}{r^2} - \frac{e_2}{2c^2} \left(\frac{\mathbf{a}_2 + (\mathbf{a}_2 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}}{r} + \frac{3(\mathbf{v}_2 \cdot \hat{\mathbf{n}})^2 - v_2^2}{r^2} \hat{\mathbf{n}} \right), \tag{16}$$

$$\mathbf{B}_1 = \boldsymbol{\nabla} \times \mathbf{A}_1 = e_2 \frac{\mathbf{v}_2 \times \hat{\mathbf{n}}}{cr^2}\,, \tag{17}$$

where $\mathbf{a}_2 = d\mathbf{v}_2/dt$ is the (present) acceleration of the charge.[15,16,17]

Thus, effects of retardation lead to terms of order v^2/c^2 in the electric field due to the conduction electrons. The total electric field inside the conductor of the current loop must still be that of eq. (12), so the surface charge density is slightly different than that which would hold in the absence of the v^2/c^2 effect. Again, the total electric self force on the surface charge is, in general, nonzero, but this is canceled by the total, nonzero (quantum) "mechanical" force that holds the surface charge at rest with respect to the current loop.

That is, the total self force on the current loop (in its rest frame) is zero, even taking into account the effect of retardation.

3 Current Loop plus External Charge

As a variant, we consider the case of an external electric charge, that is held fixed with respect to the current loop (in the latter's rest frame) by a (quantum) "mechanical" force.

The external charge induces a change in the surface charge density on the current loop, but again the total electric field inside the conductor of the loop is given by eq. (12).

The external charge then exerts a net electric force on the surface charges of the current loop, in addition to that of the v^2/c^2 part of the electric field of the conduction electrons. The electric field of the current loop, due both to the surface charges and to the v^2/c^2 field of the conduction electrons, exerts a net force on the external charge. And, (quantum)

[15]Sec. 65 of [96] first showed that in the Darwin approximation the Liénard-Wiechert potentials in the Lorenz gauge reduce to $\phi = e/r + (e/2c^2)\partial^2 r/\partial t^2$ and $\mathbf{A} = e\mathbf{v}/cr$, from which eqs. (16)-(17) also follow.

[16]See [33, 98, 99] for applications of these relations to considerations of electromagnetic momentum and energy.

[17]Note that the Liénard-Wiechert electric field (13) has a term of order v_2/c while the Darwin electric field (16) does not.

"mechanical" forces act on both the surface charges and the external charge, as the latter is fixed with respect to the current loop. The sum of all these forces is zero, and there is no net self force on the system.[18]

4 Superconducting Loop

The electric field is zero in the interior of the (super)conductor of a superconducting current loop, so in the absence of the v^2/c^2 electric field due to retardation, so the superconductor could have zero surface charge density. Taking in account, the v^2/c^2 component of the electric field, due to retardation, there will be a nonzero surface charge density such that the total electric field inside the superconduct is zero.

Then, the total electric force on the surface charge density is, in general, nonzero, but this is canceled by the total (quantum) "mechanical" force that holds the surface charge density onto the surface.

If an external charge is added to the superconducting current loop, the surface charge density is changed, but again the total electric force on the system is equal and opposite to the total (quantum) "mechanical" force.

A Appendix: The Center of Energy Theorem

A reason why there are no "bootstrap spaceships" is given by the so-called center-of-energy theorem,[19] that the total linear momentum of any isolated, stationary system is zero if the velocity of its center of mass/energy is zero.

Consider an isolated, system which is a candidate for a "bootstrap spaceship", and is initially stationary. According to the center-of-energy theorem it has zero total linear momentum.

At some time, the system could initiate internal activity that generates quasistatic electromagnetic-field momentum which is not radiated away, but which remains in the vicinity of the matter of the system. For the total momentum of the system to remain zero, there must now be some mechanical momentum in the system. Nominally, such mechanical momentum would imply that the center of mass of the matter of the system is in motion, and would be propelled in some direction.

At a later time, suppose the system stops its internal activity, such that the equal-and-opposite electromagnetic-field momentum and mechanical momentum are constant thereafter. The center of mass of the matter of the system then has a constant velocity in some direction.

If we observe the system in the (inertial) frame with that constant velocity, the system is isolated and stationary. So, according to the center-of energy theorem, the total momentum of the system should be zero in this frame. However, while the mechanical momentum of the

[18]If one neglected the surface charge on the current loop, and the (quantum "mechanical" forces thereon, as in [1], one might suppose that the external charge exerts no force on the current loop, so that nonzero force of the v^2/c^2 part of the electric field of the conduction electrons on the external charge would imply a total, nonzero self force on the current loop.

[19]See the Appendix of [100], sec. 2 of [101], and sec. I of [102].

system is zero in this frame, its electromagnetic-field momentum is nonzero, and hence the total momentum of the system is nonzero (in this frame). This contradiction implies that the above scenario is impossible.

A.1 Some Details

The mechanical behavior of a macroscopic system can be described with the aid of the (symmetric) stress-energy-momentum tensor $T^{\mu\nu}$ of the system. The total energy-momentum 4-vector of the system is given by,

$$U^\mu = (U_{\rm total}, P^i_{\rm total}\, c) = \int T^{0\mu}\, d{\rm Vol}. \tag{18}$$

As first noted by Abraham [103], at the microscopic level the electromagnetic parts of $T^{\mu\nu}$ are,

$$\begin{aligned} T^{00}_{\rm EM} &= \frac{E^2 + B^2}{8\pi} = u_{\rm EM}\,, & (19)\\ T^{0i}_{\rm EM} &= \frac{S^i}{c} = p^i_{\rm EM}\, c, & (20)\\ T^{ij}_{\rm EM} &= \frac{E^i E^j + B^i B^j}{4\pi} - \delta^{ij}\frac{E^2 + B^2}{8\pi}\,, & (21) \end{aligned}$$

in terms of the microscopic fields $\mathbf{E}$ and $\mathbf{B}$. In particular, the density of electromagnetic momentum stored in the electromagnetic field is,

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,. \tag{22}$$

The macroscopic stress tensor $T^{\mu\nu}$ also includes the "mechanical" stresses within the system, which are actually electromagnetic at the atomic level. The form (21) still holds in terms of the macroscopic fields $\mathbf{E}$ and $\mathbf{B}$ in media where $\epsilon = 1 = \mu$ such that strictive effects can be neglected. The macroscopic stresses T^{ij} are related the volume density $\mathbf{f}$ of force on the system according to,

$$f^i = \frac{\partial T^{ij}}{\partial x^j}\,. \tag{23}$$

The stress tensor $T^{\mu\nu}$ obeys the conservation law,

$$\frac{\partial T^{\mu\nu}}{\partial x_\mu} = 0, \tag{24}$$

with $x^\mu = (ct, \mathbf{x})$ and $x_\mu = (ct, -\mathbf{x})$. Once consequence of this is that the total momentum is constant for an isolated, spatially bounded system, *i.e.*,

$$\int \frac{\partial T^{\mu i}}{\partial x_\mu} = 0 = \frac{\partial}{\partial ct}\int T^{0i}\, d{\rm Vol} - \int \frac{\partial T^{ji}}{\partial x^j}\, d{\rm Vol} = \frac{dP^i_{\rm total}}{dt} - \int T^{ji}\, d{\rm Area}^j = \frac{dP^i_{\rm total}}{dt}\,. \tag{25}$$

A related result is that the total (relativistic) momentum $\mathbf{P}_{\text{total}}$ of an isolated system is proportional to the velocity $\mathbf{v}_U = d\mathbf{x}_U/dt$ of the center of mass/energy of the system [100, 101, 102],

$$\mathbf{P}_{\text{total}} = \frac{U_{\text{total}}}{c^2}\mathbf{v}_U = \frac{U_{\text{total}}}{c^2}\frac{d\mathbf{x}_U}{dt}\,, \tag{26}$$

where,

$$U_{\text{total}} = \int T^{00}\, d\text{Vol}, \tag{27}$$

$$P^i_{\text{total}} = \frac{1}{c}\int T^{0i}\, d\text{Vol}, \tag{28}$$

$$\mathbf{x}_U = \frac{1}{U_{\text{total}}}\int T^{00}\,\mathbf{x}\, d\text{Vol}. \tag{29}$$

That is, the total momentum of an isolated system is zero in that (inertial) frame in which the center of mass/energy is at rest.

B Appendix: Comment on the Electric Field in the Darwin Approximation

The part of the electric field of an electric charge e in the Darwin approximation that depends on its acceleration $\mathbf{a}$ is, according to eq. (16),

$$\mathbf{E}_{\mathbf{a},\text{Darwin}} = -e\frac{\mathbf{a} + (\mathbf{a}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}}{2c^2 r}\,. \tag{30}$$

This is possibly surprising in that Liénard-Wiechert electric field of an accelerating charge, eq. (13), depends (explicitly) on the acceleration as,

$$\mathbf{E}_{\mathbf{a},\text{L-W}} = -\frac{e}{c^2}\left[\frac{\mathbf{a} - (\mathbf{a}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}}{r}\right]_{\text{retarded}} + \mathcal{O}\left(\frac{1}{c^3}\right). \tag{31}$$

We illustrate the compatibility of the Darwin approximation with the Liénard-Wiechert electric field for the case of a charge e that moves along the x-axis with constant acceleration a, according to $x = at^2/2$. The observer is at $x = d$ on the x-axis, so that $\hat{\mathbf{n}} = \hat{\mathbf{x}}$. Then,

$$\mathbf{E}_{\mathbf{a},\text{Darwin}} = -\frac{ea}{c^2 d}\,\hat{\mathbf{x}}, \qquad \text{and} \qquad \mathbf{E}_{\mathbf{a},\text{L-W}} = 0. \tag{32}$$

However, we should compare the total electric fields before concluding that Darwin does not agree with Liénard and Wiechert. In particular, at time $t = 0$, the Darwin approximation is that,

$$\mathbf{E}_{\text{Darwin}} = \frac{e}{d^2}\left(1 - \frac{ad}{c^2}\right)\hat{\mathbf{x}}, \tag{33}$$

while the Liénard-Wiechert field is,

$$\mathbf{E}_{\text{L-W}} = e\left[\frac{\hat{\mathbf{x}} - \mathbf{v}/c}{\gamma^2 r^2(1 - \mathbf{v}\cdot\hat{\mathbf{n}}/c)^3}\right]_{\text{retarded}}. \tag{34}$$

The retarded time is $t' = t - r/c = -r/c \approx -d/c$. Then, the retarded velocity is $[v] = at' = -ad/c$ (in the $-x$ direction), the retarded Lorentz factor is $[\gamma] = 1 + \mathcal{O}(1/c^4)$, the retarded position is $[x] = at'^2/2 = ad^2/2c^2$, and the retarded distance is $[r] = d - x = d(1 - ad/2c^2)$. Using these in eq. (34), we find,

$$\mathbf{E}_{\mathrm{L-W}} \approx e\frac{1 + ad/c^2}{d^2(1 - ad/2c^2)^2(1 + ad/c^2)^3}\,\hat{\mathbf{x}} \approx \frac{e}{d^2}\left(1 - \frac{ad}{c^2}\right)\hat{\mathbf{x}} = \mathbf{E}_{\mathrm{Darwin}}. \tag{35}$$

The lesson is that when converting the Liénard-Wiechert fields from retarded time to present time, the present acceleration affects all terms, whether or not they contain explicit dependence on the retarded acceleration.

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