Can a Bicycle Speed Up by Leaning?

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

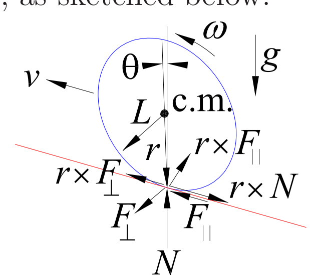
It was claimed in Prob. 6.6 of [1] that a bicycle speeds up when it is leaned (without pedaling or steering). Can this be so?

2 Solution

YES, but a naïve application of Newtonian mechanics suggests that the answer is NO.

An increase in the forward velocity of the point of contact of the bicycle with the ground suggests that the center of mass of the bicycle is accelerating in the forward direction. That would require a force on the bicycle in the forward direction, due to (static) friction at the contact points of the wheels with the ground. The torque about the center of mass of the bicycle due to this friction force would point to the right, implying that the angular velocity of the wheels would be decreasing, rather than increasing. This apparent contradiction suggests that the forward velocity of the bicycle could not change during the leaning.

To resolve this issue, it is easier to simplify the "bicycle" to a single wheel, with its center of mass at the center of the wheel, as sketched below.[1]



An early, detailed analysis of rolling without slipping of a thin disc on a horizontal surface was given by Routh in Art. 244 of [2], and was discussed by the present author in [3].

A subtle consequence of the condition of rolling without slipping is noted on p. 4 of [3]. Namely, if the disk is falling/leaning to the left (θ increasing in the figure above) while the trajectory of the disk curves to the left, then the acceleration of the center of mass along the forward direction can be negative. Then, the friction force points backwards (with also a component to the left to push the disk into a left turn), and the torque due to the friction increases the angular velocity of the disk about its axis, as required for an increasing forward

[1] The wheel leans to the left with respect to the forward velocity $\mathbf{v}$ by a small angle θ to the vertical.

It seems that the increase in the forward velocity $\mathbf{v}$ must be caused by a frictional contact force $\mathbf{F}_{\parallel}$ between the ground and the wheel, which points in the forward direction. The torque about the center of mass due to this contact force is $\mathbf{r}\times\mathbf{F}_{\parallel}$, where $\mathbf{r}$ points from the c.m. to the contact point. The direction of this torque is opposite to the angular momentum vector $\mathbf{L}$, *i.e.*, to the right, and so reduces the angular momentum, rather than increases it as would happen if the forward velocity increased during the lean.

velocity, ωr, of the point of contact on the ground of the wheel.[2] Meanwhile, the velocity of the center of mass of the bicycle decreases. We can say that the bicycle "speeds up during leaning", while angle θ is changing, if we refer to the speed of its point of contact on the ground rather than the speed of the center of mass.

A special case is possible, that the disk does not turn during the lean, but simply wobbles to the left and right while moving directly forward. In this case, there is no change in the forward velocity during the wobbling (and the gravitational potential energy liberated during the wobble goes entirely into changes of the rotational kinetic energy of the disk.

A Appendix: Comments on Angular Momentum

A.1 The System of Bicycle + Earth

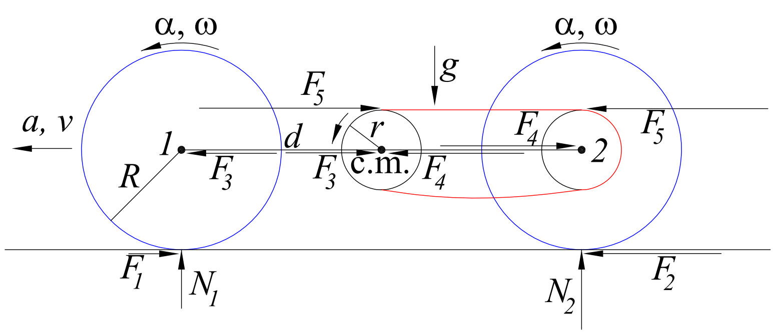
If we consider the isolated system of the bicycle plus the Earth, the total angular momentum of this system is constant. Then, pedaling of the bicycle, initially at rest with respect to the Earth, gives it nonzero angular momentum, which must be compensated by an equal and opposite change in the angular momentum of the Earth. The latter effect is very small compared to the total angular momentum of the Earth, and is reasonably neglected in all discussions of the motion of bicycles.

A.2 Analysis of the Change in Angular Momentum of the Bicycle

We first give an analysis that emphasizes the angular momenta of the two wheels of the bicycle, and then we consider the bicycle as a whole. In both of these analyses, the bicycle does not lean, and moves along a straight line.

A.2.1 Analysis of the Angular Momenta of the Two Wheels

We use a highly simplified model of the bicycle, in which the only mass is in the two wheels, each of mass m, radius R, and moments of inertia mR^2 about their centers. The centers of the wheels are distance $2d$ apart, and in this approximation the center of mass of the bicycle is at the midpoint of the line of centers of the two wheels, as sketched below.



We find that when the bicycle is accelerating forward (while the moving in a straight line without leaning), the friction force on the front wheel is backwards, while the friction force

[2]Of course, the velocity of the point on the wheel that is instantaneously in contact with the ground has zero velocity with respect to the ground.

on the real/drive wheel is forwards, and three large larger that the friction force on the front wheel.

The velocity **v** and acceleration **a** of the bicycle are both to the left in the figure. The angular velocity ω and the angular acceleration α of both wheels are counterclockwise. The condition that the wheels roll without slipping is that,

$$\omega = \frac{v}{R}, \qquad \alpha = \frac{a}{R}. \tag{1}$$

The contact forces of the two wheels with the ground have horizontal components F_i, and vertical components N_i for $i = 1, 2$, with directions as sketched in the figure, which anticipates that F_1 points to the right while F_2 points to the left.

We suppose the (massless) drive gear wheel of the bicycle, of radius $r < R$, is centered on the center of mass. It is connected by a (massless) chain to a (massless) gear wheel, also of radius r on the rear wheel, 2, of the bicycle.

The force in the horizontal strut between the center of mass and the center of wheel 1(2) is $F_{3(4)}$, and the tension in the upper segment of the chain between the drive gear wheel and the gear wheel on wheel 2 is F_5.

The force balance at the (massless) drive gear wheel is that,

$$F_3 - F_4 + F_5 = 0. \tag{2}$$

We suppress the details of the mechanism (involving a person) of the drive gear wheel.

The wheels move horizontally, and the normal forces N_i do not enter into the analysis of the motion of the two wheels of the bicycle.

For wheel 1, $\mathbf{F} = m\mathbf{a}$ is simply,

$$F_3 - F_1 = ma, \tag{3}$$

and the torque equation about the center of mass of wheel 1 is,

$$\tau_1 = RF_1 = I_1\alpha = mR^2\frac{a}{R} = mRa, \tag{4}$$

and hence, recalling eq. (2),

$$F_1 = ma, \qquad F_3 = 2ma = F_5 - F_4. \tag{5}$$

For wheel 2, $\mathbf{F} = m\mathbf{a}$ tells us that,

$$F_2 - F_4 + F_5 = ma, \tag{6}$$

and the torque equation about the center of mass of wheel 2 is,

$$\tau_2 = rF_5 - RF_2 = I_2\alpha = mR^2\frac{a}{R} = mRa, \tag{7}$$

and hence, recalling eq. (5),

$$F_2 = ma + F_5 - F_4 = 3ma, \qquad F_5 = 4ma\frac{R}{r}, \qquad F_4 = F_2 + F_5 - ma = 2ma\left(1 + \frac{2R}{r}\right). \tag{8}$$

This confirms that F_1 points to the right while $F_2 = 3F_1$ points to the right.

That is, the drive mechanism causes the rear wheel to rotate counterclockwise, leading to the reaction force F_2 that points to the left. The only torque on the front wheel is due to the frictional force F_1, which must point to the right for the angular velocity of this wheel to increase as the bicycle accelerates to the left.

A.2.2 Analysis of the Bicycle as a Whole

Considering the bicycle as a whole, the only relevant (external) forces are F_1, F_2, N_1, N_2 and the downward force of gravity, $2mg$.

The horizontal force equation is,

$$2ma = F_2 - F_1, \tag{9}$$

which is consistent with eqs. (5) and (8).

The vertical equation of motion is simply that,

$$N_1 + N_2 = 2mg. \tag{10}$$

The torque equation about the center of mass of the system is,

$$\tau_{\rm cm} = \frac{dL_{\rm cm}}{dt} = 2I\alpha = 2mR^2 \frac{a}{R} = 2mRa = (F_1 - F_2)R + (N_2 - N_1)d, \tag{11}$$

$$N_2 - N_1 = \frac{1}{d}[2mRa + (F_2 - F_1)R] = \frac{4mRa}{d}, \tag{12}$$

$$N_1 = mg - \frac{2maR}{d}, \qquad N_2 = mg + \frac{2maR}{d}. \tag{13}$$

The normal force on the front wheel is less than that on the front, as familiar with rapidly accelerating cars, for which the front end tends to rise.[3] See, for example, sec. 2.2 of [4].

This problem was suggested by Ralph Wang. Thanks to Jason Moore for e-discussions of this issue.

References

[1] M. Levi, *Why Cats Land on Their Feet, and 76 Other Physical Paradoxes and Puzzles*, (Princeton U. Press, 2012), http://kirkmcd.princeton.edu/examples/mechanics/levi_12.pdf

[2] E.J. Routh, *The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 6^{th} ed. (Macmillan, 1905; reprinted by Dover Publications, 1955), http://kirkmcd.princeton.edu/examples/mechanics/routh_advanced_rigid_dynamics.pdf

[3] A.J. McDonald and K.T. McDonald, *The Rolling Motion of a Disk on a Horizontal Plane* (Mar. 8, 2001), http://kirkmcd.princeton.edu/examples/rollingdisk.pdf

[4] K.T. McDonald, *Rocket Car* (Oct. 1, 2012), http://kirkmcd.princeton.edu/examples/rocketcar.pdf

[3] Conversely, for rapidly decelerating cars (and bicycles) the rear end tends to rise.