Momentum in a DC Circuit

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1 Problem

Discuss the electromagnetic and mechanical momentum in a DC current loop of resistance R that is powered by a battery of voltage V. The loop is at rest in the laboratory.

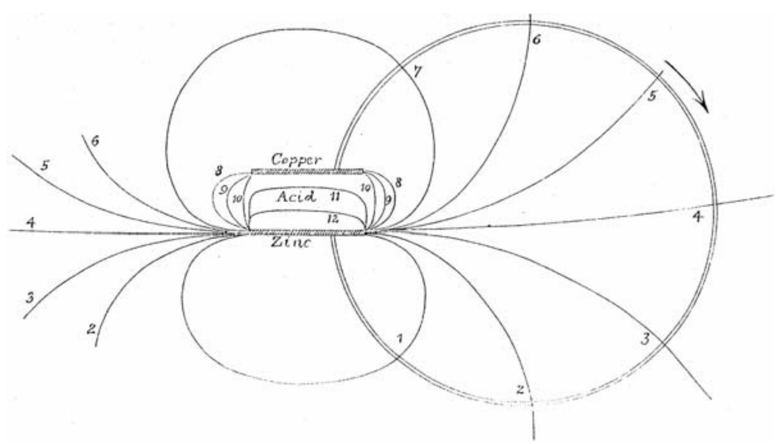
September 15, 2018. The discussion of "hidden" momentum in this note overlooks the important subtlety that in the rest frame of the loop plus battery, the flow of energy from the battery to the resistor results in a motion of the center of mass/energy in the direction from the battery to the resistor. That is, a (small!) external force would be required to hold the loop at rest in the lab frame, which frame is NOT *the center-of-mass/energy frame as tacitly assumed below.*

A better analysis in an example of this type is given in [1], whose updated version supercedes this note.

Of possible remaining interest is the computation of the electromagnetic field momentum, sec. 2.1 below. However, sec. 2.1.4 on the energy flow velocity is of doubtful merit, as already remarked there in earlier versions.

2 Solution

This problem was inspired by an extensive e-dialogue with Vladimir Hnizdo.



Because the loop is at rest in the laboratory, its total momentum should be zero. *September 15, 2018. This statement incorrectly assumes that there would be no net force on the loop if it were at rest in the lab frame.*

However, a DC current loop is a dynamical system in that its electrical resistance R causes dissipation of power at the rate $V^2/R = VI = I^2R$, where the DC current I is, of course, $I = V/R$. This power flows from the battery to the loop in a manner first well described by Poynting. As shown in the figure above by Poynting [2], the power does not flow down the wire of the loop, but rather it flows through the air/vacuum and enters the wire at right angles to its surface.[1]

Abraham noted that Poynting's vector, when divided by c^2, where c is the speed of light, represents the volume density of electromagnetic momentum [4].[2] The total electromagnetic momentum associated with the circuit is then given in terms of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ (in Gaussian units) by,

$$\mathbf{P}_{\rm EM} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol}. \tag{1}$$

The above figure suggests that the DC circuit has a nonzero electromagnetic momentum that points to the right.

Then, for the circuit to have zero total momentum, it must also possess a "hidden" mechanical momentum that points to the left.[3]

In sec. 2.1 we deduce the electromagnetic momentum of the circuit from various points of view, and in sec. 2.2 we discuss the "hidden" mechanical momentum.

2.1 Electromagnetic Momentum

A direct evaluation of the electromagnetic momentum of the circuit according to eq. (1) is cumbersome, so we begin with an indirect calculation in sec. 2.1.1. A lengthier, direct calculation is presented in sec. 2.1.2. An alternative calculation based on the concept of canonical electromagnetic momentum is given in sec. 2.1.3.

2.1.1 Indirect Calculation of the Electromagnetic Momentum

It is convenient to express the electromagnetic momentum (1) in terms of an integral of the electric potential Φ and the current density $\mathbf{J}$, following Furry [10].[4] See also [12]. For this we note that in a static situation the electric and magnetic fields obey $\mathbf{E} = -\boldsymbol{\nabla}\Phi$ and $\boldsymbol{\nabla}\times\mathbf{B} = (4\pi/c)\,\mathbf{J}$. Then, for (quasi)static examples where B varies as $1/r^3$ at large distances from the source currents,

$$\begin{aligned}
\mathbf{P}_{\rm EM} &= \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol} = -\int \frac{\boldsymbol{\nabla}\Phi\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol} = \int \frac{\Phi\boldsymbol{\nabla}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol} - \int \frac{\boldsymbol{\nabla}\times\Phi\mathbf{B}}{4\pi c}\, d\mathrm{Vol} \\
&= \int \frac{\Phi\mathbf{J}}{c^2}\, d\mathrm{Vol} - \oint \frac{d\mathbf{Area}\times\Phi\mathbf{B}}{4\pi c} \\
&= \int \frac{\Phi\mathbf{J}}{c^2}\, d\mathrm{Vol},
\end{aligned} \tag{2}$$

whenever the charges and currents are contained within a finite volume.

[1] As Sommerfeld said, p. 130 of [3]: *Conductors are nonconductors of energy. Electromagnetic energy is transported without loss only in nonconductors.*

[2] This was noted earlier, if less crisply, by Thomson [5, 6, 7], and also by Poincaré [8].

[3] For a general definition of "hidden" momentum, see [9].

[4] Additional expressions for electromagnetic momentum in static examples are discussed in [11].

In the present problem we suppose the circuit forms a circle of radius a in the x-y plane, centered on the origin. The battery of voltage V is located at $(x, y, z) = (-a, 0, 0)$ and is oriented so that the current vector at position $(a, \phi, 0)$ in a cylindrical coordinate system is,

$$\mathbf{I} = -\frac{V}{R}\hat{\boldsymbol{\phi}} = \frac{V}{R}(\hat{\mathbf{x}} \sin\phi - \hat{\mathbf{y}} \cos\phi). \tag{3}$$

We suppose that the resistance R is uniformly distributed over the circumference of the circuit, so that the electric potential along the circuit is,

$$\Phi(a, \phi, 0) = \frac{V\phi}{2\pi} \qquad (-\pi < \phi < \pi). \tag{4}$$

The electric potential varies from $-V/2$ to $V/2$ within the battery, so the potential is actually continuous in the vicinity of $\phi = \pm\pi$. If we suppose that the battery extends of the small region $|\pi - \phi| < \epsilon$, then the (azimuthal) electric field inside the battery and resistor can be expressed as,

$$E_\phi(a, \phi, 0) = \begin{cases} -\frac{V}{2(\pi-\epsilon)a} & (|\phi| < \pi - \epsilon), \\ \frac{V}{2\epsilon a} & (|\pi - \phi| < \epsilon). \end{cases} \tag{5}$$

Using eqs. (3) and (4) in (2), and expressing $\mathbf{J}\, d\mathrm{Vol}$ as $a\mathbf{I}\, d\phi$, we find the electromagnetic momentum of the circuit to be,

$$\mathbf{P}_{\mathrm{EM}} = \frac{aV^2}{2\pi c^2 R}\int_{-\pi}^{\pi} d\phi\, (\hat{\mathbf{x}} \sin\phi - \hat{\mathbf{y}} \cos\phi) = \frac{aV^2}{c^2 R}\hat{\mathbf{x}} = \frac{aVI}{c^2}\hat{\mathbf{x}} = \frac{aI^2 R}{c^2}\hat{\mathbf{x}}, \tag{6}$$

which points to the right as anticipated above.

2.1.2 Direct Calculation of the Electromagnetic Momentum

A direct evaluation of the electromagnetic momentum (1) is difficult; calculation of the magnetic field of a current loop leads to elliptic integrals. However, analytic calculations become tractable in a two-dimensional approximation of a current loop by a current cylinder [13].

We now suppose that R is the resistance of a unit length along the cylinder,[5] so that $I = V/R$ is the current per unit length that flows around its circumference due to the battery of voltage V that lies along the line $\phi = -\pi$. The magnetic field $\mathbf{B}$ is that of an infinite solenoid,

$$\mathbf{B} = \begin{cases} -\frac{4\pi I}{c}\hat{\mathbf{z}} = -\frac{4\pi V}{cR}\hat{\mathbf{z}} & (r < a), \\ 0 & (r > a). \end{cases} \tag{7}$$

The electric field $\mathbf{E}$ can be deduced from a calculation of the potential $\Phi(r, \phi, z)$. This potential now has the value $V\phi/2\pi$ for all z on the cylinder $r = a$. As this is an odd function

[5] The dimensions of R are resistance $\times$ length = length/velocity in Gaussian units.

of the angle ϕ, the potential can be represented by the expansion,

$$\Phi(r,\phi,z) = \begin{cases} \sum_{n=1}^{\infty} A_n \left(\frac{r}{a}\right)^n \sin n\phi & (r < a), \\ \sum_{n=1}^{\infty} A_n \left(\frac{a}{r}\right)^n \sin n\phi & (r > a). \end{cases} \tag{8}$$

The Fourier coefficients A_n are given by,

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{V\phi}{2\pi} \sin n\phi \, d\phi = -(-1)^n \frac{V}{n\pi}. \tag{9}$$

The radial component of the electric field is,

$$\begin{aligned} E_r = -\frac{\partial \Phi}{\partial r} &= \frac{V}{\pi a} \begin{cases} -\sum_{n=1}^{\infty} \left(\frac{-r}{a}\right)^{n-1} \sin n\phi & (r < a), \\ \sum_{n=1}^{\infty} \left(\frac{-a}{r}\right)^{n+1} \sin n\phi & (r > a), \end{cases} \\ &= \frac{aV}{\pi} \begin{cases} -\frac{\sin\phi}{r^2 + 2ar\cos\phi + a^2} & (r < a), \\ \frac{\sin\phi}{r^2 + 2ar\cos\phi + a^2} & (r > a). \end{cases} \end{aligned} \tag{10}$$

using Dwight 417.4. Similarly, the azimuthal electric field is, using Dwight 417.3,

$$\begin{aligned} E_\phi &= -\frac{1}{r}\frac{\partial \Phi}{\partial \phi} = -\frac{V}{\pi a} \begin{cases} \sum_{n=1}^{\infty} \left(\frac{-r}{a}\right)^{n-1} \cos n\phi & (r < a), \\ \sum_{n=1}^{\infty} \left(\frac{-a}{r}\right)^{n+1} \cos n\phi & (r > a), \end{cases} \\ &= -\frac{V}{\pi} \begin{cases} \frac{r + a\cos\phi}{r^2 + 2ar\cos\phi + a^2} & (r < a), \\ \frac{a}{r}\frac{a + r\cos\phi}{r^2 + 2ar\cos\phi + a^2} & (r > a). \end{cases} \end{aligned} \tag{11}$$

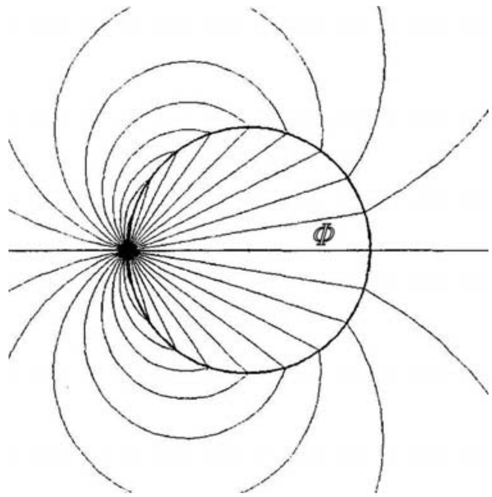
We digress briefly to discuss the equipotentials and field lines. For this, we note that,

$$\int \frac{a\sin\phi}{r^2 + 2ar\cos\phi + a^2}\, dr = \tan^{-1}\frac{r\sin\phi}{a + r\cos\phi} = -\tan^{-1}\frac{a\sin\phi}{r + a\cos\phi}\,, \tag{12}$$

so that the potential can be obtained by integrating eq. (10). Hence,

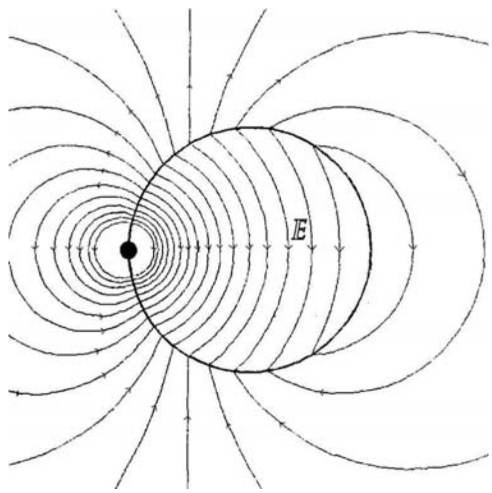
$$\Phi(r,\phi,z) = \frac{V}{\pi} \begin{cases} \tan^{-1}\frac{r\sin\phi}{a + r\cos\phi} = \theta & (r < a), \\ \tan^{-1}\frac{a\sin\phi}{r + a\cos\phi} & (r > a), \end{cases} \tag{13}$$

which is continuous at $r = a$, where θ is the azimuthal angle with respect to the x-axis measured from the location of the battery. When $r = a$, angle θ equals $\phi/2$ so that eq. (13) agrees with eq. (4). The equipotentials are shown in the figure below, from [13].

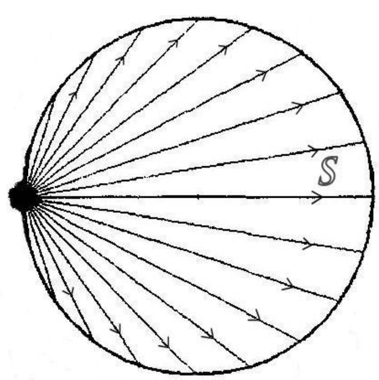


Inside the circuit the equipotentials are straight lines that emanate from the battery. Outside the circuit the equipotentials are circles that pass through the battery.

The corresponding electric field lines are shown below.



The Poynting vector, $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$, is nonzero only inside the circuit, where $\mathbf{S}$ is perpendicular to $\mathbf{E}$ and hence parallel to the equipotentials. That is, the equipotential lines inside the circuit on the upper figure on p. 4 also represent the flow of energy from the battery to the resistive circuit, as shown below as well.



A subtlety is that the electric field inside the conductor at $r = a$ is purely azimuthal, as needed to drive the current in the resistive medium. This is consistent with eq. (10) in that the average of the radial electric field at $r = a^-$ and $r = a^+$ is zero. In contrast, the azimuthal electric field (11) is continuous at $r = a$. Hence, the flow of energy into the resistor at $r = a$ is described by $\mathbf{S} = cE_\phi \hat{\boldsymbol{\phi}} \times \mathbf{B}/4\pi = VI\,\hat{\mathbf{r}}/2\pi a$, which is the power dissipated per unit length around the circumference of the resistive surface.

To evaluate the electromagnetic momentum (1), we first express the electric field for $r < a$ in rectangular coordinates,

$$\begin{aligned} \mathbf{E}(r<a) &= (E_r \cos\phi - E_\phi \sin\phi)\,\hat{\mathbf{x}} + (E_r \sin\phi + E_\phi \cos\phi)\,\hat{\mathbf{y}} \\ &= \frac{V}{\pi} \frac{r\sin\phi\,\hat{\mathbf{x}} - (a + r\cos\phi)\,\hat{\mathbf{y}}}{r^2 + 2ar\cos\phi + a^2} = E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}}. \end{aligned} \tag{14}$$

We combine eqs. (1), (7) and (14) to calculate the electromagnetic momentum per unit length along z of the cylindrical circuit as,

$$\begin{aligned} \mathbf{P}_{\rm EM} &= \int \frac{(E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}}) \times B\hat{\mathbf{z}}}{4\pi c}\, d{\rm Vol} = \int \frac{E_y B\,\hat{\mathbf{x}} - E_x B\,\hat{\mathbf{y}}}{4\pi c}\, d{\rm Vol} \\ &= -\frac{VI}{\pi c^2} \int_0^a r\,dr \int_{-\pi}^{\pi} d\phi\, \frac{(a + r\cos\phi)\,\hat{\mathbf{x}} + r\sin\phi\,\hat{\mathbf{y}}}{r^2 + 2ar\cos\phi + a^2} = \frac{2VI}{ac^2} \int_0^a r\,dr\,\hat{\mathbf{x}} \\ &= \frac{aVI}{c^2}\,\hat{\mathbf{x}}, \end{aligned} \tag{15}$$

using Dwight 859.122 [14]. This is in agreement with the indirect calculation (6) of sec. 2.1.1.

2.1.3 The Canonical Electromagnetic Momentum

The electromagnetic momentum can also be calculated using the concept of the canonical momentum of a charge that interacts with a magnetic field [15], which notion dates back to Faraday[6] and Maxwell [20]. Namely, the electromagnetic part of the momentum associated with a charge distribution ϱ that is immersed in a vector potential $\mathbf{A}$ (in the Coulomb gauge, strictly speaking) is given by,

$$\mathbf{P}_{\rm EM} = \int \frac{\varrho \mathbf{A}}{c}\, d{\rm Vol}. \tag{16}$$

For a resistive circuit to contain current I, there must be a longitudinal electric field inside the wire, and a nonzero surface charge density is needed to shape this electric field. In the case of a wire of radius $b \ll a$, the surface charge density λ per unit length along the wire is approximately,[7]

$$\lambda \approx \frac{V\phi}{\ln(b/a)} \tag{17}$$

The vector potential at the surface of the wire is approximately,

$$\mathbf{A} \approx \frac{I\ln(b/a)}{c}\hat{\mathbf{l}}. \tag{18}$$

[6]Electromagnetic momentum can be identified with the *electro-tonic* state, first discussed by Faraday in Art. 60 of [16]. Other mentions by Faraday of the electrotonic state include Art. 1661 of [17], Arts. 1729 and 1733 of [18], and Art. 3269 of [19].

[7]Compare with an "exact" calculation of the surface charge on the inner conductor of a coaxial cable [1].

The x component of the electromagnetic momentum (16) is then,

$$P_{\mathrm{EM},x} = \int \frac{\lambda A_x}{c}\, dl \approx \int \frac{V\phi}{\ln(b/a)} \frac{I \ln(b/a)}{c} \sin\phi \frac{a\, d\phi}{c} \approx \frac{aVI}{c^2}\,, \tag{19}$$

in agreement with eq. (6).

For the example of sec. 2.1.2 of a cylindrical circuit, the surface charge distribution σ per unit length in z can be related to the radial component of the electric field (10),

$$\sigma(r = a^+) = \frac{E_r(r = a^+)}{4\pi} = \frac{V \tan(\phi/2)}{8\pi^2 a}\,, \quad \sigma(r = a^-) = -\frac{E_r(r = a^-)}{4\pi} = \sigma(r = a^+), \tag{20}$$

noting that $E_r = 0$ inside the conducting cylinder.[8] The vector potential associated with the magnetic field (7) is purely azimuthal, and its value at radius $r = a$ follows from use of the relation $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ and Stokes' theorem,

$$A_\phi(r = a) = -\frac{Ba}{2} = -\frac{2\pi I a}{c}\,. \tag{21}$$

The x component of the electromagnetic momentum per unit length along z is then,

$$\begin{aligned} P_{\mathrm{EM},x} &= \int_{-\pi}^{\pi} \frac{\sigma A_x}{c}\, a\, d\phi = -\int_{-\pi}^{\pi} \frac{\sigma A_\phi \sin\phi}{c}\, a\, d\phi = \frac{aVI}{2\pi c^2} \int_{-\pi}^{\pi} \frac{\sin^2\phi}{1 + \cos\phi}\, d\phi \\ &= \frac{aVI}{2\pi c^2} \int_{-\pi}^{\pi} (1 - \cos\phi)\, d\phi = \frac{aVI}{c^2}\,, \end{aligned} \tag{22}$$

as found previously by other methods.

2.1.4 Energy-Flow Velocity

The (instantaneous) velocity $\mathbf{v}$ of the flow of energy in the electromagnetic field is sometimes taken to be the ratio of the Poynting vector $\mathbf{S} = (c/4\pi)(\mathbf{E} \times \mathbf{B})$ to the energy density $u = (E^2 + B^2)/8\pi$ in the electromagnetic field,[9]

$$\mathbf{v} = \frac{\mathbf{S}}{u} = 2c \frac{\mathbf{E} \times \mathbf{B}}{E^2 + B^2}\,. \tag{23}$$

Note that the magnitude of $\mathbf{v}$ is bounded by,[10]

$$v = |\mathbf{v}| \le c \frac{2EB}{E^2 + B^2} \le c\,, \tag{24}$$

[8] *Oct. 19, 2020.* There exists a tiny correction to the surface charge density due to the effect of centrifugal force on the conduction electrons, as discussed in sec. 4 of [21].

[9] J.J. Thomson developed the notion of field momentum density (1) essentially according to $\mathbf{p} = \mathbf{S}/c^2 = u\mathbf{v}/c^2$ [5, 6]. See also eq. (19), p. 79 of [22], and p. 6 of [23]. The idea that an energy flux vector is the product of energy density and energy flow velocity seems to be due to Umov [24] (1874), based on Euler's continuity equation [25] for mass flow, $\boldsymbol{\nabla} \cdot (\rho\mathbf{v}) = -\partial\rho/\partial t$. Poincaré applied this notion to an *électromagnétic fluide fictif* between eqs. (3) and (4) of [8] (1900). The energy-flow velocity (32) appeared on p. 392 of the textbook [26] and on p. 794 of [27]. See also [28]-[31]. Nonstandard definitions are considered in [32]-[34].

[10] *(Nov. 3, 2020).* Thanks to Oliver Johns for pointing this out.

and that the maximal $v = c$ only occurs when $E = B$ and $\mathbf{E} \perp \mathbf{B}$.

In the present example this is nonzero only inside the cylinder of radius a, where $\mathbf{E}$ is given by eq. (14) and $\mathbf{B}$ is given by eq. (7), and $\mathbf{E} \perp \mathbf{B}$. Then,

$$|\mathbf{E}| = \left| -\frac{V}{\pi} \frac{(a + r\cos\phi)\,\hat{\mathbf{x}} + r\sin\phi\,\hat{\mathbf{y}}}{r^2 + 2ar\cos\phi + a^2} \right| = \frac{V}{\pi\sqrt{r^2 + 2ar\cos\phi + a^2}} = \frac{V}{\pi r'}\,, \tag{25}$$

where $r' = \sqrt{r^2 + 2ar\cos\phi + a^2}$ is the distance from the battery to the observation point. If the resistance of a unit length along the cylinder is R_Ω is Ohms, then in Gaussian units, $R = R_\Omega/30c$, so that,

$$B = \frac{4\pi V}{Rc} = \frac{120\pi V}{R_\Omega}\,, \qquad |\mathbf{E} \times \mathbf{B}| = EB = \frac{V}{\pi r'} \frac{120\pi V}{R_\Omega}\,. \tag{26}$$

Similarly,

$$E^2 + B^2 = V^2 \left(\frac{1}{(\pi r')^2} + \left(\frac{120\pi}{R_\Omega^2} \right)^2 \right) = \frac{V^2 (R_\Omega^2 + (120\pi^2 r')^2)}{R_\Omega^2 (\pi r')^2}\,. \tag{27}$$

Then, the magnitude of the energy-flow velocity (23) is,

$$v = \frac{|\mathbf{S}|}{u} = 2c \frac{|\mathbf{E} \times \mathbf{B}|}{E^2 + B^2} = c \frac{2R_\Omega (120\pi^2 r')}{R_\Omega^2 + (120\pi^2 r')^2} \le c, \tag{28}$$

where $v = c$ on the circular arc $r' = R_\Omega/120\pi^2$ if this is less than $2a$ (and where $E = B = V/\pi r'$).

Of course, the meaning of the energy-flow velocity in a static situation is somewhat unclear; no material entity or signal propagates along lines of $\mathbf{S}$*, although the flow of (immaterial) energy from the battery to resistor results in mass transfer.*[11]

2.2 "Hidden" Mechanical Momentum

In sec. 2.1 we showed by several methods that there is a nonzero electromagnetic momentum associated with a DC circuit that is at rest in the laboratory. If this momentum is related to the kind of momentum familiar in mechanical systems, then we expect the total momentum of a system at rest to be zero. Hence, consistency between field and mechanical momenta requires that the DC circuit at rest contain nonzero mechanical momentum that is equal and opposite to the electromagnetic momentum (6).

A DC circuit does contain moving charge carriers, but it would appear that the total momentum vector associated with this motion is zero for steady currents that flow in closed loops. The challenge, then, is to identify a kind of "hidden" mechanical momentum in the DC circuit.

[11] If the battery were switched on at some time, a transient energy flow with velocity c would propagate out from the battery until the steady state fields (and field energy) are established. For an idealized discussion of transient fields in a solenoid, see [35]. For other examples of energy flow in "static" situations, see [36] and references therein. A Hertzian oscillating dipole has $v > c$ very close to the origin, as noted in sec. 4.3 of [37].

The difficulty in reconciling Newton's third law of mechanics with the physics of moving electric charges was appreciated by Ampère, who concluded that isolated moving charges do not exist, and that all electric currents must be steady [39, 40]. It is generally considered that the introduction of the Poynting vector [2] provides the desired consistency between electromagnetism and mechanics for both time-dependent currents and moving, isolated charges. However, the present example shows that the details of such consistency are subtle.

Concerns of the sort raised by the present example can be traced at least as far back as 1891 to commentary by Thomson [5, 7], but went largely unnoticed until the 1950's [41, 42]. The physical character of "hidden" mechanical momentum (and that name) were first enunciated in 1967 by Shockley and James [43], and further clarified by Coleman and Van Vleck [44].

An important clue is the factor of c^2 that appears in the denominator of eq. (6) which alerts us to the possibility of small relativistic corrections. In particular, it can be that the mechanical momentum of the charges carries in the electrical current varies slightly with position even though the current does not. This is discussed in detail for the closely related example of a batter connected to a resistor by a coaxial cable in [1].

A general discussion of "hidden" momentum is given in [9], where in sec. 4.1.4 it is argued that in a quasistatic example such as a battery plus resistor, a nonzero electromagnetic field momentum is associated with an equal and opposite "hidden" mechanical momentum. The challenge here is to give a physical model for the latter.

2.2.1 Momentum of the Moving Charges

A classical model of electric current, due to Drude [45], is that a conduction charge (electron of charge e) undergoes frequent collisions with the "lattice" of the cylindrical resistor, and that to a first approximation, all kinetic energy gained by an electron prior to a collision is transferred to the lattice during the collision. The electron then accelerates due to the azimuthal electric field inside the cylindrical resistor until the next collision, when its velocity is again reset to zero. The moving electrons in some small azimuthal portion of the cylindrical resistor can be characterized by an average "drift" velocity $v(\phi)$, which might vary slowly with azimuth if the number density $n(\phi)$ (per unit axial length) also changes, such that the average current I (per unit axial length) in the cylindrical resistor is independent of azimuth,

$$I = en(\phi)v(\phi), \qquad (\pi < \phi < \pi). \tag{29}$$

The average energy of the conduction electrons at azimuth ϕ can be written as,

$$U(\phi) = \gamma(\phi)m(\phi)c^2 + e\Phi(\phi), \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{30}$$

where $m(\phi)$ is the effective rest mass of the electron when inside the resistor, which can be different from the rest mass m_e of an electron in "empty" space where the electromagnetic field is zero.[12]

[12]The notion of effective mass of conduction charges is more familiar in quantum theory, where it was introduced in 1929 by Peierls [47] in an extension of Bloch's quantum theory of electrons in crystals [48].

An electron in an intense electromagnetic wave can also be thought of as having an effective mass different from that, m_e of a free electron in zero external field [49].

The delicate argument is that average, effective energy (30) is not actually a function of ϕ, but obeys a conservation law even though the microscopic collisions are dissipative.

If so, we can take the constant energy to be that at $\phi = 0$ where the scalar potential (4) is zero,

$$U(\phi) = U_0 = \gamma_0 m_0 c^2 = \gamma(\phi) m(\phi) c^2 + e\Phi(\phi), \qquad \gamma(\phi) m(\phi) = \gamma_0 m_0 - \frac{eV\phi}{2\pi c^2}, \tag{31}$$

where γ_0 is the Lorentz factor, and m_0 (not m_e) is the effective mass, for a charge at $\phi = 0$ where $\Phi = 0$.

To a good approximation, the drift velocity $v(\phi)$ is independent of ϕ, and its magnitude is typically smaller than 1 cm/s, such that $\gamma(\phi) \approx 1$ to very good accuracy. Then, for batteries with potential of order 1 volt, the effective mass $m(\phi)$ differs from the free-electron rest mass $m_e \approx 511$ keV/c^2 by roughly 1 eV/c^2, *i.e.*, by roughly a part per million. In the simplest classical view of an electric current inside a resistive material, we expect that the effective mass of a conduction electron is exactly m_e, but we should not necessarily expect such a view to be correct at the part-per-million level, particularly as a resistive material is a quantum system.

The net momentum (per unit axial length) of the moving charges has only an x-component,

$$\begin{aligned}
\mathbf{P}_{\rm charges} &= \int_{-\pi}^{\pi} a\, d\phi\, n(\phi)\gamma(\phi)m(\phi)\, v(\phi) \sin\phi\, \hat{\mathbf{x}} = aI \int_{-\pi}^{\pi} d\phi \left(\frac{\gamma_0 m_0}{e} - \frac{V\phi}{2\pi c^2} \right) \sin\phi\, \hat{\mathbf{x}} \\
&= -\frac{aIV}{2\pi c^2} \int_{-\pi}^{\pi} d\phi\, \phi\, \sin\phi\, \hat{\mathbf{x}} = -\frac{aIV}{c^2}\, \hat{\mathbf{x}} = -\mathbf{P}_{\rm EM}\,.
\end{aligned} \tag{32}$$

Thus, the net momentum of the charges is equal and opposite to the field momentum (6), and the total momentum of the system is zero (as expected for a system "at rest").[13]

The momentum (32) of the moving charges in the electric field of the battery is often called the "hidden" mechanical momentum of the system.

The velocity $\mathbf{v}_{\rm ce,charges}$ of the center of mass/energy of the moving charges is related by,

$$M_{\rm charges} \mathbf{v}_{\rm ce,charges} = \mathbf{P}_{\rm charges} = \frac{aIV}{c^2}\, \hat{\mathbf{x}} = -\mathbf{P}_{\rm EM}. \tag{33}$$

The velocity of the center of energy of the other matter in the system is zero, as is the velocity of the center of energy of the (static) electromagnetic fields of the system.

It seems odd that a system "at rest," with zero total momentum, could have a nonzero velocity of its center of energy, $\mathbf{v}_{\rm ce} = M_{\rm charges}\mathbf{v}_{\rm ce,charges}/M_{\rm total}$. However, we have neglected that the battery (at $x = -a$) is transferring energy to the resistor (centered at $x = 0$) at rate IV per unit axial length. As such, the mass of the battery decreases with time, while the mass of the resistor increases, at rates $dM/dt = IV/c^2$ per unit axial length. This mass transfer implies a contribution to the velocity of the center or mass/energy of the system given by,

$$\frac{dM}{dt} \Delta\mathbf{x} = \frac{aIV}{c^2}\, \hat{\mathbf{x}} = -M_{\rm charges} \mathbf{v}_{\rm ce,charges}. \tag{34}$$

[13] A version of this argument was first given on p. 215 of [50], but for a different model of the conductor. See also [51].

Hence, the velocity of the center of mass/energy of the entire system is zero, as expected,[14]

$$M_{\rm total}\mathbf{v}_{\rm ce,total} = M_{\rm charges}\mathbf{v}_{\rm ce,charges} + \frac{dM}{dt}\Delta\mathbf{x} = 0. \tag{35}$$

A final peculiarity is that although the (macroscopic) velocity of the center of mass of the system is zero, the location of the center of mass changes with time. This can be explained my a microscopic view in which with a kind of Zitterbewegung takes place, as illustrated on p. 8 of [1], which discusses a closely related example in greater detail.

2.2.2 An Alternative Computation of "Hidden" Momentum

When one seeks a general definition of "hidden" momentum of a subsystem, one is led to two expressions [9],

$$\mathbf{P}_{\rm hidden} \equiv \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{r} - \mathbf{r}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b)\cdot d\mathbf{Area} = -\int \frac{f^0}{c}(\mathbf{r} - \mathbf{r}_{\rm cm})\,d{\rm Vol}, \tag{36}$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total (relativistic) "mass," U is its total energy, $\mathbf{r}_{\rm cm}$ is its center of mass/energy, $\mathbf{v}_{\rm cm} = d\mathbf{r}_{\rm cm}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its "mass" density, u is its energy density, $\mathbf{v}_b$ is the velocity (field) of its boundary, and,

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x_\nu}\,, \tag{37}$$

is the 4-force density exerted by the subsystem on the rest of the system, with $T^{\mu\nu}$ being the stress-energy-momentum 4-tensor of the subsystem.[15]

In the present example, we consider two subsystems:

1. The macroscopic electromagnetic fields, including the electric field $\mathbf{E}$ inside the conductor, related to the current density by $\mathbf{J} = \sigma\mathbf{E}$, where σ is the electrical conductivity.

2. The rest of the system, nominally its "matter," but including the microscopic electromagnetic fields at the atomic scale.

We can regard these subsystems as unbounded, such that the first expression for the "hidden" momentum simplifies to,

$$\mathbf{P}_{\rm hidden} \equiv \mathbf{P} - M\mathbf{v}_{\rm cm}, \tag{38}$$

where the quantity $M\mathbf{v}_{\rm cm}$ is sometimes called the "overt" momentum of the subsystem.[16]

[14]The contribution (34) to $M_{\rm total}\mathbf{v}_{\rm ce,total}$ was noted in [52], but the ("hidden") momentum (32) was neglected, leading that author to the conclusion that "hidden" momentum was being "misused" by people who don't neglect it.

[15]Important qualifications are that the various quantities in eq. (36) are the result of macroscopic averaging, that the values of $\mathbf{p}$ and ρ at the boundary are the limit of those just inside the boundary, and that the volume integral of f^0 does not include possible delta-functions at the boundary.

[16]The form (38) for "hidden" momentum was advocated in eq. (78) of [53].

One consequence of the definition (38) is that the momentum of subsystem 1, the static electromagnetic field, is to be called a "hidden" momentum, as the velocity of the center of mass/energy of the field is zero (in the rest frame of the cylindrical resistor).

To apply the second from of eq. (36) to the electromagnetic subsystem, note that the top row of the stress-energy-momentum tensor has the form $T^{0\mu}_{\rm EM} = (u_{\rm EM}, c\,\mathbf{p}_{\rm EM}) = (u_{\rm EM}, \mathbf{S}/c)$, and $\mathbf{S}$ is the Poynting vector. Thus, $f^0_{\rm EM} = \partial u_{\rm EM}/\partial ct + \boldsymbol{\nabla}\cdot\mathbf{S}/c = \partial u_{\rm EM}/\partial ct - \partial u_{\rm EM}/\partial ct - \mathbf{J}\cdot\mathbf{E}/c = -\mathbf{J}\cdot\mathbf{E}/c$. Now, $\mathbf{J}\cdot\mathbf{E}\,d{\rm Vol} \to I E_\phi(a,\phi) a\, d\phi$ inside the resistor and battery, where the electric field is given by eq. (5). Hence,

$$\begin{aligned}
\mathbf{P}_{\rm hidden,EM} &= -\int \frac{f^0_{\rm EM}}{c}(\mathbf{r} - \mathbf{r}^{\rm EM}_{\rm ce})\, d{\rm Vol} = \int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\left(\mathbf{r} - \mathbf{r}^{\rm EM}_{\rm ce}\right)\, d{\rm Vol} \\
&= \int_{-\pi}^{\pi} a\, d\phi\, \frac{I E_\phi(a,\phi)}{c^2}\,(a\cos\phi - x^{\rm EM}_{\rm ce})\,\hat{\mathbf{x}} \\
&= 2\int_0^{\pi-\epsilon} \frac{a^2 IV}{2(\pi-\epsilon)ac^2}\cos\phi\, d\phi - 2\int_{\pi-\epsilon}^{\pi} \frac{a^2 IV}{2\epsilon ac^2}\cos\phi\, d\phi \to 0 + \frac{aIV}{c^2}\,\hat{\mathbf{x}} = \mathbf{P}_{\rm EM},
\end{aligned} \tag{39}$$

as $\epsilon \to 0$, noting that $\int_{-\pi}^{\pi} E_\phi(\phi)\, d\phi = 0$ for the electrostatic field.

For subsystem 2, the "matter" of the battery and resistor, it is useful to recall that for an isolated, closed system with total stress-energy-momentum tensor $T^{\mu\nu}$, the 4-divergence of the latter is zero, $\partial T^{\mu\nu}/\partial x^\nu = 0$. Considering the present example to be an isolated closed system, we then have that the stress tensor for the EM and matter subsystems are related by,

$$f^\mu_{\rm EM} = \frac{\partial T^{\mu\nu}_{\rm EM}}{\partial x^\nu} = -\frac{\partial T^{\mu\nu}_{\rm matter}}{\partial x^\nu} = -f^\mu_{\rm matter}, \tag{40}$$

where $f^\mu_{\rm matter}$ is the 4-force density exerted by the electromagnetic field on the matter subsystem.

Then, according to the second form of eq. (36),

$$\begin{aligned}
\mathbf{P}_{\rm hidden,matter} &= -\int \frac{f^0_{\rm matter}}{c}(\mathbf{r} - \mathbf{r}^{\rm matter}_{\rm ce})\, d{\rm Vol} = -\int \frac{f^0_{\rm EM}}{c}(\mathbf{r} - \mathbf{r}^{\rm matter}_{\rm ce})\, d{\rm Vol} \\
&= -\int_{-\pi}^{\pi} a\, d\phi\, \frac{I E_\phi(a,\phi)}{c^2}\,(a\cos\phi - x^{\rm matter}_{\rm ce})\,\hat{\mathbf{x}} = -\frac{aIV}{c^2}\,\hat{\mathbf{x}} = -\mathbf{P}_{\rm EM},
\end{aligned} \tag{41}$$

again using that $\int_{-\pi}^{\pi} E_\phi(\phi)\, d\phi = 0$, such that the result doesn't depend on the position of the center of mass/energy of the subsystem. Thus, we conclude that the matter subsystem contains ("hidden") momentum $-aIV\,\hat{\mathbf{x}}/c^2$ by a different argument than that which led to eq. (32).

The above argument illustrates a general result that for an isolated, closed, quasistatic system with nonzero field momentum $\mathbf{P}_{\rm EM}$ and for which the current density obeys the static form $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$, the rest of the system (the "matter" subsystem) contains "hidden" momentum equal and opposite to the field momentum (which latter is also a "hidden" momentum according to definition (36)). This relation holds whether or not the center of mass/energy of the entire system is at rest.[17]

[17] See sec. 4.1.4 of [9] for more detailed discussion.

References

[1] K.T. McDonald, *"Hidden" Momentum in a Coaxial Cable* (Mar. 28, 2002),
http://kirkmcd.princeton.edu/examples/hidden.pdf

[2] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884),
http://kirkmcd.princeton.edu/examples/EM/poynting_ptrsl_175_343_84.pdf

[3] A. Sommerfeld, *Electrodynamics* (Academic Press, 1952),
http://kirkmcd.princeton.edu/examples/EM/sommerfeld_em_52_sec17.pdf

[4] M. Abraham, *Prinzipien der Dynamik des Elektrons*, Ann. Phys. **10**, 105 (1903),
http://kirkmcd.princeton.edu/examples/EM/abraham_ap_10_105_03.pdf

[5] J.J. Thomson, *On the Illustration of the Properties of the Electric Field by Means of Tubes of Electrostatic Induction*, Phil. Mag. **31**, 149 (1891),
http://kirkmcd.princeton.edu/examples/EM/thomson_pm_31_149_91.pdf

[6] J.J. Thomson, *Recent Researches in Electricity and Magnetism* (Clarendon Press, 1893),
http://kirkmcd.princeton.edu/examples/EM/thomson_recent_researches_in_electricity.pdf

[7] K.T. McDonald, *J.J. Thomson and "Hidden" Momentum* (Apr. 30, 2014),
http://kirkmcd.princeton.edu/examples/thomson.pdf

[8] H. Poincaré, *La Théorie de Lorentz et la Principe de Réaction*, Arch. Neer. **5**, 252 (1900), http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00.pdf
Translation: *The Theory of Lorentz and the Principle of Reaction*,
http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00_english.pdf

[9] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012),
http://kirkmcd.princeton.edu/examples/hiddendef.pdf

[10] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621 (1969),
http://kirkmcd.princeton.edu/examples/EM/furry_ajp_37_621_69.pdf

[11] K.T. McDonald, *Four Expressions for Electromagnetic Field Momentum* (Apr. 10, 2006), http://kirkmcd.princeton.edu/examples/pem_forms.pdf

[12] M.G. Calkin, *Linear Momentum of the Source of a Static Electromagnetic Field*, Am. J. Phys. **39**, 513 (1971),
http://kirkmcd.princeton.edu/examples/EM/calkin_ajp_39_513_71.pdf

[13] M. Heald, *Electric fields and charges in elementary circuits*, Am. J. Phys. **52**, 522 (1984),
http://kirkmcd.princeton.edu/examples/EM/heald_ajp_52_522_84.pdf

[14] H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4$^{\text{th}}$ ed. (Macmillan, 1961), http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf

[15] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Butterworth-Heinemann, 1975), sec. 16, http://kirkmcd.princeton.edu/examples/EM/landau_ctf_71.pdf

[16] M. Faraday, *Experimental Researches in Electricity*, Phil. Trans. Roy. Soc. London **122**, 125 (1832), kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_122_163_32.pdf

[17] M. Faraday, *Experimental Researches in Electricity.—Thirteenth Series*, Phil. Trans. Roy. Soc. London **128**, 125 (1838), http://kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_128_125_38.pdf

[18] M. Faraday, *Experimental Researches in Electricity.—Fourteenth Series*, Phil. Trans. Roy. Soc. London **128**, 265 (1838), http://kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_128_265_38.pdf

[19] M. Faraday, *On the Physical Character of the Lines of Magnetic Force*, Phil. Mag. **3**, 401 (1852), kirkmcd.princeton.edu/examples/EM/faraday_pm_3_401_52.pdf

[20] Secs. 22-24 and 57 of J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865), http://kirkmcd.princeton.edu/examples/EM/maxwell_ptrsl_155_459_65.pdf

[21] K.T. McDonald, *Does Centrifugal Force Affect Conduction Electrons?* (June 25, 2019), http://kirkmcd.princeton.edu/examples/tolman.pdf

[22] O. Heaviside, *Electromagnetic Theory*, Vol. 1 (Electrician Printing Co., 1893), http://kirkmcd.princeton.edu/examples/EM/heaviside_electromagnetic_theory_1.pdf

[23] H. Bateman, *The Mathematical Analysis of Electrical and Optical Wave Motion*, (Cambridge U. Press, 1915), http://kirkmcd.princeton.edu/examples/EM/bateman_wave_motion_15.pdf

[24] N. Umow, *Ableitung der Bewegungsgleichungen der Energie in continuirlichen Körpern*, Zeit. Math. Phys. **19**, 418 (1874), http://kirkmcd.princeton.edu/examples/EM/umow_zmp_19_97,418_74.pdf http://kirkmcd.princeton.edu/examples/EM/umov_theorem.pdf

[25] L. Euler, *Principes généraux du mouvement des fluides*, Acad. Roy. Sci. Belles-Lett. Berlin (1755), http://kirkmcd.princeton.edu/examples/fluids/euler_fluids_55_english.pdf

[26] J.D. Kraus, *Electromagnetics* (McGraw-Hill, 1953, 1973), http://kirkmcd.princeton.edu/examples/EM/kraus_73.pdf

[27] M. Born and E. Wolf, *Principles of Optics*, 7th ed. (Cambridge U. Press, 1959, 1999), http://kirkmcd.princeton.edu/examples/EM/born_wolf_7ed.pdf

[28] M.A. Biot, *General Theorems on the Equivalence of Group Velocity and Energy Transport*, Phys. Rev. **105**, 1129 (1957), http://kirkmcd.princeton.edu/examples/EM/biot_pr_105_1129_57.pdf

[29] D.V. Geppert, *Energy-Transport Velocity in Electromagnetic Waves*, Proc. IEEE **53**, 1790 (1965), http://kirkmcd.princeton.edu/examples/EM/geppert_pieee_53_1790_65.pdf

[30] E.O. Schulz-DuBois, *Energy Transport Velocity of Electromagnetic Propagation in Dispersive Media*, Proc. IEEE **57**, 1748 (1969),
http://kirkmcd.princeton.edu/examples/EM/schulz-dubois_pieee_57_1748_69.pdf

[31] C.T. Sebens, *Forces on fields*, Stud. Hist. Phil. Mod. Phys. **63**, 1 (2018),
http://kirkmcd.princeton.edu/examples/EM/sebens_shpmp_63_1_18.pdf

[32] J.W. Butler, *A Proposed Electromagnetic Energy-Momentum for Charged Bodies*, Am. J. Phys. **37**, 1258 (1969), http://kirkmcd.princeton.edu/examples/EM/butler_ajp_37_1258_69.pdf

[33] K. Widin, *On the Energy Transport Velocity of the Electromagnetic Field*, Proc. IEEE **60**, 240 (1972), http://kirkmcd.princeton.edu/examples/EM/widin_pieee_60_240_72.pdf

[34] O.D. Johns, *Relativistically Correct Electromagnetic Energy Flow*, Prog. Phys. **17**, 3 (2021), http://kirkmcd.princeton.edu/examples/EM/johns_pp_17_3_21.pdf

[35] K.T. McDonald, *The Fields Outside a Solenoid with a Time-Dependent Current* (Dec. 6, 1996), http://kirkmcd.princeton.edu/examples/solenoid.pdf

[36] K.T. McDonald, *Relativity of Steady Energy Flow* (Nov. 3, 2007),
http://kirkmcd.princeton.edu/examples/1dgas.pdf

[37] K.T. McDonald, *Flow of Energy and Momentum in the Near Zone of a Hertzian Dipole* (Apr. 11, 2007), http://kirkmcd.princeton.edu/examples/hertzian_momentum.pdf

[38] K.T. McDonald, *Decomposition of Electromagnetic Fields into Electromagnetic Plane Waves* (July 11, 2010), http://kirkmcd.princeton.edu/examples/virtual.pdf

[39] A.M. Ampère, *Mémoire sur la détermination de la formule qui représente l'action mutuelle de deux portions infiniment petites de conducteurs voltaïques*, Ann. Chem. Phys. **20**, 398, 422 (1822),
http://kirkmcd.princeton.edu/examples/EM/ampere_acp_20_398_22.pdf

[40] The most extensive discussion in English of Ampère's attitudes on the relation between magnetism and mechanics is given in R.A.R. Tricker, *Early Electrodynamics, the First Law of Circulation* (Pergamon, Oxford, 1965),
http://kirkmcd.princeton.edu/examples/EM/vanbladel_rel_eng.pdf
Another historical survey is by O. Darrigol, *Electrodynamics from Ampère to Einstein* (Oxford U.P., 2000), http://kirkmcd.princeton.edu/examples/EM/darrigol_em_00.pdf
See also sec. IIA of J.D. Jackson and L.B. Okun, *Historical roots of gauge invariance*, Rev. Mod. Phys. **73**, 663 (2001),
http://kirkmcd.princeton.edu/examples/EM/jackson_rmp_73_663_01.pdf

[41] E.G. Cullwick, Nature **170**, 425 (1952),
http://kirkmcd.princeton.edu/examples/EM/cullwick_nature_170_425_52.pdf

[42] E.G. Cullwick, *Electromagnetism and Relativity* (Longmans, Green & Co., London, 1959), pp. 232-238, http://kirkmcd.princeton.edu/examples/EM/cullwick_EM_relativity_59.pdf

[43] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on Magnetic Currents*, Phys. Rev. Lett. **18**, 876 (1967), http://kirkmcd.princeton.edu/examples/EM/shockley_prl_18_876_67.pdf

[44] S. Coleman and J.H. Van Vleck, *Origin of "Hidden Momentum" Forces on Magnets*, Phys. Rev. **171**, 1370 (1968), http://kirkmcd.princeton.edu/examples/EM/coleman_pr_171_1370_68.pdf

[45] P. Drude, *Zur Elektronentheorie der Metalle*, Ann. d. Phys. **1**, 566 (1900); **3**, 369 (1900); **7**, 687 (1902), http://kirkmcd.princeton.edu/examples/EM/drude_ap_1_566_00.pdf
http://kirkmcd.princeton.edu/examples/EM/drude_ap_3_369_00.pdf
http://kirkmcd.princeton.edu/examples/EM/drude_ap_7_687_02.pdf

[46] L. Brillouin, *The Actual Mass of Potential Energy, A Correction to Classical Relativity*, Proc. Nat. Acad. Sci. **54**, 475, 1280 (1965),
http://kirkmcd.princeton.edu/examples/EM/brillouin_pnas_53_475_65.pdf
http://kirkmcd.princeton.edu/examples/EM/brillouin_pnas_53_1280_65.pdf

[47] R. Peierls, *Zur Theorie der galvanomagnetischen Effekte*, Z. Phys. **53**, 255 (1929), http://kirkmcd.princeton.edu/examples/QM/peierls_zp_53_255_29.pdf

[48] F. Bloch, *Über die Quantenmechanik der Elektronen in Kristallgittern*, Z. Phys. **52**, 555 (1928), http://kirkmcd.princeton.edu/examples/QM/bloch_zp_52_555_28.pdf

[49] T.W.B. Kibble, *Refraction of Electron Beams by Intense Electromagnetic Waves*, Phys. Rev. Lett. **16**, 1054 (1966), http://kirkmcd.princeton.edu/examples/QED/kibble_prl_16_1054_66.pdf

[50] P. Penfield, Jr and H.A. Haus, *Electrodynamics of Moving Media*, p. 215 (MIT, 1967), http://kirkmcd.princeton.edu/examples/EM/penfield_haus_chap7.pdf

[51] K.T. McDonald, *"Hidden" Momentum in a Current Loop*, (June 30, 2012), http://kirkmcd.princeton.edu/examples/penfield.pdf

[52] T.H. Boyer, *Concerning hidden momentum*, Am. J. Phys. **76**, 190 (2008), http://kirkmcd.princeton.edu/examples/EM/boyer_ajp_76_190_08.pdf

[53] V. Hnizdo, *Hidden mechanical momentum and the field momentum in stationary electromagnetic and gravitational systems*, Am. J. Phys. **65**, 515 (1997), http://kirkmcd.princeton.edu/examples/EM/hnizdo_ajp_65_515_97.pdf